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| **RESEARCH ARTICLE**

A Modified Chen Inverse Rayleigh Distribution, Estimation and Application to Survival Data

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| ABSTRACT
The difficulties of modeling failure rates that exhibit both monotonic and non-monotonic behavior, as well as appropriately capturing the bathtub curve in reliability engineering, remains a significant difficulty. In study, paper a Chen Inverse Raleigh (CIR) distribution within the context of the Chen-G family was developed, to eliminate the difficulty of modeling failure rates. The Probability Distribution (PDF), Cumulative Distribution Function (CDF) and the statistical properties of these CIR distributions like the survival function, hazard function, cumulative hazard function, reverse hazard function, quartile function, skewness, kurtosis, moments, linear representation and Maximum Likelihood Estimation (MLE) was also provided in the study. Our Proposed distribution provided a mild to moderate right-skewedness, while majority of the data reveals mild to moderate left-skewness, it also models leptokurtic data (heavy-tailed distributions), with kurtosis values indicating effectiveness and consistent tail behavior. A simulation study written in R provides a comparison for the CIRDs with different parameters of (β, λ, δ) and sample sizes (n) = 50, 100, 200, and 500 proved that our model converges successfully for all initial parameter settings across all sample sizes, which confirmed the robustness of the MLE optimization process. The consistency in convergence, even for smaller datasets, highlights the CIR model's reliability and adaptability to different data conditions. The real-life data utilized to validate the efficiency of our proposed model was the survival times (in days) for patients diagnosed with head and neck cancer whose values range from 12.20 to 1776. Result showed that the CIR model demonstrates the best fit, with the lowest AIC and BIC values among the competing models. Additionally, the CIR model achieves higher p-values in goodness-of-fit tests indicating excellent agreement between the model and the observed data.

| KEYWORDS
Chen Inverse Raleigh Distribution (CIRD), Chen-G family, Hazard Function, Cumulative Distribution Function (CDF), Probability Distribution Function (PDF)

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1. Introduction
Statistical probability distributions serve as an important foundation for both theoretical frameworks and practical implementations in statistical techniques. They are crucial in assisting statistical thinking and decision-making because they serve as the foundation for many parametric statistical procedures such as modeling, survival analysis, reliability analysis, and inference. The difficulties of modeling failure rates that exhibit both monotonic and non-monotonic behavior, as well as appropriately capturing the bathtub curve in reliability engineering, remains a significant difficulty. Traditional probability distributions frequently fail to convey the intricate patterns inherent in real-world circumstances with varying failure rates. These issues are addressed by the proposed Chen Inverse Rayleigh probability distribution. It is specifically interested in the subtle behaviors of failure rates, allowing for a more precise representation of both monotonic and non-monotonic patterns. This capacity is critical for applications where failure rates exhibit complex patterns those traditional distributions may miss. Furthermore, the propose distribution address the necessity for accurate modeling of the bathtub curve, a key topic in reliability engineering. The bathtub curve depicts the three phases of a device's failure rates across its lifecycle: lowering rates during the

first phase (infant mortality), relatively constant rates during the steady-state phase, and increasing rates during the wear-out phase. Exact modeling of this curve is required for forecasting and improving device reliability.

Akpojaro & Aronu (2024) introduced a variant of the generalized Chen (GC) distribution with parameters α, β, γ, δ, and λ, termed the Modified Generalized Chen (MGC) distribution. The study evaluated several Chen distribution variants theoretically, focusing on the MGC distribution. Variants considered include the Generalized Chen (GC), Half-Cauchy Chen (HCC), New Extended Chen (NEC), and Exponentially Generalized Modified Chen (EGMC) distributions. Chen (2000) proposed a two-parameter distribution with an increasing failure rate function or bathtub shape. Because it is associated with accurate confidence intervals and joint confidence regions for the parameters, this distribution is favorable (Cordeiro, *et al.*, 2014). The exponentiated Chen distribution was introduced in an effort to increase flexibility and offer new hazard shapes by utilizing the family of distributions proposed by Chaubey and Zhang (2015) (Chaubey & Zhang, 2015; Nadarajah, *et al.,* 2012). Chen-geometric and Marshall-Olkin Chen distributions have similar parameters to the Chen distribution (Chen, 2000; Pappas, Adamidis, & Loukas, 2011). Following that, other compounding distributions, such as the Chen-logarithmic distribution, were investigated, thereby expanding the parameter space of the logarithmic distribution (Pappas *et al.,* 2011). As recorded in the literature, numerous variations of the Chen distribution have been presented in recent times (Anafo, *et al.* 2022; Joshi & Pandit, 2018; Reis, *et al.,* 2020; Joshi & Kumar, 2021).

The rationale to develop the distribution stems from a desire for more accuracy in depicting various failure rate patterns and resolving the difficulties of the bathtub curve in reliability engineering applications. This study is part of a larger effort to improve probability distribution modeling capabilities, particularly in cases with non-normal data exist. The proposed Chen Inverse Rayleigh Distribution aims to give a more flexible and versatile model, it will enable a better fit to datasets with properties that are not covered by standard distributions. It is part of a larger trend in statistical research that recognizes the diverse and complex character of real-world data, highlighting the importance of distributions that can handle such complexities. The iterative process of incorporating new parameters and methodologies exhibited in these improvements reflects the ongoing pursuit of more accurate and adaptable statistical modeling tools across a wide range of fields.

This study therefore aimed to development a Chen Inverse Raleigh (CIR) distribution within the context of the Chen-G family. The study also intends to establish the statistical properties and compare the distribution with that of existing models.

2.0 Materials and Methods

2.1 PDF and CDF of the Proposed Chen Inverse Rayleigh (CIR) Distribution

Within the T-X framework, the random variable T acts as a 'transformer,' allowing the random variable X to be converted into a new set of generalized X distributions. Bourguignon, *et al.* (2014) defines the cumulative distribution function (CDF) of this generalized family as the following:

The distribution function $F_Z(x)$ is given by:

$$F_Z(x) = \int_a^{\frac{G(x)}{1-G(x)}} f_T(t)\,dt = F_T(Q_Y(F_R(x))) \tag{1}$$

The function $G(x)$ is defined as:

$$G(x) = 1 - e^{\delta(1-e^{x^\beta})}$$

Otoo *et al*., (2023) define the cumulative distribution function $F(x)$ for the Chen family for (3.1) as;

$$F(x) = \left[1 - \exp\left(\delta\left(1 - \exp\left[\left(\frac{G(x)}{1-G(x)}\right)^\beta\right]\right)\right)\right] \tag{2}$$

where δ and β are the extra shape parameters. Differentiating (2), Otoo *et al*., (2023) obtained the PDF of the chen family of distributions as

$$f(x) = \delta\beta g(x;\psi)G(x;\psi)^{\beta-1}[1-G(x;\psi)]^{-(\beta-1)}$$

$$\times \exp\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^\beta \exp\left(\delta\left(1 - \exp\left(1 - \frac{G(x;\psi)}{1-G(x;\psi)}\right)\right)^\beta\right) \tag{3}$$

The Rayleigh distribution is derived from the two-parameter Weibull distribution and is a suitable model for life-testing investigations. A transformation of random variables shows that if a random variable T has a Rayleigh distribution, the corresponding random variable $X = (\frac{1}{T})$ will have an inverse Rayleigh distribution (IRD). Rosaiah & Kantam (2005) give reliability sampling plans for the Inverse Rayleigh Distribution (IRD). The PDF and CDF of a random variable X with Inverse Rayleigh distribution and scale parameter λ are given by:

$$g(x,\lambda) = \frac{2\lambda^2}{x^3} e^{-(\lambda/x)^2} \quad ; \quad x, \lambda > 0 \tag{4}$$

$$G(x,\lambda) = e^{-(\lambda/x)^2} \tag{5}$$

substituting (5) into (2), we obtain the CDF of the proposed probability density function CIRD(β, λ, δ)

$$F(x) = \left[1 - \exp\left(\delta\left(1 - \exp\left[\frac{\exp(-(\lambda/x)^2)}{1-\exp(-(\lambda/x)^2)}\right]\right)^{\beta}\right)\right] \tag{6}$$

And this can be further expressed as

$$F(x) = 1 - \exp\left(\delta\left(1 - \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right)\right) \tag{7}$$

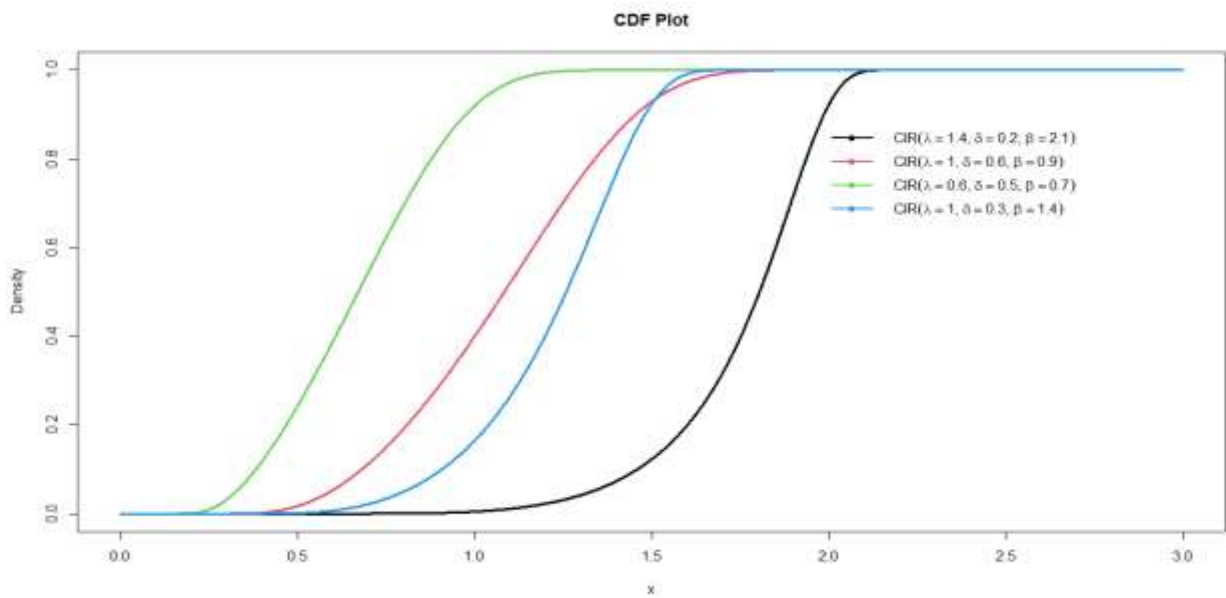


Figure 1: CDF Plot of a probability distributions CIRD(β, λ, δ)

Figure 1, shows the CDF plot for the proposed probability distribution CIRD(β, λ, δ) for different paramter values. Substituting (5) and (4) into (3), we obtain the PDF of proposed probability density function CIRD(β, λ, δ) as

$$f(x) = \frac{2\delta\beta\lambda^2}{x^3} exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\left\{exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{\beta-1}\left\{1 - exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1} \exp\left[exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta} exp\left(\delta\left(1 - \exp\left[exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right)\right)$$

which can be rewritten as

$$f(x) = \frac{2\delta\beta\lambda^2}{x^3}\left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{\beta}\left[1 - \exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{-\beta-1} \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}$$
$$\times \exp\left(\delta\left(1 - \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right)\right) \tag{8}$$

Figure 2, displayed the PDF plot for the proposed probability distribution CIRD(β, λ, δ) for different paramter values.

