Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Double Intuitionistic Separation Axioms in Double Intuitionistic Topological Spaces

Asmaa Ghasoob Raoof
Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Iraq
Corresponding Author: Asmaa Ghasoob Raoof, **E-mail**: asmaa.g.raoof@tu.edu.iq

| ABSTRACT
Our goal in this paper is to give a comprehensive study of S**.** Bayhan and D**.** Coker [5] paper entitled "On separation axioms in intuitionistic topological spaces" I give generalizations to the Double intuitionistic separation axioms named Double intuitionistic T_0–space, Double intuitionistic T_1-space, Double intuitionistic T_2-space, Double intuitionistic T_3–space, Double intuitionistic T_4-space, Double intuitionistic $\mathrm{Ragular}$ –space, Double intuitionistic Normal – space. Also, give an example for each type and work on presenting the characteristics and relationships between them through the given theorems, Finally, I studied the relations among several types of Double intuitionistic separation axioms in Double intuitionistic topological spaces (DITS) with some fundamental properties and examples of the opposite are not true for these relationships.

| KEYWORDS
$\mathrm{DIT}_0-\mathrm{space}$, $\mathrm{DIT}_1-\mathrm{space}$, $\mathrm{DIT}_2-\mathrm{space}$, $\mathrm{DIT}_3-\mathrm{space}$, $\mathrm{DIT}_4-\mathrm{space}$, $\mathrm{DIR}-\mathrm{space}$, DIN-space.

| ARTICLE INFORMATION

ACCEPTED: 12 April 2025 **PUBLISHED:** 24 May 2025 **DOI:** 10.32996/jmss.2025.6.2.5

1. Introduction

The thought of usual topological spaces, their sorts and simple perceptions was advanced by step-by-step fuzzy topological & several kinds of double fuzzy semi-closed sets [1-2]. There concept of intuitionistic fuzzy sets in intuitionistic fuzzy sets was given by Atanassov, Krassimir, T. [3]. Bayhan, Sadik [4-6] and Coker, Dogan investigated on separation axioms in intuitionistic topological spaces. The concept of intuitionistic sets and the topology of intuitionistic sets was first by Coker, Dogan [7,8]. Jasim, T.H., Lafta, H. I., & Abdullah, S. M. [9] presented some types of separation axioms for (k, m) -generalized fuzzy-(resp, fuzzy)-closed sets in DFTS. In 2015, Kandil, A., El-Sheikh, S.A., Yakout, M., Hazza, S.A. [10] laid out some types of compactness in double topological spaces. Kandil, A., O. Tantawy, A. E., & Wafaie, M. [11,12] presented in flou (intuitionistic) topological spaces & compact spaces. Khedr, F. H., Sayed, O. R., & Mohamed, S. R. [13] studied some generalized fuzzy separation axioms. In 2023, P. Suganya and J. Arul Jesti now study separation axioms in intuitionistic topological spaces [14]. Prova, Tamanna Tasnim, and Md Sahadat Hossain later used the concept to define separation axioms in intuitionistic topological spaces [15], and Saleh, S., introduces the concept of intuitionistic fuzzy separation axioms [19]. After that, Raoof, Gh., A. & Jassim, T. H. [16,17] was introduced into double intuitionistic topological spaces. In 2019, Roshmi, Rupaya, and M.S. Hossain [18] investigated the properties of separation axioms in bitopological spaces. In (2025), Yaseen S. R., Asmaa Gh. R. Shadia Majeed Noori [20] introduced a new class of closed sets in fuzzy neutrosophic topology. The conceptual framework of the fuzzy set was first discovered in Zane'**s** paper in 1965 [21]**.** The result of this article is a new kind of space in (DITS) Double intuitionistic separation axioms space, that lies between the class of Double intuitionisticT_0 (respectively, Double intuitionistic T_1, Double intuitionistic T_2, Double intuitionistic T_3, Double intuitionistic T_4, Double intuitionistic Regular, Double intuitionistic Normal) spaces, we also showed the basic characteristics of these types and extracted the relationships between them.

2. Préliminaires

Throughout this section, we mention the concepts and notations that we shall use in this paper.

Definition 2.1 [7],[8] Let $X \neq \emptyset$, and let θ and μ be intuitionistic sets (Is, for short) taking the formula $\theta = \langle\theta_1, \theta_2\rangle$; $\mu = \langle\mu_1, \mu_2\rangle$ respectively, let $\{\theta_j: j \in J\}$ be a random assortment of Is in X, where $\theta_j = \langle\theta_j^{(1)}, \theta_j^{(2)}\rangle$ then,

1) $\widetilde{\emptyset} = \langle\emptyset, X\rangle$; $\widetilde{X} = \langle X, \emptyset\rangle$.

2) $\theta \subseteq \mu$, if and only if $\theta_1 \subseteq \mu_1$ & $\theta_2 \supseteq \mu_2$.

3) The complement of θ is denoted by θ^c and defined by $\theta^c = \langle\theta_2, \theta_1\rangle$.

4) $\cup\,\theta_j = \langle\cup\,\theta_j^{(1)}, \cap\,\theta_j^{(2)}\rangle$; $\cap\,\theta_j = \langle\cap\,\theta_j^{(1)}, \cup\,\theta_j^{(2)}\rangle$.

5) $\theta - \mu = \theta \cap \mu^c$.

6) $\theta = \mu$ if and only if is $\theta \subseteq \mu$ and $\mu \subseteq \theta$.

Definition 2.2 [16], [17] Let $\mathcal{X} \neq \emptyset$.

1) A Double intuitionistic open set (DIOS) that is an ordered pair $(\mathscr{g}, \ell) = (\langle\mathscr{g}_1, \mathscr{g}_2\rangle, \langle\ell_1, \ell_2\rangle) \in \mathrm{PI}(\mathcal{X}) \times \mathrm{PI}(\mathcal{X})$ such that $\mathscr{g} \subseteq \ell$.

2) Double I$(\mathcal{X}) = \{(\mathscr{g}, \ell) \in \mathrm{PI}(\mathcal{X}) \times \mathrm{PI}(\mathcal{X}), \mathscr{g} \subseteq \ell\}$. The family of DIOS consists of everyone on $\mathcal{X}$.

3) Let $\psi_1, \psi_2 \subseteq \mathrm{PI}(\mathcal{X})$. The Double product of ψ_1&ψ_2, expressed by $\psi_1 \times \psi_2 = \{(\mathscr{g}, \ell): \mathscr{g} \in \psi_1, \ell \in \psi_2, \mathscr{g} \subseteq \ell\}$

4) The DIOS $(\langle\mathcal{X}, \emptyset\rangle, \langle\mathcal{X}, \emptyset\rangle) = (\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}})$ the universal DIOS, the empty DIOS $(\widetilde{\emptyset}, \widetilde{\emptyset}) = (\langle\emptyset, \mathcal{X}\rangle, \langle\emptyset, \mathcal{X}\rangle)$ the empty DIOS is represented by the DITS.

5) ADIOS $(\mathscr{g}, \ell)$ is called a finite DIOS provided ℓ is a finite DIOS.