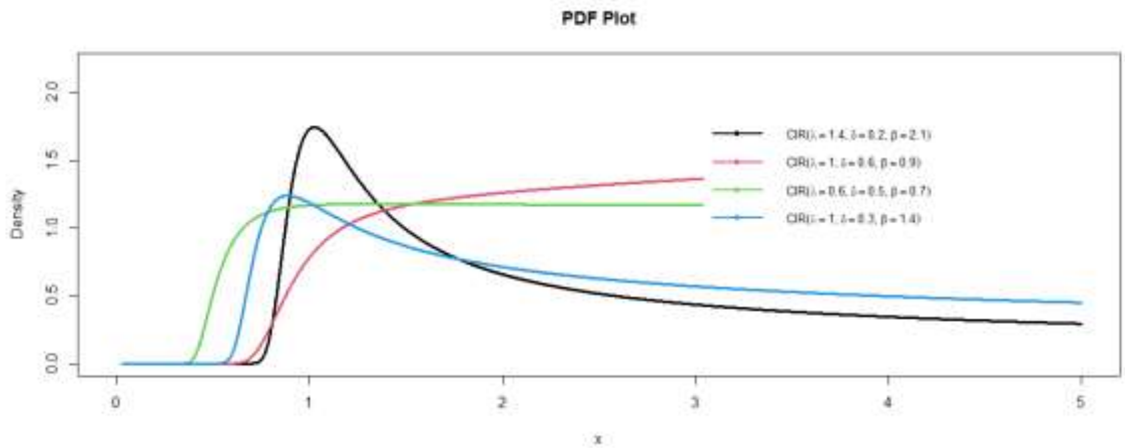


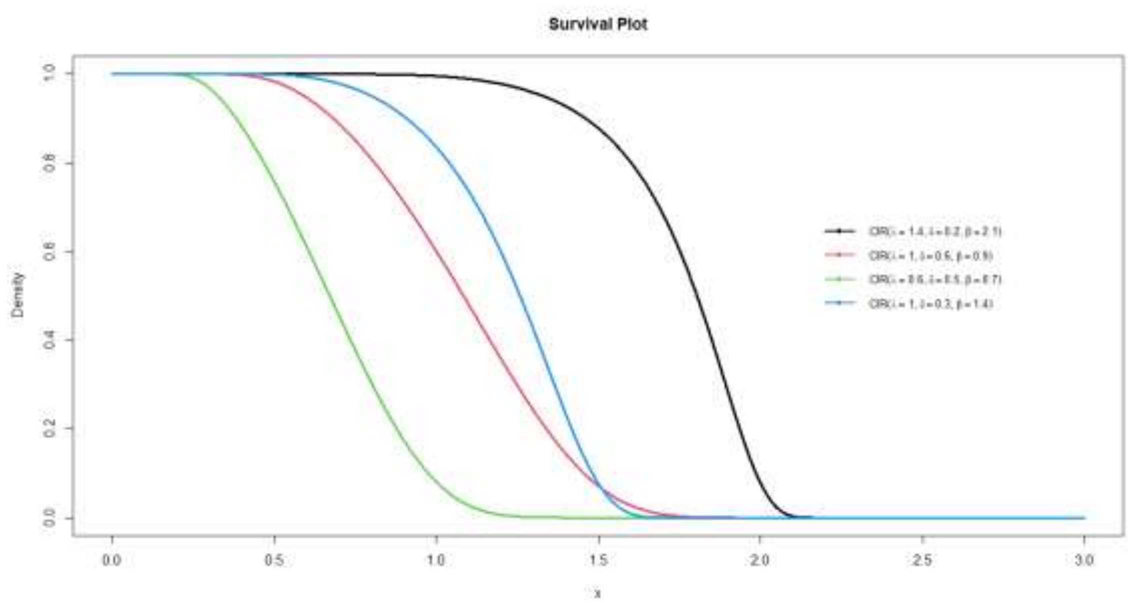
Figure 2: PDF Plot of a probability distribution CIRD(β, λ, δ)

2.2 Survival Function of the CIR Distribution

The survival function, $S_X(x)$, for a continuous random variable X that follows the probability density function CIRD(β, λ, δ) is obtained by substituting (7) into (9).

$$S_X(x) = 1 - F(x) \tag{9}$$

So therefore, the survival function is expressed as;



$$S_X(x) = \exp\left(\delta\left(1 - \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right)\right) \tag{10}$$

Figure 3: Survival Plot of a probability distributions CIRD(β, λ, δ)

Figure 3 displayed the survival plot for the proposed probability distribution CIRD(β, λ, δ) for different paramter values.

2.3: Hazard Function of CIR Distribution

The mathematical expression for hazard function $h_X(x)$ for any probability distribution function is given as:

$$h_X(x) = f_X(x)/S_X(x). \tag{11}$$

To obtain the $h_X(x)$ the probability density function CIRD(β, λ, δ), equations (8) and (10) are subtituted into (11)

$$h_X(x) = \frac{2\delta\beta\lambda^2}{x^3}\left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{\beta}\left[1-\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{(-\beta-1)} \exp\left[\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right]^{-\beta} \tag{12}$$

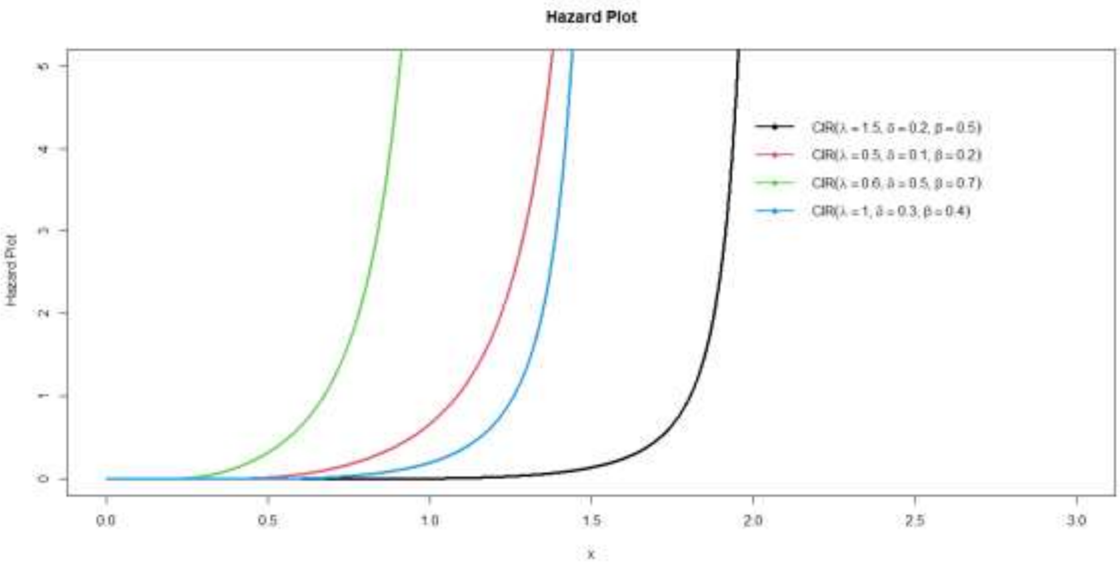


Figure 4: Hazard Plot of a Probability Distributions CIRD(β, λ, δ)

Figure 4, shows the hazard plot for the proposed probability distribution CIRD(β, λ, δ) for different paramter values.

2.4 Cumulative Hazard Function of CIR Distribution

The cumulative hazard function of a random X denoted as $H(x) = \Delta h(x)$, is given by:

$$H(x) = -\log(S(x)). \tag{13}$$

The cumulative hazard function for the probability density function CIRD(β, λ, δ) is obtained by substituting (10) into (13)

$$H(x) = -\log\left(\exp\left(\delta\left(1-\exp\left[\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right]\right)\right)\right) \tag{14}$$

$$H(x) = -\delta\left(1-\exp\left[\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right]\right) \tag{15}$$

2.5: Quantile Function, Median, Skewness, Kurtosis and Mode of CIRD

2.5.1 Quantile of CIRD

If X is a random variable that has CIRD $(x, \beta, \lambda, \delta)$ and let $Q_X(p)$ denote the quantile function for the CIRD, such that $0 \le p \le 1$, then $Q_X(p)$ is given by:

$$Q_X(p; \beta, \lambda, \delta) = \frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\left(\ln\left(1-\frac{\ln(1-p)}{\delta}\right)\right)^{1/\beta}}\right)}} \tag{16}$$

Proof: Let p = F(X)

$$p = 1-\exp\left(\delta\left(1-\exp\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right)\right)$$

$$\exp\left(\delta\left(1-\exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right)\right)\right)=1-p$$

$$\delta\left(1-\exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right)\right)=\ln(1-p)$$

$$1-\exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right)=\frac{\ln(1-p)}{\delta}$$

$$-\exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right)=1-\frac{\ln(1-p)}{\delta}$$

$$\exp\left(\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}\right)=\frac{\ln(1-p)}{\delta}-1$$

$$\left(\exp\left(\frac{\lambda}{x}\right)^2-1\right)^{-\beta}=\ln\left(1-\frac{\ln(1-p)}{\delta}\right)$$

$$\left(\frac{1}{\exp\left(\frac{\lambda}{x}\right)^2-1}\right)^{\beta}=\ln\left(1-\frac{\ln(1-p)}{\delta}\right)$$

$$\exp\left(\frac{\lambda}{x}\right)^2-1=\frac{1}{\ln\left(1-\frac{\ln(1-p)}{\delta}\right)^{1/\beta}}$$

$$\exp\left(\frac{\lambda}{x}\right)^2=1+\frac{1}{\ln\left(1-\frac{\ln(1-p)}{\delta}\right)^{1/\beta}}$$

$$\left(\frac{\lambda}{x}\right)^2=\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(1-p)}{\delta}\right)^{1/\beta}}\right)$$

$$Q_X(p)=\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(1-p)}{\delta}\right)^{1/\beta}}\right)}} \quad (17)$$

We get the first three quantiles, $Q_1=Q(1/4)$, $Q_1=Q(1/2)$ and $Q_3=Q(3/4)$, by putting $p=0.25$,$p=0.5$ and $p=0.75$ into X_p, respectively. Quantiles also help calculate the distribution's skewness, median and kurtosis.

2.5.2: Median of CIRD

To obtain, the median of the probability density function CIRD (x,β,λ,δ), we substitute p=0.5 in (3.17), we have

$$Q_X(0.5)=\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(1-0.5)}{\delta}\right)^{1/\beta}}\right)}} \quad (18)$$

2.5.3: Skewness of CIRD

The skewness and kurtosis coefficients are frequently computed using distribution moments. However, in this study, we will use skewness (S) metrics based on quantiles proposed by Galton(1983).