6) Let $(\mathscr{g}, \ell), (\mathscr{k}, \mathcal{S}) \in$ DI$(\mathcal{X})$, then:

1) $(\mathscr{g}, \ell)^c = (\ell^c, \mathscr{g}^c) = (\langle\ell_1, \ell_2\rangle^c, \langle\mathscr{g}_1, \mathscr{g}_2\rangle^c) = (\langle\ell_2, \ell_1\rangle, \langle\mathscr{g}_2, \mathscr{g}_1\rangle)$.

2) $(\mathscr{g}, \ell) \setminus (\mathscr{k}, \mathcal{S}) = ((\mathscr{g} \setminus \mathcal{S}), (\ell \setminus \mathscr{k})) = (\langle\mathscr{g}_1, \mathscr{g}_2\rangle, \langle\ell_1, \ell_2\rangle) \setminus (\langle\mathscr{k}_1, \mathscr{k}_2\rangle, \langle\mathcal{S}_1, \mathcal{S}_2\rangle) = (\langle\mathscr{g}_1, \mathscr{g}_2\rangle \setminus \langle\mathcal{S}_1, \mathcal{S}_2\rangle), (\langle\ell_1, \ell_2\rangle, \langle\mathscr{k}_1, \mathscr{k}_2\rangle)$.

7) Let $\{(\mathscr{g}_{1\alpha}, \mathscr{g}_{2\alpha}): \alpha \in \Lambda\} \subseteq$ DI$(\mathcal{X})$, then $\cup_{\alpha \in \Lambda} (\mathscr{g}_{1\alpha}, \mathscr{g}_{2\alpha}) = < \cup_{\alpha \in \Lambda} \mathscr{g}_{1\alpha}, \cap_{\alpha \in \Lambda} \mathscr{g}_{2\alpha} >$ as well as $\cap_{\alpha \in \Lambda} (\mathscr{g}_{1\alpha}, \mathscr{g}_{2\alpha}) = < \cap_{\alpha \in \Lambda} \mathscr{g}_{1\alpha}, \cup_{\alpha \in \Lambda} \mathscr{g}_{2\alpha} >$. It is understood that any component of ψ is called DIOS in $\mathcal{X}$. DIOS, which is a double intuitionistic open set is the complement of a double intuitionistic closed set (DICS).

Definition 2.3 [10], [16], [17] Let $(\mathcal{X}, \psi)$ be a DITS and $(\mathscr{g}, \ell) \in$ DI$\mathcal{X}$, then

1) Double intuitionistic interior (DI interior, for short) of $(\mathscr{g}, \ell)$ is the DIOS s.t int $(\mathscr{g}, \ell) = \cup \{(\mathscr{k}, \mathcal{S}): (\mathscr{k}, \mathcal{S}) \in \psi$ & $(\mathscr{k}, \mathcal{S}) \subseteq (\mathscr{g}, \ell)\}$.

2) Double intuitionistic closure (DI closure, for short) of $(\mathscr{g}, \ell)$ denoted by cl$(\mathscr{g}, \ell)$ or $\overline{(\mathscr{g}, \ell)}$ is the DIOS s.t cl $(\mathscr{g}, \ell) = \cap\{(\mathscr{k}, \mathcal{S}) : (\mathscr{k}, \mathcal{S}) \in \psi^c$ and $(\mathscr{g}, \ell) \subseteq (\mathscr{k}, \mathcal{S})\}$.

Definition 2.4 [8] Let $X \neq \emptyset$, IS in Ꞧ. It is expressed in the form $Ꞧ = \langle Ꞧ_1, Ꞧ_2\rangle$ where $Ꞧ_1$ and $Ꞧ_2$ are separate subsections of X. So $Ꞧ_1$ is called members of Ꞧ, while $Ꞧ_2$ it is called nonmembers of Ꞧ.

Definition 2.5 [7],[10],[11] Let $X \neq \emptyset$. The collection η of DS(X) is termed a double topology on X when it satisfies the following axioms:

1) $\underline{\emptyset}$, $\underline{X} \in \eta$.

2) If $\underline{M}$, $\underline{N} \in \eta$, then $\underline{M} \cap \underline{N} \in \eta$.

3) If $\{\underline{M}_\alpha: \alpha \in \Lambda\} \subseteq \eta$, then $\cup_{\alpha \in \Lambda} \underline{M}_\alpha \in \eta$. The pair called the combination (X, η) a double topological space (DTS, for short).

Definition 2.6 [11], [12], [16], [17], [19] Let $X \neq \emptyset$, then:

1) $\mathfrak{J}(X) = \{\underline{\emptyset}, \underline{X}\}$ then $\mathfrak{J}$ is a DT(X) and is an indiscrete double topology. $(X, \mathfrak{J})$ is called an indiscrete double space.

2) Ŋ (X) = $p(X) \times p(X)$ (power set (PS) of X's) is a DTS, which is said to be a discrete double topology space.

Definition 2.7 [16], [17] Let $\mathcal{X} \neq \emptyset$, $(\tilde{q}, \tilde{q}) \in \mathcal{X}$ be a fixed element in $\mathcal{X}$, and let $(\mathscr{g}, \ell) = (\langle\mathscr{g}_1, \mathscr{g}_2\rangle, \langle\ell_1, \ell_2\rangle)$ be DIOS. The DIOS $(\tilde{q}, \tilde{q})$ is defined by $(\tilde{q}, \tilde{q}) = (\langle\{q\}, \{q\}^c\rangle, \langle\{q\}, \{q\}^c\rangle)$ a Double intuitionistic point (Double I-point, for short) in $\mathcal{X}$.

Definition 2.8 [16], [17] Let $(\mathcal{X}, \psi)$ be a DITS and У be a ≠ subset of $\mathcal{X}$. Then,

$\psi_У = \{(\mathscr{g}, \ell) \cap (У, У): (\mathscr{g}, \ell) \in \psi\}$ is a DIT(ψ). The DITS $(У, \psi_У)$ is the DIT subspace of $(\mathcal{X}, \psi)$.

Definition 2.9 [16], [17] Let$(\mathcal{X},\psi)$ a DITS. $(\gamma_1, \gamma_2) = \{(\mathbb{G}_{1\alpha}, \mathbb{G}_{2\alpha}) : \alpha \in \Lambda\} \subseteq$ DI$(\mathcal{X})$ is called a Double cover of $\mathcal{X}$ if $(\widetilde{\mathcal{X}},\widetilde{\mathcal{X}}) = \cup_{\alpha\in\Lambda}(\mathbb{G}_{1\alpha}, \mathbb{G}_{2\alpha})$, then (γ_1, γ_2) is called DIO cover (DI-open cover, for short). If $(\gamma_1, \gamma_2) \subseteq \Psi^c$, then (γ_1,γ_2) is called DI-closed cover (DIC cover, for short).

Definition 2.10 [4], [5], [6], [15], [18], [19]

Let (X,T) be a topological spaces. Then the space (X,T) is called:

1) T_0 − space iff ∀ a pair of distinct points z ,c ∈X, there is either an open set containing z but not c or an open set containing c but not z.

2) T_1 − space iff ∀ a pair of distinct points j, i ∈ X, ∃ the open set containing j but not i and an open set in$i$ containing j but not i.