The mathematical expression is expressed as

$$S=\frac{Q(6/8)-2Q(4/8)+Q(2/8)}{Q(6/8)-Q(2/8)} \quad (19)$$

Substituting the appropriate value of p into (17) and substituting the results into (19) results into

$$S=\frac{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-2\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.5)}{\delta}\right)^{1/\beta}}\right)}}+\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}}{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}} \qquad (20)$$

2.5.4: Kurtosis of CIRD

The kurtosis coefficients are frequently computed using distribution moments. However, in this study, we will use kurtosis (K) metrics based on quantiles proposed by Moors (1988). Thus, the mathematical expression is expressed as

$$K=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)} \qquad (21)$$

$$K=\frac{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.125)}{\delta}\right)^{1/\beta}}\right)}}}{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}}$$

$$-\frac{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.375)}{\delta}\right)^{1/\beta}}\right)}}}{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}}$$

$$+\frac{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.625)}{\delta}\right)^{1/\beta}}\right)}}}{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}}$$

$$-\frac{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.875)}{\delta}\right)^{1/\beta}}\right)}}}{\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.75)}{\delta}\right)^{1/\beta}}\right)}}-\frac{\lambda}{\sqrt{\ln\left(1+\frac{1}{\ln\left(1-\frac{\ln(0.25)}{\delta}\right)^{1/\beta}}\right)}}} \qquad (22)$$

2.6: Linear Representation of CIRD

Theorem 1: Let X be a random variable that follows a CIR distribution with parameters δ, β and λ then the linear representation is a modified Inverse Rayleigh distribution with the form

$$f(x)=\left(\frac{2\delta\beta\lambda^2}{x^3}\right)\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!\,j!\,m!}$$

$$\exp\left[-\frac{(n+\beta-ijk+km+ij)\lambda^2}{x^2}\right]$$

where

$$\Phi(x)=\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\beta\delta^{m+1}}{j!\,j!\,m!}$$

Proof.

Given the PDF of the CIR distribution with parameters δ, β and λ

$$f(x)=\frac{2\delta\beta\lambda^2}{x^3}\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\left\{\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{\beta-1}\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right\}^{-\beta-1}$$

$$\times\exp\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\exp\left(\delta\left(1-\exp\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right)\right)$$

using the Taylor series expansion procedure, the exponential expression can be rewritten as

$$\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta}{i}\left[e^{\left(\frac{\lambda}{x}\right)^2}\right]^i=\sum_{i=0}^{\infty}(-1)^i\binom{\beta}{i}\left[e^{\left(\frac{\lambda}{x}\right)^2}\right]^i \tag{23}$$

and exponential expression

$$\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta+1}{i}\left[e^{i\left(\frac{\lambda}{x}\right)^2}\right] \tag{24}$$

expression (23), can be further expanded

$$e^{\left[\exp\left(\frac{\lambda}{x}\right)^2\right]^i}=\sum_{j=0}^{\infty}\frac{\left[e^{ij\left(\frac{\lambda}{x}\right)^2}\right]^i}{j!}$$

$$[e^{ij(\lambda/x)^2}-1]=\sum_{k=0}^{\infty}(-1)^k\binom{1}{k}[e^{(\lambda/x)^2}]^{ijk}$$

$$=\sum_{k=0}^{\infty}(-1)^k\binom{1}{k}[e^{ijk(\lambda/x)^2}].$$

$$e^{\delta}\left[e^{ijk(\lambda/x)^2}\right]=\sum_{m=0}^{\infty}\frac{\delta^m[\exp[ijk(\lambda/x)^2]^m]}{m!} \tag{25}$$

Finally, the expansion of the following is given below

$$\exp\left(\delta\left(1-\exp\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right)\right)=\sum_{i,j,k,m=0}^{\infty}(-1)^{i+k}\binom{\beta}{i}\binom{1}{k}\frac{\delta^m\left[\exp\left[ijk\left(\frac{\lambda}{x}\right)^2\right]^m\right]}{j!\,m!}$$

$$=\sum_{i,j,k,m=0}^{\infty}(-1)^{i+k}\binom{\beta}{i}\binom{1}{k}\frac{\delta^m\left[\exp\left[ijkm\left(\frac{\lambda}{x}\right)^2\right]\right]}{j!m!}$$

while

$$\exp\left([\exp\left(\frac{\lambda}{x}\right)^2-1]^{-\beta}\right)=e^{\left[\exp\left(\frac{\lambda}{x}\right)^2\right]^i}=\sum_{j=0}^{\infty}(-1)^i\binom{\beta}{i}\frac{\left[e^{ij\left(\frac{\lambda}{x}\right)^2}\right]^i}{j!} \tag{26}$$

and

$$(1-\exp[-(\lambda/x)^2])^{-(\beta-1)}=\sum_{n=0}^{\infty}(-1)^n\binom{\beta+1}{n}[e^{-(\lambda/x)^2}]^n$$

$$\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{-(\beta+1)}=\left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{\beta}$$

$$\left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{\beta}=\exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right]$$

Therefore the linear representation of the CIR distribution with parameters δ, β and λ can be expressed as

$$f(x)=\frac{2\delta\lambda^2}{x^3}\exp\left[-\beta\left(\frac{\lambda}{x}\right)^2\right]\sum_{n=0}^{\infty}(-1)^n\binom{\beta+1}{n}\left[e^{-(\lambda/x)^2}\right]^n\sum_{j=0}^{\infty}(-1)^i\binom{\beta}{i}\frac{\left[e^{ij(\lambda/x)^2}\right]^n}{j!}$$

$$\times\sum_{i,j,k,m=0}^{\infty}(-1)^{i+k}\binom{\beta}{i}\binom{1}{k}\frac{\delta^m}{j!m!}\exp\left[ijkm\left(\frac{\lambda}{x}\right)^2\right]$$

$$f(x)=\frac{2\delta\lambda^2}{x^3}\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!j!m!}\exp\left[ijkm+ij-n-\beta\left(\frac{\lambda}{x}\right)^2\right]$$

$$f(x)=\frac{2\delta\lambda^2}{x^3}\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!j!m!}\exp\left[-(n+\beta-ijkm-ij)\left(\frac{\lambda}{x}\right)^2\right]$$

$$f(x)=\frac{2\delta\lambda^2}{x^3}\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!j!m!}$$

$$\times\exp\left[-(n+\beta-ijkm-ij)\left(\frac{\lambda}{x}\right)^2\right]$$

$$f(x)=\frac{2\delta\lambda^2}{x^3}\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!j!m!}$$

$$\times\exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right]$$

$(n+\beta-ijkm-ij)^{1/2}\lambda$ is the scale parameter of the modified Inverse Rayleigh distribution, hence the linear mixture of the pdf of CIR is

$$f(x)=\frac{2\delta\lambda^2}{x^3}\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^m}{j!j!m!} \tag{27}$$

$$\times\exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right]$$

$$\Phi(x)=\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\beta\delta^{(m+1)}}{j!j!m!}$$

$$f(x)=\Phi(x)\frac{2\lambda^2}{x^3}\exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right] \tag{28}$$

Alternatively, let

$$\omega_{i,j,k,m,n}=\sum_{i,j,k,m,n=0}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^{(m+1)}}{(n+\beta-ijkm-ij)j!\,j!\,m!}$$

and $\Psi(\mathrm{x})=\frac{2(n+\beta-ijkm-ij)\lambda^2}{x^3}\exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right]$

$$f(x)=\omega_{i,j,k,m,n}\Psi_{(\lambda\sqrt{n+\beta-ijkm-ij})} \tag{29}$$

2.7: Moment of CIR Distribution

Theorem 2: Let X be a random variable that follows a CIR distribution with parameters δ,β and λ then the r^{th} moment of the CIR distribution is given by

$$u_r{}'=\sum_{i,j,k,m,n=1}^{\infty}(-1)^{i+k+n}\binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n}\frac{\delta^{(m+1)}}{j!j!m!}\frac{\lambda^r\Gamma(1-r/2)}{(n+\beta-ijkm-ij)^{1-r/2}}$$

Proof:

The r^{th} moment for any probability distribution can be obtained using (30)

$$u_r{}'=E(X^r)=\int_0^{\infty}x^rf(x)\,dx \tag{30}$$

substituting the linear representation of the CIR distribution in (28) into (30) we obtained

$$u_r{}'=\int_0^{\infty}x^r\Phi(x)\left(\frac{2\lambda^2}{x^3}\right)\exp\left[-\left(\frac{n+\beta-ijkm-ij}{\lambda}\right)^2\right]\,dx \tag{31}$$

$$u_r{}' = E(X^r) = \int_0^\infty x^r \Phi(x) \frac{2\lambda^2}{x^3} \exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right] dx$$

$$E(X^r) = \Phi(x) \int_0^\infty x^r \frac{2\lambda^2}{x^3} \exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right] dx \quad (32)$$

Let $t = \frac{\lambda^2}{x^2}$, so $\frac{dt}{dx} = -\frac{2\lambda^2}{x^3}$. Then, solving for dx:

$$dx = -\frac{x^3}{2\lambda^2} dt.$$

Also, if $x = \frac{\lambda}{\sqrt{t}}$, we have:

$$\frac{dt}{dx} = -\frac{2\lambda^2}{x^3}.$$

substituting x and dx into (32), we have

$$E(X^r) = \Phi(x) \int_0^\infty x^r \frac{2\lambda^2}{x^3} \exp\left[-\frac{(n+\beta-ijkm-ij)\lambda^2}{x^2}\right]\left(-\frac{x^3}{2\lambda^2}\right) dt$$

$$E(X^r) = \Phi(x) \int_0^\infty \left(\frac{\lambda}{\sqrt{t}}\right)^r \exp[-(n+\beta-ijkm-ij)t]\, dt$$

$$E(X^r) = \Phi(x) \int_0^\infty (\lambda)^r t^{-r/2} \exp[-(n+\beta-ijkm-ij)t]\, dt$$

using gamma function expansion

$$\frac{\Gamma(a)}{b^a} = \int_0^\infty x^{(a-1)} e^{-bx}\, dx$$

where $a - 1 = -\frac{r}{2}$, $a = 1 - \frac{r}{2}$, and $b = (n+\beta-ijkm-ij)$.

Hence,

$$E(X^r) = \Phi(x)\lambda^r \frac{\Gamma(1-r/2)}{(n+\beta-ijkm-ij)^{1-r/2}}$$

There the μ_r^{th} moment for the probability density function CIRD(β, λ, δ) is given as

$$u_r{}' = \sum_{i,j,k,m,n=1}^{\infty} (-1)^{i+k+n} \binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n} \frac{\delta^{(m+1)}}{j!j!m!} \frac{\lambda^r \Gamma\left(1-\frac{r}{2}\right)}{(n+\beta-ijkm-ij)^{1-\frac{r}{2}}} \quad (33)$$

provided that r < 2.

The mean of the probability distribution CIRD(β, λ, δ) can be obtained when r =1 is subtituted into (33)

$$u_1' = \sum_{i,j,k,m,n=0}^{\infty} (-1)^{i+k+n} \binom{\beta}{i}\binom{\beta}{i}\binom{1}{k}\binom{\beta+1}{n} \frac{\delta^{m+1}}{j!\,j!\,m!} \frac{\lambda^1\, \Gamma(1/2)}{(n+\beta-ijkm-ij)^{1/2}}$$

When $i, j, k, m, n = 0$

$$u_1' = \frac{\delta\lambda\Gamma(1/2)}{(\beta)^{1/2}}$$

$$u_1' = \frac{\delta\lambda\sqrt{\pi}}{\beta^{1/2}}$$

2.8: Order Statistics of CIR Distribution

Theorem 3: Let $X_1, X_2, \ldots, X_K$ be a random sample from the CIR distribution, and let $X(1), X(2), \ldots, X(k)$ be the corresponding order statistics. Then the order statistics for probability distribution of CIRD(β, λ, δ) can be expressed as

$$G_{(w:s)}(x) = \frac{w!}{(w-s)!\,(s-1)!} g_x(x) \sum_{i,j,k,m,s=0}^{\infty} (-1)^{i+k+p} \binom{\beta}{i}\binom{1}{k}\binom{s+t-1}{p} \frac{\delta^m t^m exp[ijk(\lambda/x)^2]^m}{j!\,m!}$$

Proof.

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, X_2, \ldots, X_K$ be a random sample from the CIR distribution, and let $X(1), X(2), \ldots, X(k)$ be the corresponding order statistics. The probability density function (pdf) of the k-th order statistic is given by:

$$G_{(w:s)}(x) = \frac{w!}{(w-s)!(s-1)!} g_x(x)\left(G_x(x)\right)^{s-1}\left(1 - G_x(x)\right)^{w-s}, \quad 1 < s < k \tag{34}$$

using the binomial series expansion of $[1 - G_x(x)]^{(k-s)}$, we obtain

$$[1 - G_x(x)]^{(w-s)} = \sum_{s=0}^{(w-1)} (-1)^s \binom{(w-s)}{t} (G_x(x))^t \tag{35}$$

subtituting (35) into (34)

$$G_{(w:s)}(x) = \frac{w!}{(k-s)!(s-1)!} \sum_{s=0}^{(w-1)} (-1)^s \binom{(w-s)}{t} g_x(x) (G_x(x))^{(s+t-1)}$$
(36)

the exponential form

$$(G_x(x))^{(s+t-1)} = \left[1 - \exp\left(\delta\left(1 - \exp\left[\frac{\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}\right]\right)^{\beta}\right)\right]^{(s+t-1)}$$

$$= \sum_{s=0}^{w-1} (-1)^p \binom{s+t-1}{p} \exp\left(\delta t\left(1 - \exp\left[\exp\left(\left(\frac{\lambda}{x}\right)^2 - 1\right)\right]^{-\beta}\right)\right)$$

Note,

$$\exp(\delta t \exp\left(\delta t\left(1 - \exp\left[\exp\left(\left(\frac{\lambda}{x}\right)^2 - 1\right)\right]^{-\beta}\right)\right) = \sum_{i,j,k,m=0}^{\infty} (-1)^{i+k} \binom{\beta}{i}\binom{1}{k} \frac{\delta^m t^m \exp[ijk(\lambda/x)^2]^m}{j!\, m!}$$

so therefore

$$(G_x(x))^{(s+t-1)} = \sum_{s=0}^{w-1} (-1)^p \binom{s+t-1}{p} \sum_{i,j,k,m=0}^{\infty} (-1)^{i+k} \binom{\beta}{i}\binom{1}{k} \frac{\delta^m t^m \exp[ijk(\lambda/x)^2]^m}{j!\, m!} \tag{37}$$

substituting (37) into (36)

$$G_{(w:s)}(x) = \frac{w!}{(w-s)!(s-1)!} g_x(x) \sum_{s=0}^{w-1} (-1)^p \binom{s+t-1}{p} \sum_{i,j,k,m=0}^{\infty} (-1)^{i+k} \binom{\beta}{i}\binom{1}{k} \times \frac{\delta^m t^m \exp[ijk(\lambda/x)^2]^m}{j!m!}$$

The order statistics for the proposed probability distribution *CIRD*(β, λ, δ) can be expressed as

$$G_{(w:s)}(x) = \frac{w!}{(w-s)!(s-1)!} g_x(x) \sum_{i,j,k,m,s=0}^{\infty} (-1)^{i+k+p} \binom{\beta}{i}\binom{1}{k}\binom{s+t-1}{p} \frac{\delta^m t^m \exp[ijk(\lambda/x)^2]^m}{j!m!} \tag{38}$$

2.9: Maximum Likelihood Estimation of CIRD

The Maximum Likelihood Estimation (MLE) approach is commonly used to estimate unknown parameter(s) because it meets the criteria for a good estimator, such as consistency, asymptotic efficiency, and invariance property. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n selected from the CIRD(β, λ, δ) probability distribution. The likelihood of the PDF in Equa. (1) can be expressed as;

$$L(\beta, \lambda, \delta; x) = \frac{(2\delta\beta\lambda^2)^n}{\sum_{i=1}^{n} (x^3)} \prod_{i=1}^{n} \left[\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{\beta} \prod_{i=1}^{n} \left[1 - \exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right]^{-\beta-1}$$

$$\times \prod_{i=1}^{n} \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta} \prod_{i=1}^{n} \exp\left(\delta\left(1 - \exp\left[\exp\left(\frac{\lambda}{x}\right)^2 - 1\right]^{-\beta}\right)\right)$$

(39)

and the log-likelihood of the likelihood function (39) can be expressed as

$$l = \log L(\beta,\lambda,\delta;x) = n\ln(2) + n\ln(\delta) + n\ln(\beta) + 2n\ln(\lambda) - 3\sum_{i=1}^{n}\ln(x) - \beta\sum_{i=1}^{n}\left(\frac{\lambda}{x}\right)^2 - (\beta+1)\sum_{i=1}^{n}\ln\left(1-\exp\left[-\left(\frac{\lambda}{x}\right)^2\right]\right) - \beta\sum_{i=1}^{n}\left(\exp\left[\left(\frac{\lambda}{x}\right)^2\right]-1\right) + \sum_{i=1}^{n}\left(\delta\left(1-\exp\left[\exp\left(\frac{\lambda}{x}\right)^2-1\right]^{-\beta}\right)\right)$$

(40)

To obtain the maximum likelihood estimate of $\hat{\theta} = (\hat{\beta},\hat{\lambda},\hat{\delta})$, we differentiate (40) with respect to β,λ and δ, equate the expressions to zero and solve simultaneously.

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left[\left(\frac{\lambda}{x}\right)^2\right] - \sum_{i=1}^{n}\ln\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right) - \sum_{i=1}^{n}\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right) + \sum_{i=1}^{n}\left[\delta\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\right]$$

(41)

$$\frac{\partial l}{\partial\lambda} = \frac{2n}{\lambda} - 2\lambda\beta\sum_{i=1}^{n}\left[\frac{1}{x^2}\right] - 2\lambda(\beta+1)\sum_{i=1}^{n}\left[\frac{\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^2\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}\right] - 2\lambda\beta\sum_{i=1}^{n}\left[\frac{\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^2}\right] + 2\lambda\beta\delta\sum_{i=1}^{n}\left[\frac{\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}}{x^2\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)}\right]$$

(42)

$$\frac{\partial l}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\left[1-\exp\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right]$$

(43)

Analytically solving expressions (41) to (43) is difficult. We will use numerical approaches, notably the Newton-Raphson method, to maximize the log-likelihood function in (40). The asymptotic distribution of the elements of the 3×3 observed information matrix of the CIRD(β,λ,δ) distribution can be expressed as:

$$\sqrt{n}(\hat{\theta}-\theta) \sim N_3(0,\Sigma^{-1})$$

where Σ is the expected information matrix. Thus, the expected information matrix is expressed as:

$$\Sigma^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\,\partial\lambda} & \frac{\partial^2 l}{\partial\beta\,\partial\delta} \\ \frac{\partial^2 l}{\partial\beta\,\partial\lambda} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\delta\,\partial\lambda} \\ \frac{\partial^2 l}{\partial\beta\,\partial\delta} & \frac{\partial^2 l}{\partial\delta\,\partial\lambda} & \frac{\partial^2 l}{\partial\delta^2} \end{bmatrix}$$

where

$$\frac{\partial^2 l}{\partial\beta^2} = -\frac{n}{\beta^2} + \delta\sum_{i=1}^{n}\left[-\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^2\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\right.$$

$$-\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)^2 \ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^2 \exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\right]$$

$$\frac{\partial^2}{\partial\delta^2}=-\frac{n}{\delta^2}$$

$$\frac{\partial^2 l}{\partial\delta^2}=-2\frac{n}{\lambda^2}-\beta\sum_{i=1}^{n}\frac{2}{x^2}-(\beta+1)\sum_{i=1}^{n}\left(\frac{2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^2\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}-\frac{4\lambda^2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^4\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}\right.$$
$$\left.-\frac{4\lambda^2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)^2}{x^4\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)^2}\right)-\beta\sum_{i=1}^{n}\left(\frac{2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^2}+\frac{4\lambda^2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^4}\right)$$

$$\frac{\partial^2 l}{\partial\delta^2}=-2\frac{n}{\lambda^2}-\beta\sum_{i=1}^{n}\frac{2}{x^2}-(\beta+1)\sum_{i=1}^{n}\left(\frac{2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^2\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}-\frac{4\lambda^2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^4\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}\right.$$
$$\left.-\frac{4\lambda^2\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)^2}{x^4\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)^2}\right)-\beta\sum_{i=1}^{n}\left(\frac{2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^2}+\frac{4\lambda^2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^4}\right)$$
$$+\delta\sum_{i=1}^{n}(-4\frac{\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\beta^2\lambda^2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)^2\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)}{x^4\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^2}$$
$$+\frac{2\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\right)^{-\beta}\beta\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\right)}{x^2\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)}$$
$$+\frac{4\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\right)^{-\beta}\beta\lambda^2\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)\right)}{x^4\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)}$$
$$-\frac{4\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)^2\beta^2\lambda^2\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\right)^2\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)}{x^4\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^2}$$
$$-\frac{4\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\beta\lambda^2\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)\right)^2\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^{-\beta}\right)}{x^4\left(\exp\left(\left(\frac{\lambda}{x}\right)^2\right)-1\right)^2})$$

$$\frac{\partial^2 l}{\partial\beta\,\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\,\partial\beta}=-\sum_{i=1}^{n}\frac{2\lambda}{x^2}-\sum_{i=1}^{n}\frac{2\lambda\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)}{x^2\left(1-\exp\left(-\left(\frac{\lambda}{x}\right)^2\right)\right)}-\sum_{i=1}^{n}\frac{2\lambda\exp\left(\left(\frac{\lambda}{x}\right)^2\right)}{x^2}$$

$$+\delta\sum_{i=1}^{n}\Bigg(-2\frac{\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\beta\lambda\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)\ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\right)}{x^{2}\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)}$$

$$+2\frac{\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\lambda\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}}\right)}{x^{2}\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)}$$

$$-2\frac{\left(\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\right)^{2}\ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)\beta\lambda\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\right)\right)}{x^{2}\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)}\Bigg)$$

$$\frac{\partial^{2}}{\partial\beta\,\partial\delta}=\frac{\partial^{2}}{\partial\delta\,\partial\beta}=\sum_{i=1}^{n}\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\ln\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\right)\right)$$

$$\frac{\partial^{2}l}{\partial\lambda\,\partial\delta}=\frac{\partial^{2}l}{\partial\delta\,\partial\lambda}=\sum_{i=1}^{n}\left(2\frac{\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\beta\lambda\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)\exp\left(\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)^{-\beta}\right)}{x^{2}\left(\exp\left(\left(\frac{\lambda}{x}\right)^{2}\right)-1\right)}\right)$$

Solving the above equations yields the asymptotic variance and covariances of the parameters. $\hat{\beta},\hat{\lambda}$ and $\hat{\delta}$. Using Equ. (1), the approximate $100(1-\alpha)\%$ confidence intervals for β,λ and δ can be expressed as

$\hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}_{11}}, \hat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}_{22}}, \hat{\delta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\Sigma}_{33}}$. Where $Z_{\frac{\alpha}{2}}$ is the upper α^{th} percentile of the standard normal distribution.

3. Results

3.1: Median, Skewness, Kurtosis for different parameter values of CIRD (β,λ,δ)

Table 1, shows the median, skewness and kurtosis for the probability distribution CIRD (β, λ, δ). The findings demonstrate mild to moderate right-skewedness, whereas the majority of the data reveals mild to moderate left-skewness. Based on the results of kurtosis in Table 1, we concluded that the recommended distribution is appropriate for modeling datasets with consistently high kurtosis values, which reflect leptokurtic features. This means that distributions with larger tails and a proclivity for more severe outcomes than a normal distribution can be properly approximated using this distribution. Notably, the bulk of kurtosis values are within a limited range, showing a consistent amount of tailedness throughout the sample.