3) T_2 − space iff ∀a pair of distinct points$v$, w ∈ X, ∃ an open set $\mathcal{U}$ and $\mathcal{V}$ such that $v \in \mathcal{U}$, $w \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

4) Regular- space iff ∀ closed set F ⊆ X and each point $d \notin$ F, ∃ IOXS $\mathcal{U}$ and $\mathcal{V}$ such that $d \in$ U, F ⊆ $\mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

5) Normal-space iff ∀ pair of closed disjoint subsets $\mathcal{F}$and $\mathcal{E}$ of X, ∃ IOXS $\mathcal{U}$and $\mathcal{V}$ such that $\mathcal{F} \subseteq \mathcal{U}$, $\mathcal{E} \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

6) T_3 − space iff is regular and T_1 − space.

7) T_4 − space iff is normal and T_1 − space.

Definition 2.11 [16], [17] In a DITS, noted as $(\mathcal{X},\psi)$, there's a subset from ψ called β. This subset β serves as a Double I-basis open set (DIBOS, for short) for ψ ▪

1) β ⊆ ψ.

2) ▪ $(\pi, o) \in \psi$; $(\pi, o) = \cup_j (g_j, \ell_j)$; $(g_j, \ell_j) \in \beta$ ▪j.

Theorem 2.12 [16], [17] If $(\mathcal{X},\psi)$ a DITS and $(g, \ell), (k, \mathcal{S}) \in$ DI$(\mathcal{X})$. The following properties are hold:

1) (g, ℓ) ▪▪cl (g, ℓ).

2) (g, ℓ) is DI-closed ▪ cl$(g, \ell) = (g,\ell)$.

3) If (g, ℓ) ▪▪$(k, \mathcal{S})$ then cl (g,ℓ) ▪ cl $(k, \mathcal{S})$.

Proposition 2.13 [16], [17] Let $(\mathcal{X},\psi)$ is a DI-compact space. Then every DI-closed set is DI-compact.

3. Double Intuitionistic T_0 − space, Double Intuitionistic T_1 − space,Double Intuitionistic T_2 − space in DITS.

This section delves into new concepts of Double intuitionistic separation axioms. It analyzes their interrelationships, providing detailed proofs and examples for clarity.

Definition 3.1 Let (χ, φ) be DITS, then (χ, φ) is said to be Double intuitionistic $T_0 - \text{space}$ $(\text{DIT}_0 - \text{S}, \text{for short})$ ▪▪▪$(\tilde{g}, \tilde{g}), (\tilde{f}, \tilde{f}) \in \chi$▪, $(\tilde{g}, \tilde{g}) \neq (\tilde{f}, \tilde{f}), \exists\, (u_1, v_1) \in \varphi$, $((\tilde{g}, \tilde{g}) \in (u_1, v_1)$ ▪▪$(\tilde{f}, \tilde{f}) \notin (u_1, v_1)) \vee ((\tilde{g}, \tilde{g}) \notin (u_1, v_1)$▪▪ $(\tilde{f}, \tilde{f}) \in (u_1, v_1))$.

Example 3.2 Let χ ={i, j, k}; φ = {$(\widetilde{\emptyset},\widetilde{\emptyset})$,$(\widetilde{\chi}, \widetilde{\chi})$, $(\mho_1, \mho_2)$, $(\mho_3, \mho_4)$, $(\mho_5, \widetilde{\chi})$, $(\mho_6, \mho_7)$} where $(\mho_1,\mho_2) = (\langle\{j\},\{i\}\rangle, \langle\{i,k\},\{j\}\rangle)$, $(\mho_3,\mho_4) = (\langle\{j\},\{k\}\rangle, \langle\{i,j\},\{k\}\rangle)$, $(\mho_5,\widetilde{\chi}) = (\langle\{i,j\},\emptyset\rangle, \langle\chi,\emptyset\rangle)$, $(\mho_6, \mho_7) = (\langle\emptyset,\{j,k\}\rangle, \langle\{i\},\{j,k\}\rangle)$. Since $(\tilde{i},\tilde{i}) = (\langle\{i\},\{j,k\}\rangle, \langle\{i\},\{j,k\}\rangle)$, $(\tilde{j}, \tilde{j}) = (\langle\{j\},\{i,k\}\rangle, \langle\{j\},\{i,k\}\rangle)$, $(\tilde{k},\tilde{k}) = (\langle\{k\},\{i,j\}\rangle, \langle\{k\},\{i,j\}\rangle)$, then $(\tilde{i},\tilde{i}) \neq (\tilde{j},\tilde{j}) \to \exists$ DIOS(χ), $(\tilde{i},\tilde{i}) \in (\mho_1,\mho_2)$ & $(\mho_5, \widetilde{\chi})$▪▪ $(\tilde{j},\tilde{j}) \notin (\mho_1,\mho_2)$. $(\tilde{i}, \tilde{i}) \neq (\tilde{k}, \tilde{k}) \to \exists$ DIOS(χ), $(\tilde{i}, \tilde{i}) \in (\mho_1, \mho_2)$ & $(\mho_5, \widetilde{\chi})$▪▪ $(\tilde{k},\tilde{k}) \notin (\mho_1,\mho_2)$ and $(\mho_5, \widetilde{\chi})$, $(\tilde{j}, \tilde{j}) \neq (\tilde{k},\tilde{k}) \to$ ∃ DIOS(χ), $(\tilde{j},\tilde{j}) \in (\mho_3, \mho_4)$▪▪ $(\tilde{k}, \tilde{k}) \notin (\mho_3, \mho_4)$ and $(\mho_5,\widetilde{\chi})$. Therefore is satisfy, (χ, φ) is $\text{DIT}_0 - \text{S}$.

Now, I present the theorem that gives equivalent parts for the definition $\text{DIT}_0 - \text{S}$.

Theorem 3.3 Let (χ, φ) is $\text{DIT}_0 - \text{Space}$ if and only if $\text{cl}_\varphi\{(\tilde{g}, \tilde{g})\} \neq \text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$,▪▪$(\tilde{g}, \tilde{g}), (\tilde{f}, \tilde{f}) \in \chi$▪, $(\tilde{g}, \tilde{g}) \neq (\tilde{f}, \tilde{f})$.

Proof (▪) Let χ be $\text{DIT}_0 - \text{S}$, to prove that $\text{cl}_\varphi\{(\tilde{g},\tilde{g})\} \neq \text{cl}_\Psi\{(\tilde{f},\tilde{f})\}$,▪▪$(\tilde{g}, \tilde{g}), (\tilde{f}, \tilde{f}) \in \chi$, $(\tilde{g}, \tilde{g}) \neq (\tilde{f}, \tilde{f})$. Since χ is $\text{DIT}_0 - \text{S}$ and $(\tilde{g}, \tilde{g}) \neq (\tilde{f}, \tilde{f}) \to \exists\, (u_1, v_1) \in \varphi$; $((\tilde{g}, \tilde{g}) \in (u_1, v_1)$▪▪ $(\tilde{f}, \tilde{f}) \notin (u_1, v_1)) \vee ((\tilde{g}, \tilde{g}) \notin (u_1, v_1)$▪▪ $(\tilde{f}, \tilde{f}) \in (u_1, v_1))$. Let $(\tilde{g}, \tilde{g}) \in (u_1, v_1)$ ▪ $(\tilde{f}, \tilde{f}) \notin (u_1, v_1) \to (\tilde{g}, \tilde{g}) \in (u_1, v_1)$ ▪$(\tilde{f}, \tilde{f}) \in \chi^c$. Hence χ^c is DICS, since (u_1, v_1) is DIOS in χ $\to \{(\tilde{f},\tilde{f})\}$ ▪ $\chi^c \to \text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$▪▪ $\text{cl}_\varphi(\chi^c) = \chi^c$ (by theorem 2.14 (2)), so $\{(\tilde{f},\tilde{f})\}$ ▪ χ^c▪▪$\binom{\tilde{g}}{,\tilde{g}} \in (u_1, v_1) \to \{(\tilde{g},\tilde{g})\} \not\subseteq \chi^c$,$\text{cl}_\varphi\{(\tilde{g},\tilde{g})\} \not\subseteq \chi^c$. So $\text{cl}_\varphi\{(\tilde{g},\tilde{g})\} \neq \text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$. By similar means if I take $(\tilde{g},\tilde{g}) \in (u_1, v_1)$ ▪ $(\tilde{f},\tilde{f}) \notin (u_1, v_1)$.

Conversely ▪▪) suppose that χ is not $\text{DIT}_0 - \text{Space}$, $\exists\, ((\tilde{g}, \tilde{g}), (\tilde{f},\tilde{f})) \in \chi$; ▪$(u_1, v_1) \in \chi$, $(\tilde{g}, \tilde{g}) \in (u_1, v_1) \to (\tilde{f},\tilde{f}) \in (u_1, v_1)$ (by definition $\text{DIT}_0 - \text{S}$). Let $(\tilde{w},\tilde{w}) \in \chi$; $(\tilde{w}, \tilde{w}) \in \text{cl}_\varphi\{(\tilde{g}, \tilde{g})\} \to$ ▪▪$(u_1, v_1) \in \varphi$, $(\tilde{w},\tilde{w}) \in (u_1, v_1)$ ▪▪▪(u_1, v_1) ▪$\cap \{(\tilde{g},\tilde{g})\} \neq (\widetilde{\emptyset},\widetilde{\emptyset})$. But $(u_1, v_1) \cap \{(\tilde{g},\tilde{g})\} \neq (\widetilde{\emptyset},\widetilde{\emptyset}) \to (\tilde{g},\tilde{g}) \in (u_1, v_1)$ (since the only Double I-element it $\{(\tilde{g}, \tilde{g})\}$ is $(\tilde{g},\tilde{g})$. So every DIS contain $(\tilde{w},\tilde{w})$ must contain $(\tilde{g}, \tilde{g})$. Thus, I have the following two statements; Every DIOS(χ) must contain $(\tilde{g},\tilde{g})$ and every DIOS in χ that contains must contain $(\tilde{f},\tilde{f})$. Therefore, every DIOS in χ, $(\tilde{w},\tilde{w})$ must contain $(\tilde{f},\tilde{f}) \to$▪ $(u_1, v_1) \in \varphi$, $(\tilde{w},\tilde{w}) \in (u_1, v_1)$ ▪▪$(u_1, v_1) \cap \{(\tilde{f},\tilde{f})\} \neq (\widetilde{\emptyset},\widetilde{\emptyset}) \to (\tilde{w},\tilde{w}) \in \text{cl}_\varphi\{(\tilde{f},\tilde{f})\} \to$▪ $(\tilde{w},\tilde{w}) \in \text{cl}_\varphi\{(\tilde{g},\tilde{g})\} \to (\tilde{w},\tilde{w}) \in \text{cl}_\varphi\{(\tilde{f},\tilde{f})\} \to \text{cl}_\varphi\{(\tilde{g},\tilde{g})\}$ ▪▪$\text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$ by a similar proof $\text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$ ▪ $\text{cl}_\varphi\{(\tilde{g},\tilde{g})\}$. Therefore $\text{cl}_\varphi\{(\tilde{g},\tilde{g})\}$ ▪ $\text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$. This is a contradiction (since $\text{cl}_\varphi\{(\tilde{g},\tilde{g})\} \neq \text{cl}_\varphi\{(\tilde{f},\tilde{f})\}$). Hence χ is $\text{DIT}_0 - \text{S}$.

Theorem 3.4 Let (χ , φ) be $\mathrm{DIT_0}$ − Space and (Ɏ, $\varphi_{Ɏ}$) is Double intuitionistic topological subspace of (χ , φ), then (Ɏ, $\varphi_{Ɏ}$) is $\mathrm{DIT_0}$ − S.

Proof Let $(\tilde{g},\tilde{g}),(\tilde{f},\tilde{f}) \in$ Ɏ; $(\tilde{g},\tilde{g}) \neq (\tilde{f},\tilde{f}) \rightarrow (\tilde{g},\tilde{g}),(\tilde{f},\tilde{f}) \in \chi$ (since Ɏ • χ) • • Since χ is $\mathrm{DIT_0} - \mathrm{S} \rightarrow \exists(u_1, v_1) \in \varphi$; $((\tilde{g},\tilde{g}) \in (u_1,v_1)$ • $(\tilde{f},\tilde{f}) \notin (u_1,v_1)) \vee ((\tilde{g},\tilde{g}) \notin (u_1,v_1)$ • • $(\tilde{f},\tilde{f}) \in (u_1,v_1)) \rightarrow (u_1,v_1)$ • ∩ Ɏ ∈ $\varphi_{Ɏ}$ (by definition 2.8); $((\tilde{g},\tilde{g}) \in (u_1,v_1) \cap$ Ɏ • $(\tilde{f},\tilde{f}) \notin (u_1,v_1) \cap$ Ɏ) ∨ $((\tilde{g},\tilde{g}) \notin (u_1,v_1) \cap$ Ɏ • $(\tilde{f},\tilde{f}) \in (u_1,v_1) \cap$ Ɏ). Therefore, (Ɏ, $\varphi_{Ɏ}$) is $\mathrm{DIT_0}$ − S.

Theorem 3.5 Let (χ , φ) & (Ɏ,φ*) be two DITS. The Double I-product space χ × Ɏ is $\mathrm{DIT_0}$ − Space if and only if each χ & Ɏ are $\mathrm{DIT_0}$ − Space.