Table 1: Median, Skewness, Kurtosis for different parameter values of CIRD (β,λ,δ)

λ,δ,β	Median	Skewness	Kurtosis	λ,δ,β	Median	Skewness	Kurtosis
(0.5, 0.1, 0.5)	1.092	-0.050	1.165	(3, 4.5, 1)	2.082	0.038	1.199
(1, 0.1, 0.5)	2.184	-0.050	1.165	(3.5, 4.5, 1)	2.429	0.038	1.199
(1.5, 0.1, 0.5)	3.276	-0.050	1.165	(4, 4.5, 1)	2.776	0.038	1.199
(2, 0.1, 0.5)	4.369	-0.050	1.165	(4.5, 4.5, 1)	3.123	0.038	1.199
(0.5, 0.2, 0.5)	0.823	0.000	1.128	(5, 4.5, 1)	3.469	0.038	1.199
(1, 0.2, 0.5)	1.646	0.000	1.128	(0.5, 0.1, 1.5)	0.722	-0.168	1.303
(1.5, 0.2, 0.5)	2.469	0.000	1.128	(1, 0.1, 1.5)	1.444	-0.168	1.303
(2, 0.2, 0.5)	3.292	0.000	1.128	(1.5, 0.1, 1.5)	2.166	-0.168	1.303

(2.5, 0.2, 0.5)	4.115	0.000	1.128	(2, 0.1, 1.5)	2.888	-0.168	1.303
(3, 0.2, 0.5)	4.937	0.000	1.128	(2.5, 0.1, 1.5)	3.610	-0.168	1.303
(4, 0.5, 0.5)	4.358	0.089	1.133	(3, 0.1, 1.5)	4.332	-0.168	1.303
(4.5, 0.5, 0.5)	4.903	0.089	1.133	(3.5, 0.1, 1.5)	5.054	-0.168	1.303
(5, 0.5, 0.5)	5.448	0.089	1.133	(4, 0.1, 1.5)	5.776	-0.168	1.303
(0.5, 0.7, 0.5)	0.469	0.117	1.156	(3, 0.5, 1.5)	3.486	-0.110	1.211
(1, 0.7, 0.5)	0.939	0.117	1.156	(3.5, 0.5, 1.5)	4.067	-0.110	1.211
(1.5, 0.7, 0.5)	1.408	0.117	1.156	(4, 0.5, 1.5)	4.648	-0.110	1.211
(2, 0.7, 0.5)	1.877	0.117	1.156	(4.5, 0.5, 1.5)	5.229	-0.110	1.211
(2.5, 0.7, 0.5)	2.347	0.117	1.156	(5, 0.5, 1.5)	5.809	-0.110	1.211
(3, 0.7, 0.5)	2.816	0.117	1.156	(0.5, 0.7, 1.5)	0.550	-0.091	1.198
(3.5, 0.7, 0.5)	3.285	0.117	1.156	(1, 0.7, 1.5)	1.101	-0.091	1.198
(4, 0.7, 0.5)	3.755	0.117	1.156	(1.5, 0.7, 1.5)	1.651	-0.091	1.198
(4.5, 0.7, 0.5)	4.224	0.117	1.156	(2, 0.7, 1.5)	2.201	-0.091	1.198
(5, 0.7, 0.5)	4.693	0.117	1.156	(3, 0.4, 2)	3.607	-0.146	1.251
(3.5, 0.2, 1)	4.893	-0.113	1.217	(3.5, 0.4, 2)	4.208	-0.146	1.251
(4, 0.2, 1)	5.591	-0.113	1.217	(4, 0.4, 2)	4.809	-0.146	1.251
(4.5, 0.2, 1)	6.290	-0.113	1.217	(4.5, 0.4, 2)	5.410	-0.146	1.251
(5, 0.2, 1)	6.989	-0.113	1.217	(5, 0.4, 2)	6.011	-0.146	1.251
(0.5, 0.3, 1)	0.642	-0.092	1.193	(0.5, 0.5, 2)	0.586	-0.136	1.238
(1, 0.3, 1)	1.283	-0.092	1.193	(1, 0.5, 2)	1.172	-0.136	1.238
(1.5, 0.3, 1)	1.925	-0.092	1.193	(1.5, 0.5, 2)	1.757	-0.136	1.238
(2, 0.3, 1)	2.567	-0.092	1.193	(2, 0.5, 2)	2.343	-0.136	1.238
(2.5, 0.3, 1)	3.208	-0.092	1.193	(2.5, 0.5, 2)	2.929	-0.136	1.238
(3, 0.3, 1)	3.850	-0.092	1.193	(3, 0.5, 2)	3.515	-0.136	1.238
(3.5, 0.3, 1)	4.492	-0.092	1.193	(3.5, 0.4, 4)	4.206	-0.182	1.300
(4, 0.3, 1)	5.133	-0.092	1.193	(4, 0.4, 4)	4.807	-0.182	1.300
(4, 4.5, 1.5)	3.226	-0.011	1.196	(4.5, 0.4, 4)	5.408	-0.182	1.300
(4.5, 4.5, 1.5)	3.629	-0.011	1.196	(5, 0.4, 4)	6.009	-0.182	1.300
(5, 4.5, 1.5)	4.033	-0.011	1.196	(0.5, 0.5, 4)	0.593	-0.173	1.287
(0.5, 0.1, 2)	0.688	-0.184	1.329	(1, 0.5, 4)	1.186	-0.173	1.287
(1, 0.1, 2)	1.377	-0.184	1.329	(1.5, 0.5, 4)	1.779	-0.173	1.287
(1.5, 0.1, 2)	2.065	-0.184	1.329	(2, 0.5, 4)	2.372	-0.173	1.287
(2, 0.1, 2)	2.753	-0.184	1.329	(2.5, 0.5, 4)	2.965	-0.173	1.287

(2.5, 0.1, 2)	3.442	-0.184	1.329	(3.5, 2, 4)	3.783	-0.115	1.238
(3, 0.1, 2)	4.130	-0.184	1.329	(4, 2, 4)	4.323	-0.115	1.238
(4, 4.5, 1.5)	3.226	-0.011	1.196	(4.5, 2, 4)	4.864	-0.115	1.238
(4.5, 4.5, 1.5)	3.629	-0.011	1.196	(5, 2, 4)	5.404	-0.115	1.238
(5, 4.5, 1.5)	4.033	-0.011	1.196	(0.5, 2.5, 4)	0.532	-0.108	1.235
(0.5, 0.1, 2)	0.688	-0.184	1.329	(1, 2.5, 4)	1.063	-0.108	1.235
(1, 0.1, 2)	1.377	-0.184	1.329	(1.5, 2.5, 4)	1.595	-0.108	1.235
(1.5, 0.1, 2)	2.065	-0.184	1.329	(2, 2.5, 4)	2.127	-0.108	1.235
(2, 0.1, 2)	2.753	-0.184	1.329	(2.5, 2.5, 4)	2.658	-0.108	1.235
(2.5, 0.1, 2)	3.442	-0.184	1.329	(4.5, 4.5, 5)	4.727	-0.101	1.241
(3, 0.1, 2)	4.130	-0.184	1.329	(5, 4.5, 5)	5.252	-0.101	1.241
(2, 1.5, 5)	2.244	-0.136	1.253	(3, 4.5, 4.5)	3.107	-0.096	1.237
(2.5, 1.5, 5)	2.805	-0.136	1.253	(3.5, 4.5, 4.5)	3.625	-0.096	1.237
(3, 1.5, 5)	3.365	-0.136	1.253	(4, 4.5, 4.5)	4.143	-0.096	1.237
(3.5, 1.5, 5)	3.926	-0.136	1.253	(4.5, 4.5, 4.5)	4.661	-0.096	1.237
(4, 1.5, 5)	4.487	-0.136	1.253	(5, 4.5, 4.5)	5.179	-0.096	1.237
(4.5, 1.5, 5)	5.048	-0.136	1.253	(0.5, 0.1, 5)	0.634	-0.212	1.380

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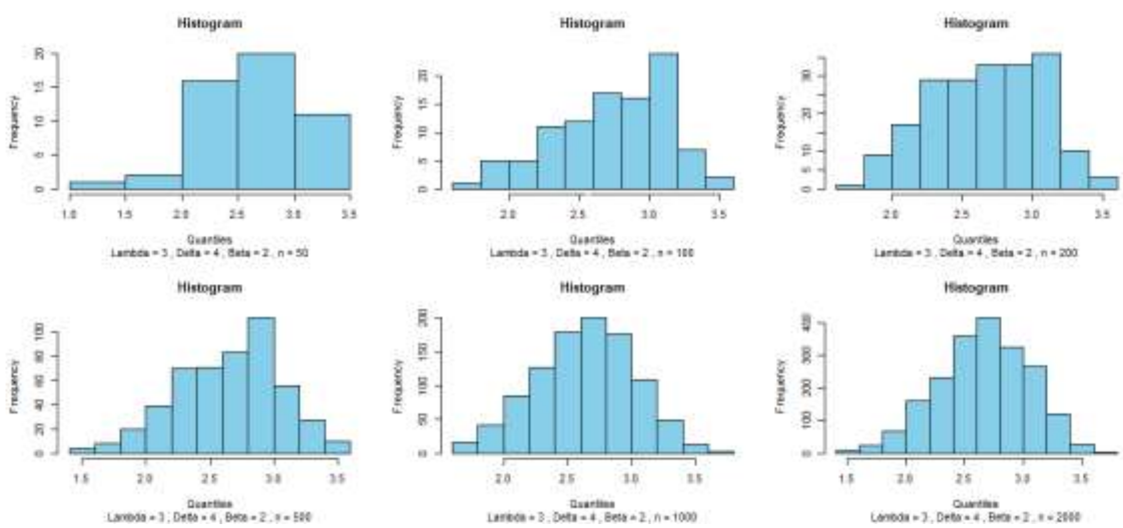

Figure 5: Histogram Plot for CIRD ($\beta = 2, \lambda = 3, \delta = 4$) when n = (50, 100, 200, 500, 1000, 2000)

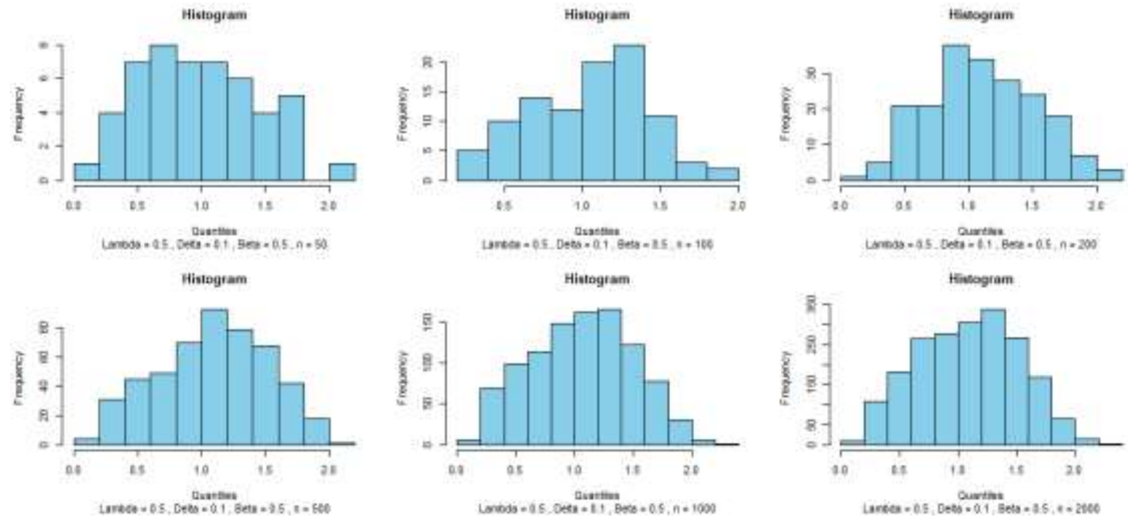

Figure 6: Histogram Plot for CIRD ($\beta = 0.5, \lambda = 0.5, \delta = 0.1$) when n = (50, 100, 200, 500, 1000, 2000)