Proof (•) Let $(\tilde{g_1},\tilde{g_1}),(\tilde{g_2},\tilde{g_2}) \in \chi$, $(\tilde{g_1},\tilde{g_1}) \neq (\tilde{g_2},\tilde{g_2})$ and $(\tilde{f_1},\tilde{f_1}),(\tilde{f_2},\tilde{f_2}) \in$ Ɏ, $(\tilde{f_1},\tilde{f_1}) \neq (\tilde{f_2},\tilde{f_2}) \rightarrow ((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}),(\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \in$ χ ×Ɏ, $(\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \neq (\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})$. Since χ × Ɏ are $\mathrm{DIT_0}$ − S → ∃DIBOS $(u_1,v_1) \times (u_2,v_2) \in \varphi_{\chi \times Ɏ}$.$((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \in (u_1,v_1) \times (u_2,v_2))$ • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \notin (u_1,v_1) \times (u_2,v_2))$ • ∨ $(\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \notin (u_1,v_1) \times (u_2,v_2))$ • • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \in (u_1,v_1) \times (u_2,v_2)) \rightarrow \exists\,(u_1,v_1) \in \varphi$; $((\tilde{g_1},\tilde{g_1}) \in (u_1,v_1)$ • • $(\tilde{g_2},\tilde{g_2}) \notin (u_1,v_1)) \vee ((\tilde{g_1},\tilde{g_1}) \notin (u_1,v_1)$ • • $(\tilde{g_2},\tilde{g_2}) \in (u_1,v_1)) \rightarrow$ χisDIT$_0$ − S, $\exists(u_2,v_2) \in \varphi^*$;$((\tilde{f_1},\tilde{f_1}) \in (u_2,v_2)$ • • $(\tilde{f_2},\tilde{f_2}) \notin (u_2,v_2))$ • ∨$((\tilde{f_1},\tilde{f_1}) \notin (u_2,v_2)$ • $(\tilde{f_2},\tilde{f_2}) \in (u_2,v_2))$ •. Therefore Ɏ is DITS.

Conversely • •) Let $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1})),((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \in \chi \times$ Ɏ; $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \neq ((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \rightarrow ((\tilde{g_1},\tilde{g_1}),(\tilde{g_2},\tilde{g_2})) \in \chi$ • • • $(\tilde{g_1},\tilde{g_1}) \neq (\tilde{g_2},\tilde{g_2})$ • • • $((\tilde{f_1},\tilde{f_1}),(\tilde{f_2},\tilde{f_2})) \in$ Ɏ • • • $((\tilde{f_1},\tilde{f_1}) \neq (\tilde{f_2},\tilde{f_2})$. Since χ is $\mathrm{DIT_0}$ − S, so → $\exists\,(u_1,v_1) \in \varphi$; $((\tilde{g_1},\tilde{g_1}) \in (u_1,v_1)$ • • • • $(\tilde{g_2},\tilde{g_2}) \notin (u_1,v_1)) \vee ((\tilde{g_1},\tilde{g_1}) \notin (u_1,v_1)$ • • $(\tilde{g_2},\tilde{g_2}) \in (u_1,v_1))$. Since Ɏ is $\mathrm{DIT_0}$ − S, so $\exists\,(u_2,v_2) \in \varphi^*$; $((\tilde{f_1},\tilde{f_1}) \in (u_2,v_2)$ • $(\tilde{f_2},\tilde{f_2}) \notin (u_2,v_2)) \vee (\tilde{f_1},\tilde{f_1}) \notin (u_2,v_2)$ • • $(\tilde{f_2},\tilde{f_2}) \in (u_2,v_2))$. So there exists $((u_1,v_1) \times (u_2,v_2))$ is Double I-basic open set, $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \in (u_1,v_1) \times (u_2,v_2)$ • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \notin ((u_1,v_1) \times (u_2,v_2)) \vee ((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \notin (u_1,v_1) \times (u_2,v_2))$ • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \in ((u_1,v_1) \times (u_2,v_2))$. Hence χ × Ɏ is $\mathrm{DIT_0}$ − Space.

Definition 3.6 Let (χ , φ) a DITS, then (χ , φ) it is supposed to be Double intuitionistic $\mathrm{T_1}$ − space ($\mathrm{DIT_1}$ − S, for short) • • • • $(\tilde{g},\tilde{g}),(\tilde{f},\tilde{f}) \in \chi$, $(\tilde{g},\tilde{g}) \neq (\tilde{f},\tilde{f})$, $\exists\,(u_1,v_1),(u_2,v_2) \in \varphi$, $((\tilde{g},\tilde{g}) \in (u_1,v_1)$ • • $(\tilde{f},\tilde{f}) \notin (u_1,v_1))$ • • • $((\tilde{g},\tilde{g}) \notin (u_2,v_2)$ • • $(\tilde{f},\tilde{f}) \in (u_2,v_2))$.

Example 3.7 Let χ = {𝑎, 𝒷}; φ = $\{(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\chi},\tilde{\chi}),(\mathcal{S}_1,\tilde{\chi}),(\mathcal{S}_1^c,\mathcal{S}_1^c),(\tilde{\emptyset},\mathcal{S}_1^c)\}$ where $(\mathcal{S}_1,\tilde{\chi}) = (\langle\{a\},\{b\}\rangle, \langle\chi,\emptyset\rangle)$, $((\mathcal{S}_1^c,\mathcal{S}_1^c) = (\langle\{b\},\{a\}\rangle,\langle\{b\},\{a\}\rangle)$ and $(\tilde{\emptyset},\mathcal{S}_1^c) = (\langle\emptyset,\chi\rangle,\langle\{b\},\{a\}\rangle)$. Since $(\tilde{a},\tilde{a}) = (\langle\{a\},\{b\}\rangle, \langle\{a\},\{b\}\rangle)$, $(\tilde{b},\tilde{b}) = (\langle\{b\},\{a\}\rangle, \langle\{b\},\{a\}\rangle) \in \chi$; then $(\tilde{a},\tilde{a}) \neq (\tilde{b},\tilde{b}) \rightarrow \exists\,(\mathcal{S}_1,\tilde{\chi}),(\mathcal{S}_1^c,\mathcal{S}_1^c) \in \varphi$ such that $(\tilde{a},\tilde{a}) \in (\mathcal{S}_1,\tilde{\chi})$ • • • $(\tilde{b},\tilde{b}) \notin (\mathcal{S}_1,\tilde{\chi})$ • • $(\tilde{a},\tilde{a}) \notin (\mathcal{S}_1^c,\mathcal{S}_1^c)$ • • $(\tilde{b},\tilde{b}) \in (\mathcal{S}_1^c,\mathcal{S}_1^c)$. Therefore, (χ , φ) is $\mathrm{DIT_1}$ − Space.

Theorem 3.8 Let (χ , φ) is $\mathrm{DIT_1}$ − Space • $\{(\tilde{g},\tilde{g})\}$ is Double I-closed set, for each $(\tilde{g},\tilde{g}) \in \chi$.

Proof (•) Let $(\tilde{f},\tilde{f}) \in \{(\tilde{g},\tilde{g})\}$, $(\tilde{g},\tilde{g_1}) \neq (\tilde{g_2},\tilde{g_2})$. Since χ it is $\mathrm{DIT_1}$ − S, then thereexist (u_1,v_1)., $\left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right) \in \varphi$; $(\tilde{g},\tilde{g}) \in (u_1,v_1)$ • • $(\tilde{f},\tilde{f}) \notin (u_1,v_1)$ • • $(\tilde{g},\tilde{g}) \notin \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right)$ • • • $(\tilde{f},\tilde{f}) \in \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right)$. So $\{(\tilde{g},\tilde{g})\} \cap \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right) = (\tilde{\emptyset},\tilde{\emptyset}) \rightarrow \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right)$ • • $\{(\tilde{g},\tilde{g})\}^c$ • • $(\tilde{f},\tilde{f}) \in \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right) \rightarrow \left(u_{(\tilde{f},\tilde{f})_2}, v_{(\tilde{f},\tilde{f})_2}\right)$ • • • $\{(\tilde{g},\tilde{g})\}^c$ for each $(\tilde{f},\tilde{f}) \in \{(\tilde{g},\tilde{g})\}^c$. Hence $\{(\tilde{g},\tilde{g})\}^c$ is DIOS. Therefore, $\{(\tilde{g},\tilde{g})\}$ is DICS, for each$(\tilde{g},\tilde{g}) \in \chi$.

Conversely • •) Let $(\tilde{g},\tilde{g})$ • $(\tilde{f},\tilde{f}) \in \chi$; $(\tilde{g},\tilde{g}) \neq (\tilde{f},\tilde{f}) \rightarrow \{(\tilde{g},\tilde{g})\}$, $\{(\tilde{f},\tilde{f})\}$ are DICS in χ. Then $\{(\tilde{g},\tilde{g})\}^c$, $\{(\tilde{f},\tilde{f})\}^c$ are DIOS in χ. Say (u_1,v_1) • • $\{(\tilde{f},\tilde{f})\}^c$, (u_2,v_2) • • $\{(\tilde{g},\tilde{g})\}^{\cdot c} \rightarrow (\tilde{g},\tilde{g}) \in (u_1,v_1)$ • $(\tilde{f},\tilde{f}) \notin (u_1,v_1)$ • • $(\tilde{g},\tilde{g}) \notin (u_2,v_2)$ • $(\tilde{f},\tilde{f}) \in (u_2,v_2)$. Therefore, (χ , φ) is $\mathrm{DIT_1}$ − Space.

Theorem 3.9 Let (χ , φ) be $\mathrm{DIT_1}$ − Space and (Ɏ, $\varphi_{Ɏ}$) is Double intuitionistic topological subspace of (χ , φ), then (Ɏ, $\varphi_{Ɏ}$) is $\mathrm{DIT_1}$ − S.

Proof: Let $(\tilde{g},\tilde{g}),(\tilde{q},\tilde{q}) \in$ Ɏ; $(\tilde{g},\tilde{g}) \neq (\tilde{f},\tilde{f})$, so $(\tilde{g},\tilde{g}),(\tilde{f},\tilde{f}) \in \chi$ • • Since χ is $\mathrm{DIT_1}$ − S, $\exists(u_1,v_1)$, $(u_2,v_2) \in \varphi$; $((\tilde{g},\tilde{g}) \in (u_1,v_1)$ • $(\tilde{f},\tilde{f}) \notin (u_1,v_1))$ • • $((\tilde{g},\tilde{g}) \notin (u_2,v_2)$ • • • $(\tilde{f},\tilde{f}) \in (u_2,v_2)) \rightarrow (u_1,v_1) \cap$ Ɏ • (u_2,v_2) • ∩ Ɏ ∈ $\varphi_{Ɏ}$ (by definition 2.8); $((\tilde{g},\tilde{g}) \in (u_1,v_1) \cap$ Ɏ • • $(\tilde{f},\tilde{f}) \notin (u_1,v_1)$ • ∩ Ɏ) • • $((\tilde{g},\tilde{g}) \notin (u_2,v_2) \cap$ Ɏ • • • $(\tilde{f},\tilde{f}) \in (u_2,v_2) \cap$ Ɏ)). Therefore, (Ɏ, $\varphi_{Ɏ}$) is $\mathrm{DIT_1}$ − S.

Theorem 3.10 Let (χ , φ) & (Ɏ,φ*) be two DITS. Then the Double I-product space χ ×Ɏ is $\mathrm{DIT_1}$ − S if and only if each χ & Ɏ are $\mathrm{DIT_1}$ − Space.

Proof (•) Let $(\tilde{g_1},\tilde{g_1}),(\tilde{g_2},\tilde{g_2}) \in \chi$, $(\tilde{g_1},\tilde{g_1}) \neq (\tilde{g_2},\tilde{g_2})$ and $(\tilde{f_1},\tilde{f_1}),(\tilde{f_2},\tilde{f_2}) \in$ Ɏ, $(\tilde{f_1},\tilde{f_1}) \neq (\tilde{f_2},\tilde{f_2}) \rightarrow ((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}),(\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \in \chi \times$ Ɏ, $(\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \neq (\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}))$. Since χ × Ɏ is $\mathrm{DIT_1}$ − S → $\exists\,(u_1,v_1),(u_2,v_2) \in \varphi_{\chi\times Ɏ}$ $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \in (u_1,v_1)$ • • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \notin (u_1,v_1))$ • • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \in (u_2,v_2)$ • • • $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \notin (u_2,v_2))$, • that means there exist Double • • • basis open sets $((u_1,v_1)_1 \times (u_2,v_2)_1)$, $((u_1,v_1)_2 \times (u_2,v_2)_2) \in \varphi_{\chi\times Ɏ}$. $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \in ((u_1,v_1)_1 \times (u_2,v_2)_1)$ • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \notin$ • • $((u_1,v_1)_1 \times (u_2,v_2)_1)$ • • $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1}) \notin ((u_1,v_1)_2 \times (u_2,v_2)_2)$ • • $((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2}) \in ((u_1,v_1)_2 \times (u_2,v_2)_2) \rightarrow \exists\,(u_1,v_1)_1,(u_1,v_1)_2$ • $\in \varphi$; • $((\tilde{g_1},\tilde{g_1}) \in (u_1,v_1)_1$ • • $(\tilde{g_2},\tilde{g_2}) \notin (u_1,v_1)_1)$ • $((\tilde{g_1},\tilde{g_1}) \notin (u_2,v_2)_2$ • • $(\tilde{g_2},\tilde{g_2}) \in (u_2,v_2)_2)$ • • So • • χ is DIT$_1$-Space • $\exists((u_2,v_2)_1$ •, $(u_2,v_2)_2) \in \varphi$ • • • • $(\tilde{f_1},\tilde{f_1}) \in$ • $((u_2,v_2)_1$ • • $(\tilde{f_2},\tilde{f_2}) \notin ((u_2,v_2)_2)$ • • $((\tilde{f_1},\tilde{f_1}) \notin ((u_2,v_2)_2$ • • $(\tilde{f_2},\tilde{f_2}) \in ((u_2,v_2)_2)$. Hence, Ɏ is $\mathrm{DIT_1}$ − Space.