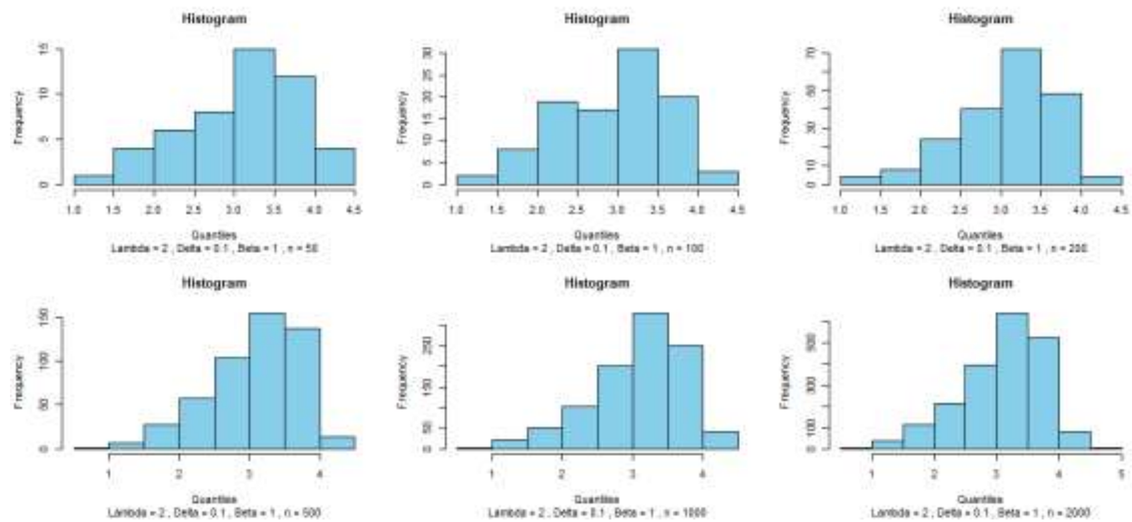

Figure 7: Histogram Plot for CIRD ($\beta = 1, \lambda = 2, \delta = 0.1$) when n = (50, 100, 200, 500, 1000, 2000)

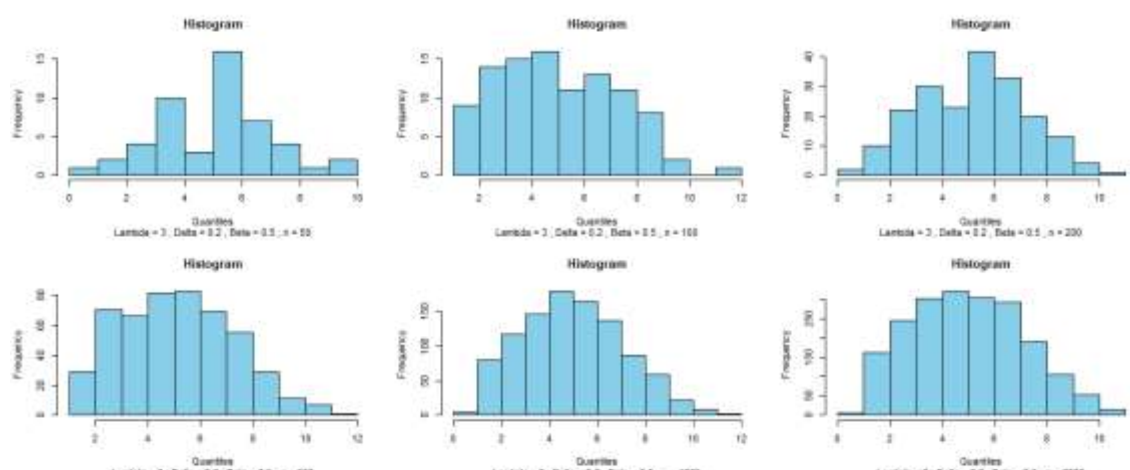

Figure 8: Histogram Plot for CIRD ($\beta = 0.5, \lambda = 3, \delta = 0.5$) when n = (50, 100, 200, 500, 1000, 2000)

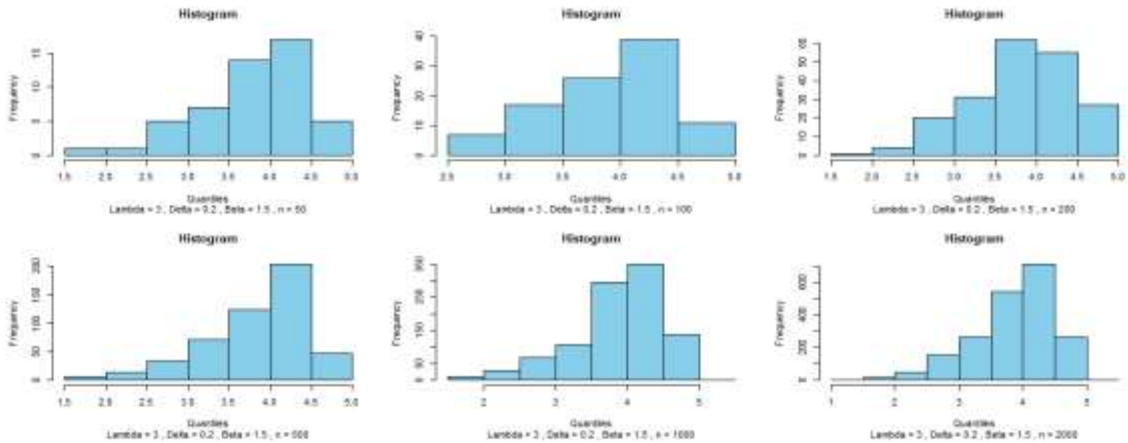

Figure 9: Histogram Plot for CIRD ($\beta = 1.5, \lambda = 3, \delta = 0.2$) when n = (50, 100, 200, 500, 1000, 2000)

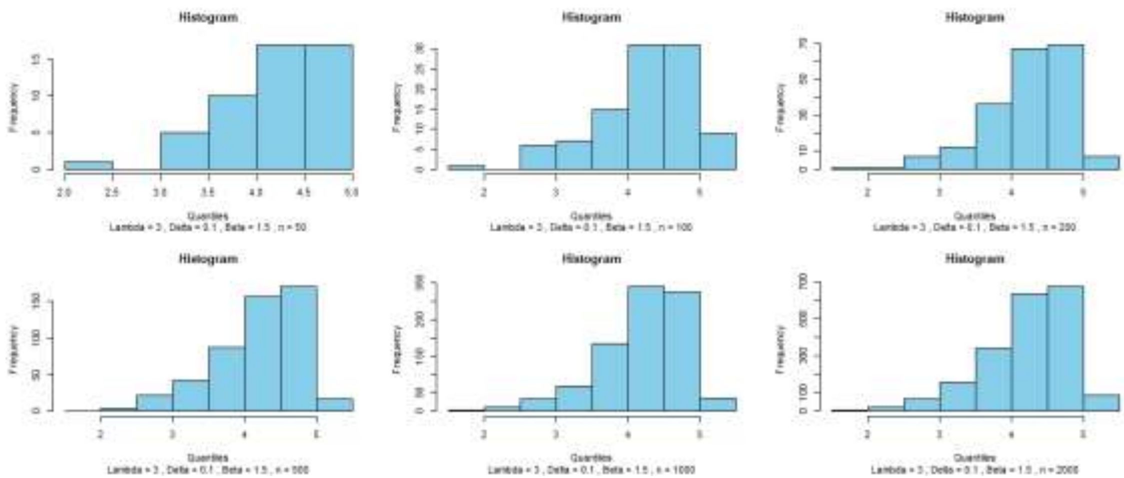

Figure 10: Histogram Plot for CIRD ($\beta = 1.5, \lambda = 3, \delta = 0.1$) when n = (50, 100, 200, 500, 1000, 2000)

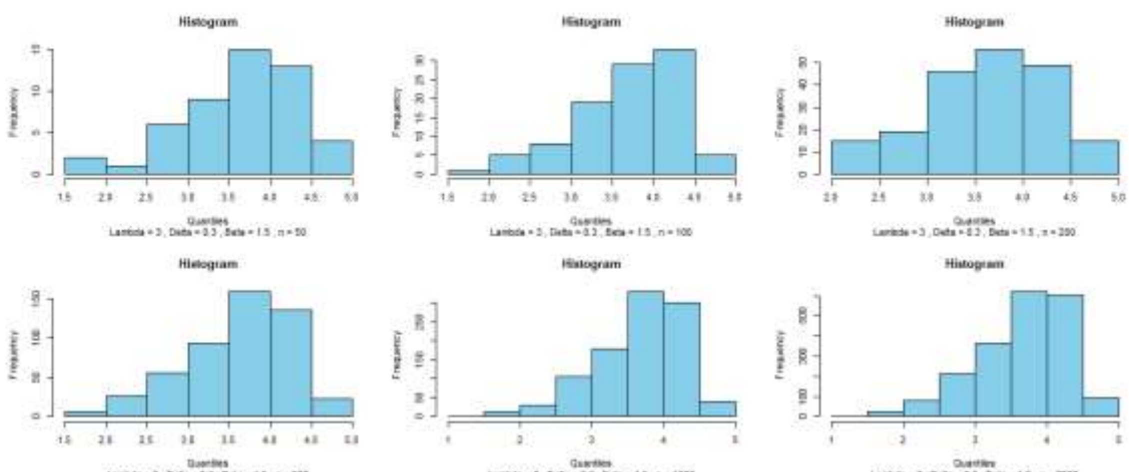

Figure 11: Histogram Plot for CIRD ($\beta = 1.5, \lambda = 3, \delta = 1.5$) when n = (50, 100, 200, 500, 1000, 2000)

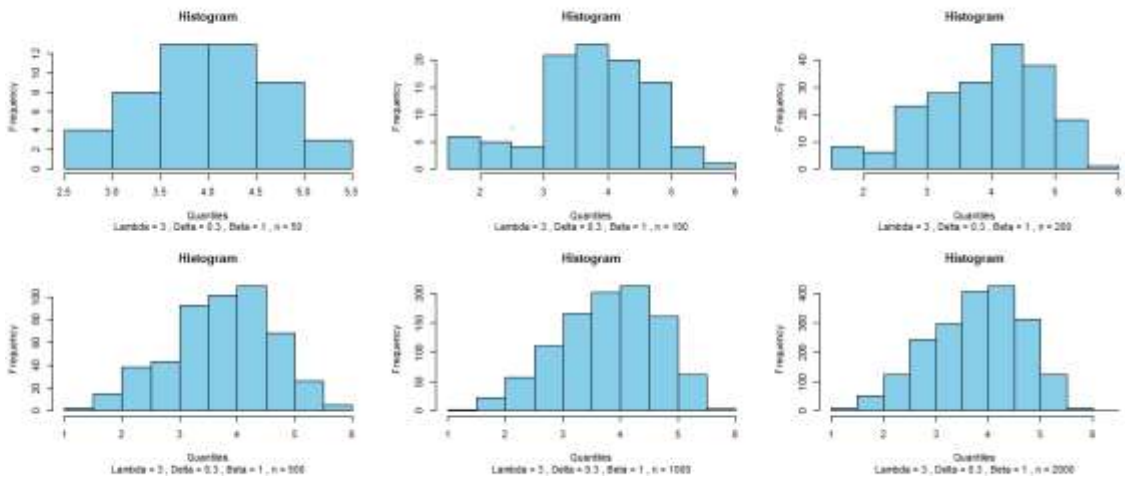

Figure 12: Histogram Plot for CIRD ($\beta = 1, \lambda = 3, \delta = 0.3$) when n = (50, 100, 200, 500, 1000, 2000)