Conversely • • •) Let $((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1})),((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \in \chi \times$ Ɏ; `$((\tilde{g_1},\tilde{g_1}),(\tilde{f_1},\tilde{f_1})) \neq ((\tilde{g_2},\tilde{g_2}),(\tilde{f_2},\tilde{f_2})) \rightarrow (\tilde{g_1},\tilde{g_1}),((\tilde{g_2},\tilde{g_2}) \in \chi$ • • $(\tilde{g_1},\tilde{g_1}) \neq ((\tilde{g_2},\tilde{g_2})$ • • $((\tilde{f_1},\tilde{f_1}),(\tilde{f_2},\tilde{f_2})) \in$ Ɏ • • $(\tilde{f_1},\tilde{f_1}) \neq (\tilde{f_2},\tilde{f_2})$. Since χ is $\mathrm{DIT_1}$ − S, there exists $((u_1,v_1)_1,(u_1,v_1)_2 \in \varphi$ •; $((\tilde{g_1},\tilde{g_1}) \in$

$(u_1, v_1)_1 \cdot\cdot\ (\widetilde{g_2}, \widetilde{g_2}) \notin (u_1, v_1)_1) \cdot\cdot\ ((\tilde{g}_1, \tilde{g}_1) \notin (u_1, v_1)_2 \cdot (\widetilde{g_2}, \widetilde{g_2}) \in (u_1, v_1)_2)$. Since Ɐ is $\mathrm{DIT}_1 - \mathrm{S}$, then there exist $((u_2, v_2)_1, \cdot (u_2, v_2)_2) \cdot \in \varphi^*; ((\tilde{f}_1, \tilde{f}_1) \in (u_2, v_2)_1 \cdot\cdot (\tilde{f}_2, \tilde{f}_2) \notin (u_2, v_2)_1) \cdot\cdot\ ((\tilde{f}_1, \tilde{f}_1) \notin (u_2, v_2)_2 \cdot\cdot\ (\tilde{f}_2, \tilde{f}_2) \in (u_2, v_2)_2) \cdot$ there exists Double$\cdot\cdot\cdot$basis open sets $((u_1, v_1)_1 \times (u_2, v_2)_1); \left((\tilde{g}_1, \tilde{g}_1), (\tilde{f}_1, \tilde{f}_1)\right) \in ((u_1, v_1)_1 \times (u_2, v_2)_1) \cdot\cdot\ ((\widetilde{g_2}, \widetilde{g_2}), (\tilde{f}_2, \tilde{f}_2)) \notin ((u_1, v_1)_1 \times (u_2, v_2)_1) \cdot\cdot\cdot \left((\tilde{g}_1, \tilde{g}_1), (\tilde{f}_1, \tilde{f}_1)\right) \notin ((u_1, v_1)_2 \times (u_2, v_2)_2) \cdot\cdot\ ((\widetilde{g_2}, \widetilde{g_2}), (\tilde{f}_2, \tilde{f}_2)) \in ((u_1, v_1)_2 \times (u_2, v_2)_2)$. Therefore, $\chi \times \text{Ɐ}$ is DIT_1 $-$Space.

Definition 3.11 Let (χ, φ) a DITS, then(χ, φ) is assumed to be Double intuitionistic $\mathrm{T}_2 - \mathrm{space}(\mathrm{DIT}_2 -$
$\mathrm{S}, \text{for short}) \cdot\cdot\cdot\cdot (\tilde{p}, \tilde{p}), (\tilde{ł}, \tilde{ł}) \in \chi$, $(\tilde{p}, \tilde{p}) \neq (\tilde{ł}, \tilde{ł})\ \exists\ (\upsilon_1, V_1), (\upsilon_2, V_2) \in \varphi$, $(\tilde{p}, \tilde{p}) \in (\upsilon_1, V_1) \cdot\cdot (\tilde{ł}, \tilde{ł}) \in (\upsilon_2, V_2) \cdot\cdot (\upsilon_1, V_1) \cdot \cap (\upsilon_2, V_2) = (\tilde{\emptyset}, \tilde{\emptyset})$.

Example 3.12 Let $\chi = \{x, y\}$; $\varphi = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{\chi}, \tilde{\chi}), (\tilde{\emptyset}, \omega_1), (\tilde{\emptyset}, \omega_2), (\tilde{\emptyset}, \omega_3), (\tilde{\emptyset}, \omega_3^c), (\tilde{\emptyset}, \omega), (\tilde{\emptyset}, \omega_2^c), (\tilde{\emptyset}, \omega_4), (\omega_1, \tilde{\chi}), (\omega_3, \tilde{\chi}), (\omega_2, \tilde{\chi}), (\omega_3^c, \tilde{\chi}), (\omega_1^c, \tilde{\chi}), (\omega_2^c, \tilde{\chi}), (\omega_4, \tilde{\chi}), (\omega_3, \omega_1), (\omega_1, \omega_1), (\omega_3, \omega_3), (\omega_2, \omega_2), (\omega_3^c, \omega_3^c), (\omega_1^c, \omega), (\omega_2^c, \omega_2^c), (\omega_4, \omega_4), (\omega_3^c, \omega_2), (\omega_1^c, \omega_1), (\omega_1^c, \omega_2), (\omega_1^c, \omega_3^c), (\omega_1^c, \omega_4), (\omega_2^c, \omega_1), (\omega_2^c, \omega_3), (\omega_2^c, \omega_2), (\omega_2^c, \omega_4), (\omega_4, \omega_1), (\omega_4, \omega_2), (\tilde{\emptyset}, \tilde{\chi})\}$ where $(\tilde{\emptyset}, \omega_1) = (\langle\emptyset, \chi\rangle, \langle\{x\}, \emptyset\rangle)$, $(\tilde{\emptyset}, \omega_2) = (\langle\emptyset, \chi\rangle, \langle\{y\}, \emptyset\rangle)$, $(\tilde{\emptyset}, \omega_3) = (\langle\emptyset, \chi\rangle, \langle\{x\}, \{y\}\rangle)$, $(\tilde{\emptyset}, \omega_3^c) = (\langle\emptyset, \chi\rangle, \langle\{y\}, \{x\}\rangle)$, $(\tilde{\emptyset}, \omega_1^c) = (\langle\emptyset, \chi\rangle, \langle\emptyset, \{x\}\rangle)$, $(\tilde{\emptyset}, \omega_2^c) = (\langle\emptyset, \chi\rangle, \langle\emptyset, \{y\}\rangle)$, $(\tilde{\emptyset}, \omega_4) = (\langle\emptyset, \chi\rangle, \langle\emptyset, \emptyset\rangle)$, $(\omega_1, \tilde{\chi}) = (\langle\{x\}, \emptyset\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_3, \tilde{\chi}) = (\langle\{x\}, \{y\}\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_2, \tilde{\chi}) = (\langle\{y\}, \emptyset\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_3^c, \tilde{\chi}) = (\langle\{y\}, \{x\}\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_1^c, \tilde{\chi}) = (\langle\emptyset, \{x\}\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_2^c, \tilde{\chi}) = (\langle\emptyset, \{y\}\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_4, \tilde{\chi}) = (\langle\emptyset, \emptyset\rangle, \langle\chi, \emptyset\rangle)$, $(\omega_3, \omega_1) = (\langle\{x\}, \{y\}\rangle, \langle\{x\}, \emptyset\rangle)$, $(\omega_1, \omega_1) = (\langle\{x\}, \emptyset\rangle, \langle\{x\}, \emptyset\rangle)$, $(\omega_3, \omega_3) = (\langle\{x\}, \{y\}\rangle, \langle\{x\}, \{y\}\rangle)$, $(\omega_2, \omega_2) = (\langle\{y\}, \emptyset\rangle, \langle\{y\}, \emptyset\rangle)$, $(\omega_3^c, \omega_3^c) = (\langle\{y\}, \{x\}\rangle, \langle\{y\}, \{x\}\rangle)$, $(\omega_1^c, \omega_1^c) = (\langle\emptyset, \{x\}\rangle, \langle\emptyset, \{x\}\rangle)$, $(\omega_2^c, \omega_2^c) = (\langle\emptyset, \{y\}\rangle, \langle\emptyset, \{y\}\rangle)$, $(\omega_4, \omega_4) = (\langle\emptyset, \emptyset\rangle, \langle\emptyset, \emptyset\rangle)$, $(\omega_3^c, \omega_2) = (\langle\{y\}, \{x\}\rangle, \langle\{y\}, \emptyset\rangle)$, $(\omega_1^c, \omega_1) = (\langle\emptyset, \{x\}\rangle, \langle\{x\}, \emptyset\rangle)$, $(\omega_1^c, \omega_2) = (\langle\emptyset, \{x\}\rangle, \langle\{y\}, \emptyset\rangle)$, $(\omega_1^c, \omega_3^c) = (\langle\emptyset, \{x\}\rangle, \langle\{y\}, \{x\}\rangle)$, $(\omega_1^c, \omega_4) = (\langle\emptyset, \{x\}\rangle, \langle\emptyset, \emptyset\rangle)$, $(\omega_2^c, \omega_1) = (\langle\emptyset, \{y\}\rangle, \langle\{x\}, \emptyset\rangle)$, $(\omega_2^c, \omega_3) = (\langle\emptyset, \{y\}\rangle, \langle\{x\}, \{y\}\rangle)$, $(\omega_2^c, \omega_2) = (\langle\emptyset, \{y\}\rangle, \langle\{y\}, \emptyset\rangle)$, $(\omega_2^c, \omega_4) = (\langle\emptyset, \{y\}\rangle, \langle\emptyset, \emptyset\rangle)$, $(\omega_4, \omega_1) = (\langle\emptyset, \emptyset\rangle, \langle\{x\}, \emptyset\rangle)$, $(\omega_4, \omega_2) = (\langle\emptyset, \emptyset\rangle, \langle\{y\}, \emptyset\rangle)$ and $(\tilde{\emptyset}, \tilde{\chi}) = (\langle\emptyset, \chi\rangle, \langle\chi, \emptyset\rangle)$. Let $(\tilde{x}, \tilde{x}) = (\langle\{x\}, \{y\}\rangle, \langle\{x\}, \{y\}\rangle)$, $(\tilde{y}, \tilde{y}) = (\langle\{y\}, \{x\}\rangle, \langle\{y\}, \{x\}\rangle) \in \chi$; then $(\tilde{x}, \tilde{x}) \neq (\tilde{y}, \tilde{y})$, since $(\omega_3, \omega_3), (\omega_3^c, \omega_3^c) \in \varphi$ such that $(\omega_3, \omega_3) \cap (\omega_3^c, \omega_3^c) = (\tilde{\emptyset}, \tilde{\emptyset})$, $(\tilde{x}, \tilde{x}) \in (\omega_3, \omega_3) \cdot\cdot\ (\tilde{y}, \tilde{y}) \in (\omega_3^c, \omega_3^c)$. Hence (χ, φ) is $\mathrm{DIT}_2 - \mathrm{Space}$.