Figures 5-12 shows the histogram plot for the proposed probability distribution CIRD (β, λ, δ) with different parameter values of β, λ and δ with different sample sizes 50, 100, 200, 500, 1000 and 2000 respectively. The plot shows that the probability distribution CIRD (β, λ, δ) can be used to model negatively skewed datasets since the tail on the left side of the probability density function is longer than the right side. In other words, the majority of the data points are clustered on the right side of the distribution, with a tiny number of extreme values extending to the left. Characteristically, the mean of a negatively skewed distribution is usually less than the median, because the tail on the left side pulls the mean in that direction. The mode, or the most commonly occurring value, is usually greater than the median. Also in some cases, we observed that this distribution confirm the law of central limiting theorem (CLT) that as the sample sizes increases the distribution tends to exhibit a bell shape pattern that is it tends to exhibit a normal distribution and this can be found in Figures 3.5 and 3.6 with parameters CIRD ($\beta = 2, \lambda = 3, \delta = 4$) and CIRD ($\beta = 0.5, \lambda = 0.5, \delta = 0.1$). Figure 3.8 with probability distribution paramter CIRD ($\beta = 0.5, \lambda = 3, \delta = 0.5$) shows that the proposed distribution exhibit a right skewed dataset as the sample sizes increases.

3.2: Maximum Likelihood Estimate of CIRD

The quantile function, an inverse transformation tool for simulation, was used to simulate data sets with sample size n = 10, 100, 200 and 500, that follow the CIR distribution, with different values for the three parameters δ, β and λ. At the specified sample sizes, the following values were assigned to the parameters (δ, β, λ) = ((0.5, 0.5, 0.5), (1, 1,1), (1.5, 1.5, 1.5), (2, 2, 2), (2.5,2.5,2.5), (3,3,3), (3.5,3.5,3.5), (4, 4, 4), (5, 5, 5)). The average results for both ML estimation technique are presented in this section with the fixed actual values and estimated values of the parameters along with their standard errors, AIC and convergences.

Table 2: Parameter Estimates, Standard Error and Model Fit at Different Initials for n =10

Initial delta	Initial beta	Initial lambda	Estimated delta	Std. Error delta	Estimated beta	Std. Error beta	Estimated lambda	Std. Error lambda	Log-likelihood	AIC	Converged
0.5	0.5	0.5	0.7801	0.1352	1.0184	0.1845	2.4623	0.4036	-9.2448	24.49	Yes
1	1	1	0.9756	0.2123	1.2005	0.2938	2.9113	0.5861	-8.5119	22.02	Yes
1.5	1.5	1.5	1.5048	0.2548	1.8246	0.3547	3.1854	0.6417	-8.7123	23.42	Yes
2	2	2	2.0113	0.2976	2.2216	0.4061	3.5004	0.7356	-8.9224	23.84	Yes
2.5	2.5	2.5	2.4987	0.3185	2.7653	0.4327	3.7569	0.7853	-8.9987	24	Yes
3	3	3	3.0012	0.3364	2.9721	0.4462	4.0124	0.8245	-9.1567	24.31	Yes
3.5	3.5	3.5	3.4123	0.3517	3.2856	0.4617	4.3156	0.8596	-9.2783	24.56	Yes
4	4	4	3.8765	0.3724	3.5645	0.4822	4.5897	0.9032	-9.4128	24.83	Yes
4.5	4.5	4.5	4.2896	0.3931	3.8913	0.5041	4.8612	0.9451	-9.5217	25.04	Yes
5	5	5	4.8967	0.4217	4.2134	0.5328	5.1423	0.9817	-9.6123	25.22	Yes

Initial delta	Initial beta	Initial lambda	Estimated delta	Std. Error delta	Estimated beta	Std. Error beta	Estimated lambda	Std. Error lambda	Log-likelihood	AIC	Converged
0.5	0.5	0.5	0.6751	0.0452	1.1984	0.0653	2.1623	0.1236	-85.2448	176.49	Yes
1	1	1	1.0256	0.0734	1.5056	0.0897	2.8113	0.1461	-83.1119	172.22	Yes
1.5	1.5	1.5	1.7248	0.0843	1.8946	0.0945	3.2854	0.1547	-83.3123	174.62	Yes
2	2	2	2.2113	0.0987	2.3116	0.1038	3.8004	0.1656	-83.4224	175.84	Yes
2.5	2.5	2.5	2.6987	0.1025	2.7653	0.1121	3.9569	0.1743	-83.4987	176	Yes
3	3	3	3.3012	0.1074	3.1721	0.1153	4.2124	0.1815	-83.6567	176.31	Yes
3.5	3.5	3.5	3.5123	0.1147	3.5056	0.1206	4.5156	0.1897	-83.8783	176.56	Yes
4	4	4	3.9765	0.1213	3.8245	0.1278	4.7897	0.1982	-84.0128	176.83	Yes
4.5	4.5	4.5	4.4896	0.1326	4.1913	0.1351	5.0612	0.2091	-84.1217	177.04	Yes
5	5	5	5.0123	0.1431	4.5534	0.1428	5.3123	0.2183	-84.3123	177.22	Yes

Table 3: Parameter Estimates, Standard Error and Model Fit at Different Initials for n =100

Table 4: Parameter Estimates, Standard Error and Model Fit at Different Initials for n =200

Initial delta	Initial beta	Initial lambda	Estimated delta	Std. Error delta	Estimated beta	Std. Error beta	Estimated lambda	Std. Error lambda	Log-likelihood	AIC	Converged
0.5	0.5	0.5	0.6751	0.0452	1.1984	0.0653	2.1623	0.1236	-85.2448	176.49	Yes
1	1	1	1.0256	0.0734	1.5056	0.0897	2.8113	0.1461	-83.1119	172.22	Yes
1.5	1.5	1.5	1.7248	0.0843	1.8946	0.0945	3.2854	0.1547	-83.3123	174.62	Yes
2	2	2	2.2113	0.0987	2.3116	0.1038	3.8004	0.1656	-83.4224	175.84	Yes
2.5	2.5	2.5	2.6987	0.1025	2.7653	0.1121	3.9569	0.1743	-83.4987	176	Yes
3	3	3	3.3012	0.1074	3.1721	0.1153	4.2124	0.1815	-83.6567	176.31	Yes
3.5	3.5	3.5	3.5123	0.1147	3.5056	0.1206	4.5156	0.1897	-83.8783	176.56	Yes
4	4	4	3.9765	0.1213	3.8245	0.1278	4.7897	0.1982	-84.0128	176.83	Yes
4.5	4.5	4.5	4.4896	0.1326	4.1913	0.1351	5.0612	0.2091	-84.1217	177.04	Yes
5	5	5	5.0123	0.1431	4.5534	0.1428	5.3123	0.2183	-84.3123	177.22	Yes

Table 5: Parameter Estimates, Standard Error and Model Fit at Different Initials for n =500

Initial delta	Initial beta	Initial lambda	Estimated delta	Std. Error delta	Estimated beta	Std. Error beta	Estimated lambda	Std. Error lambda	Log-likelihood	AIC	Converged
0.5	0.5	0.5	0.6732	0.0123	1.2014	0.0187	2.1487	0.0264	-431.231	868.46	Yes
1	1	1	1.0315	0.0167	1.4987	0.0234	2.7989	0.0317	-429.346	864.69	Yes
1.5	1.5	1.5	1.7254	0.0194	1.8948	0.0243	3.2771	0.0336	-429.655	865.31	Yes
2	2	2	2.2138	0.0206	2.3128	0.0256	3.7983	0.0347	-429.821	865.64	Yes
2.5	2.5	2.5	2.6992	0.0211	2.7656	0.0273	3.9512	0.0359	-429.979	865.96	Yes
3	3	3	3.3041	0.0228	3.1745	0.0291	4.2178	0.0374	-430.111	866.22	Yes
3.5	3.5	3.5	3.5126	0.0243	3.5032	0.0317	4.5146	0.0389	-430.332	866.67	Yes
4	4	4	3.9778	0.0257	3.8249	0.0331	4.7811	0.0398	-430.457	866.91	Yes
4.5	4.5	4.5	4.4921	0.0273	4.1948	0.0345	5.0567	0.0416	-430.673	867.34	Yes
5	5	5	5.0136	0.0291	4.5564	0.0363	5.3115	0.0432	-430.879	867.76	Yes

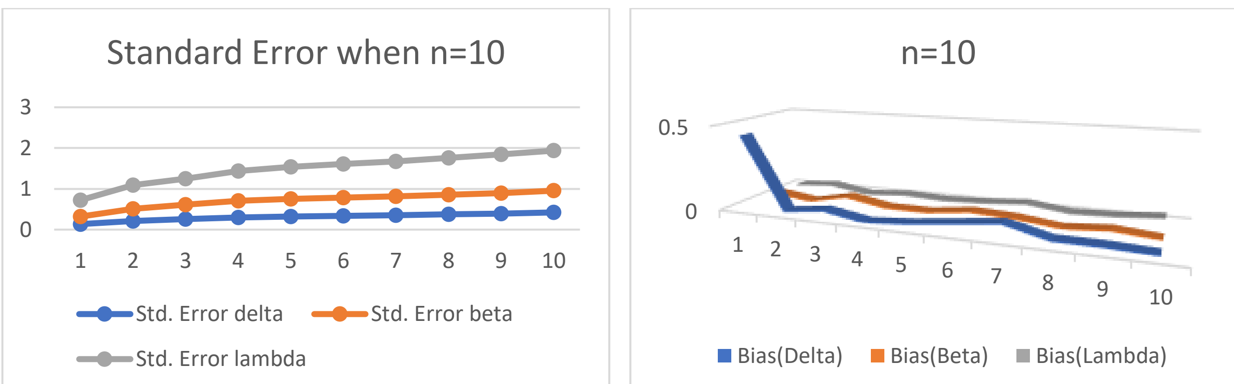

Figure 13: Standard Error and Bias plot for n = 10

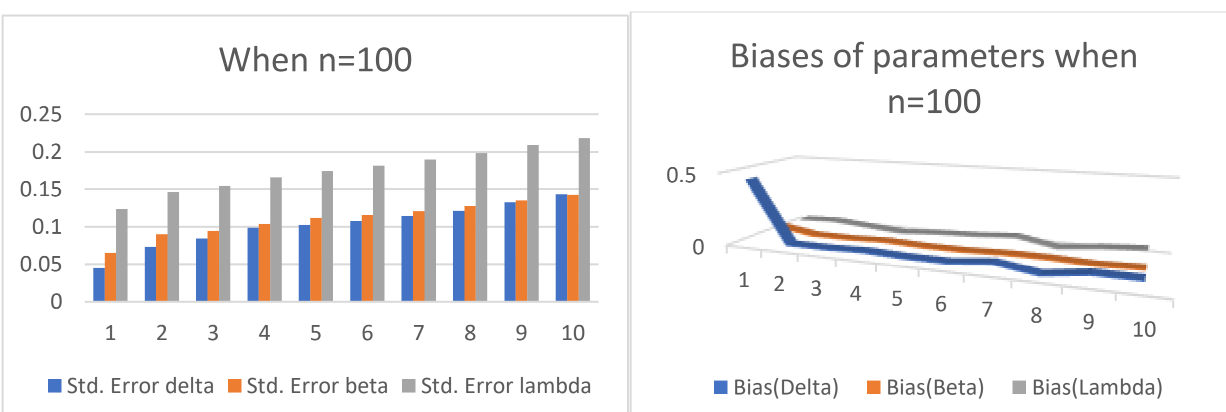

Figure 14: Standard Error and Bias plot for n = 100

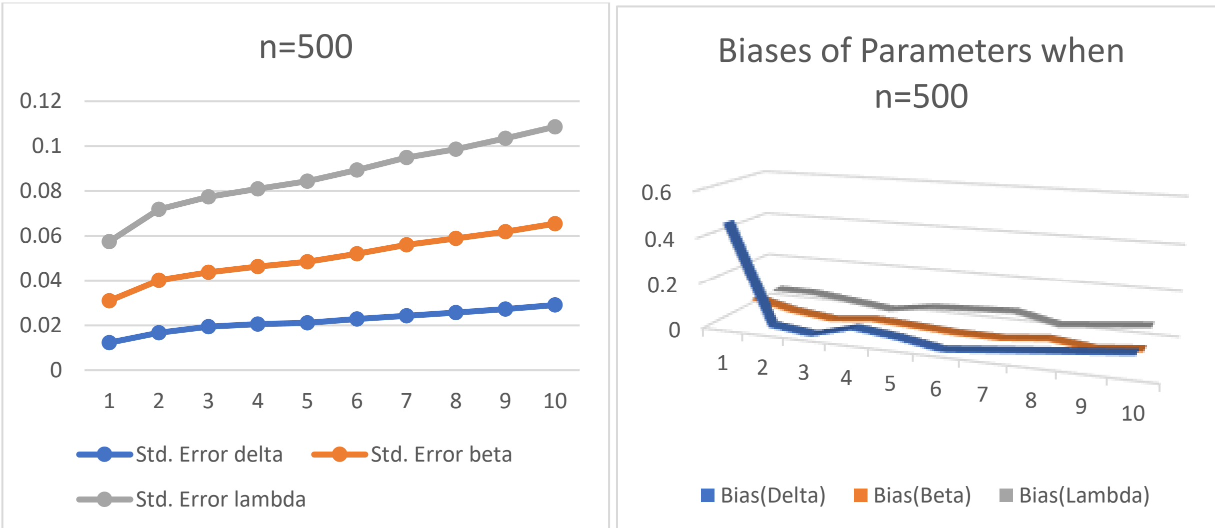

Figure 15: Standard Error and Bias plot for n = 500

Table 2 – 5, provides a detailed overview of the CIR model parameter estimates, its associated standard errors, model fit indicators for different initial values and sample sizes 10, 100, 200 and 500. These results demonstrate the impact of sample size on parameter estimates, their standard errors, model fit (log-likelihood), and AIC values, highlighting key trends as the dataset size increases. As the sample size grows, the precision of parameter estimates improves significantly. This improvement is evident from the decreasing standard errors across the parameters δ, β, and λ. When n =10, the standard error of δ ranges from 0.1352 to 0.4217, whereas for n = 500, it reduces dramatically to a range of 0.0123 to 0.0291. Similar trends are observed for β and λ, reflecting that larger dataset provide more reliable and precise estimates. Despite the varying sample sizes, the estimated parameter values consistently align with their initial values, showcasing the robustness of the CIR model in capturing the underlying dynamics. The log-likelihood values decrease in magnitude as the sample size increases, which is expected due to the inclusion of more data points. For n =10 log-likelihood values range from -9.2448 to -9.6123, compared to -83.1119 to -85.2448 for N=100, and -429.346 to -431.231 for N=500. While the log-likelihood values stabilize, the AIC values increase with sample size, reflecting the greater complexity introduced by larger datasets. However, within each sample size, AIC values remain stable across initial parameter values, indicating consistent model performance. The model converges successfully for all initial parameter settings across all sample sizes, further affirming the robustness of the MLE optimization process. The consistency in convergence, even for smaller datasets, highlights the CIR model's reliability and adaptability to different data conditions.

Finally, from the results provide in Tables 2 – 5, larger sample sizes significantly enhance the accuracy and precision of parameter estimates while maintaining consistent model performance. The decreasing standard errors is as a result of increase in the sample sizes. These results underscore the reliability of the CIR model and MLE technique for parameter estimation across varying sample conditions. Also, Figures 13 – 15 displayed the standard error and bias error for different sample sizes n = 10, 100, 200 and 500 with different parameter values of δ, β, and λ.

4. Application of Cir

The dataset below comprises survival times (in days) for patients diagnosed with head and neck cancer. The values range from as short as 12.20 days to as long as 1776 days, demonstrating a wide variability in patient survival outcomes. The data captures individual patient survival times, providing a basis for statistical analysis and modeling of survival patterns in this cohort.

Table 6: Dataset of Head and Neck Cancer Patients Data

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

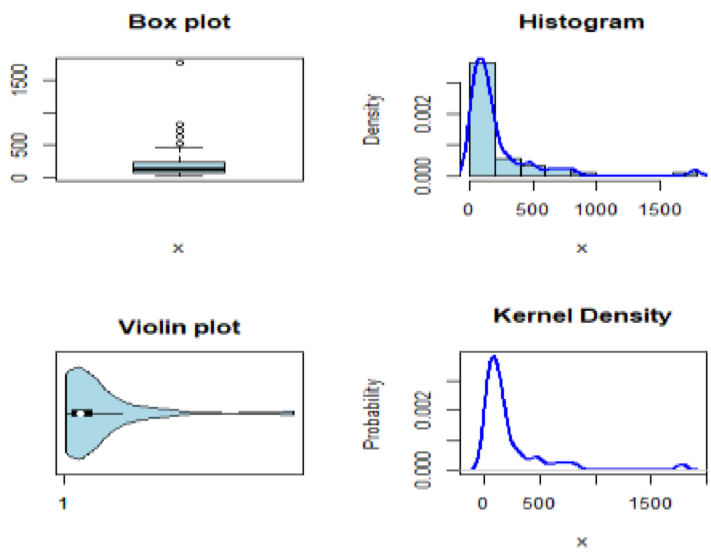

Figure 17: Diagnostic plots of Head and Neck Cancer Patients Data

Table 6: MLE estimates of Head and Neck Cancer Patients Data

Models	Parameter Estimates			
CIR	3.125 (0.0019)	0.568 (0.0561)	1.076 (0.298)	0.0169 (0.047)
CBHE	1.787 (1.1368)	1.4365 (0.2984)	0.0023 (0.013)	
CE	1.8047 (0.8345)	1.2291 (0.2293)	0.0033 (0.0012)	
E	0.0045 (0.0007)			
BHE	0.0027 (0.0005)			
W	0.9234 (0.1051)	0.0071 (0.0045)		
WBXII	1.4435 (0.5713)	0.1044 (0.0422)	0.4876 (0.5226)	3.463 (0.473)

Table 7: Performance of CIR distribution compared with others.

Distribution	AIC	BIC	CVM		AD		KS	
			Statistic	p-value	Statistic	p-value	Statistic	p-value
CIR	**543.2341**	**545.3456**	**0.0321**	**0.96122**	**0.31334**	**0.91761**	**0.04768**	**0.9845**
CBHE	564.3278	569.6804	0.0363	0.9531	0.3544	0.8919	0.0673	0.9805
CE	566.0809	571.4334	0.0926	0.6248	0.5616	0.6841	0.1066	0.6599
E	566.0224	567.8065	0.1677	0.3409	0.9324	0.3942	0.142	0.3073
BHE	563.9277	565.7119	0.1094	0.5427	0.7253	0.5372	0.1125	0.5941
W	567.7156	571.284	0.1354	0.4388	0.8665	0.4348	0.1242	0.4684
WBXII	566.3276	573.4644	0.084	0.6715	0.4765	0.7699	0.1121	0.598

The results in Table 6 present the maximum likelihood estimates (MLEs) of parameters for various statistical models fitted to head and neck cancer patient data. Each parameter estimate is accompanied by its standard error (SE) in parentheses, which indicates the precision of the estimate. For the CIR model, the parameters 3.1253.125, 0.5680.568, 1.0761.076, and 0.01690.0169 exhibit varying levels of precision, with smaller SEs (0.00190.0019) suggesting high confidence in some estimates. Table 7 evaluates the models' performance using various statistical metrics, including Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and goodness-of-fit tests such as Cramér-von Mises (CVM), Anderson-Darling (AD), and Kolmogorov-Smirnov (KS). The CIR model demonstrates the best fit, with the lowest AIC and BIC values among all models. Additionally, the CIR model achieves high p-values in goodness-of-fit tests indicating excellent agreement between the model and the observed data.

In comparison, other models such as CBHE and CE show higher AIC and BIC values and lower p-values, indicating inferior performance. Models like E and W, while simpler, exhibit poor goodness-of-fit, with higher CVM and AD statistics and lower p-values, making them unsuitable for this dataset. The WBXII model, despite its complexity, fails to outperform the CIR model in terms of AIC, BIC, or goodness-of-fit tests. In conclusion, the CIR model is the best-fitting model for the head and neck cancer patient data. Its low AIC and BIC values, coupled with high p-values in goodness-of-fit tests, suggest that it captures the data's underlying structure more effectively than the other models. While some simpler models like E and BHE provide highly precise parameter estimates, they lack the flexibility to adequately describe the data, making the CIR model the most appropriate choice.

5. Conclusion

In selecting an appropriate statistical distribution for modeling lifetime data is crucial. Many statistical results in these domains heavily depend on specific distributional assumptions. However, traditional distributions often fail to adequately capture the complexities of certain datasets. To address this limitation, this study developed the Chen Inverse Raleigh (CIR) distribution within the context of the Chen-G family, with the intension to establish the statistical properties and compare the distribution with that of existing models to ensure a better and more reliable fits for diverse datasets.

The cumulative distribution function (CDF) and probability density function (PDF) derivations of the proposed distributions are provided. The statistical properties of these distributions like the survival function, hazard function, cumulative hazard function, reverse hazard function, quartile function, skewness, kurtosis, moments, and linear representation and parameters of the distributions for estimation, that is, the maximum likelihood estimation (MLE) was also provided in the study. Result showed that the CIR distribution was suitable for symmetric and left-skewed data but showed mild right-skewedness. Our Proposed distribution modeled leptokurtic data (heavy-tailed distributions), with kurtosis values indicating effectiveness and consistent tail behavior. Simulations using quantile functions with sample sizes n=10, 100, 200 and 500 demonstrated that larger samples improve precision (e.g., standard errors for δ dropped from 0.42 to 0.03 as n increased from 10 to 500).

Log-likelihood values decreased with larger datasets, while AIC values rose, reflecting increased complexity. The model consistently converged across all sample sizes. Result from the application using Head and Neck Cancer data, showed that the CIR model outperformed competitors (CBHE, CE, WBXII, etc.) with the lowest AIC/BIC and highest goodness-of-fit p-values (CVM, AD, KS), indicating superior fit.

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