Remark 3.13 Every $\mathrm{DIT}_2 - \mathrm{S}$ is $\mathrm{DIT}_1 - \mathrm{S}$. But the reverse implications do not hold and the following example shows that:

Example 3.14 Let $\chi = \{\alpha, \beta\}$; $\varphi = \{(\tilde{\emptyset}, \tilde{\emptyset}), (\tilde{\chi}, \tilde{\chi}), (\mathcal{P}_1, \tilde{\chi}), (\mathcal{P}_2, \mathcal{P}_2), (\mathcal{P}_1^c, \mathcal{P}_2)\}$ where $(\mathcal{P}_1, \tilde{\chi}) = (\langle\{\alpha\}, \emptyset\rangle, \langle\chi, \emptyset\rangle)$, $(\mathcal{P}_2, \mathcal{P}_2) = (\langle\{\beta\}, \{\alpha\}\rangle, \langle\{\beta\}, \{\alpha\}\rangle)$ and $(\mathcal{P}_1^c, \mathcal{P}_2) = (\langle\emptyset, \{\alpha\}\rangle, \langle\{\beta\}, \{\alpha\}\rangle)$. Since $(\tilde{\alpha}, \tilde{\alpha}) = (\langle\{\alpha\}, \{\beta\}\rangle, \langle\{\alpha\}, \{\beta\}\rangle)$, $(\tilde{\beta}, \tilde{\beta}) = (\langle\{\beta\}, \{\alpha\}\rangle, \langle\{\beta\}, \{\alpha\}\rangle) \in \chi$; then $(\tilde{\alpha}, \tilde{\alpha}) \neq (\tilde{\beta}, \tilde{\beta}) \rightarrow \exists\ (\mathcal{P}_1, \tilde{\chi}), (\mathcal{P}_2, \mathcal{P}_2) \in \varphi$ such that $((\tilde{\alpha}, \tilde{\alpha}) \in (\mathcal{P}_1, \tilde{\chi}) \cdot\cdot\ (\tilde{\beta}, \tilde{\beta}) \notin (\mathcal{P}_1, \tilde{\chi})) \cdot\cdot\ (\tilde{\alpha}, \tilde{\alpha}) \notin (\mathcal{P}_2, \mathcal{P}_2) \cdot\cdot\ (\tilde{\beta}, \tilde{\beta}) \in (\mathcal{P}_2, \mathcal{P}_2))$. Clear, (χ, φ) is $\mathrm{DIT}_1 - \mathrm{S}$. But (χ, φ) is not $\mathrm{DIT}_2 - \mathrm{S}$, since if $(\tilde{\alpha}, \tilde{\alpha}) \neq (\tilde{\beta}, \tilde{\beta})$ take $(\mathcal{P}_1, \tilde{\chi}), (\mathcal{P}_2, \mathcal{P}_2)$. But $(\mathcal{P}_1, \tilde{\chi}) \cap (\mathcal{P}_2, \mathcal{P}_2) = (\mathcal{P}_1^c, \mathcal{P}_2) \neq (\tilde{\emptyset}, \tilde{\emptyset})$. Hence $\mathrm{DIT}_1 - \mathrm{S} \nrightarrow \mathrm{DIT}_2 - \mathrm{S}$.

Remark 3.15 If (χ, φ) it is $\mathrm{DIT}_2 - \mathrm{S}$, then not essential that the space is $\mathrm{DIT}_1 - \mathrm{S}$ and $\mathrm{DIT}_0 - \mathrm{S}$ (i.e., $\mathrm{DIT}_2 - \mathrm{S} \rightarrow \mathrm{DIT}_1 - \mathrm{S} \rightarrow \mathrm{DIT}_0 - \mathrm{S}$).

Example 3.16 The space $(\chi, \Im(\chi))$ if χ is any DIS containing more than one Double-element, then $(\chi, \Im(\chi))$ is not $\mathrm{DIT}_0 - \mathrm{S}$, so that it's not $\mathrm{DIT}_1 - \mathrm{S}$ and not $\mathrm{DIT}_2 - \mathrm{S}$.

Theorem 3.17 The property of being $\mathrm{DIT}_2 - \mathrm{S}$ is a hereditary property.

Proof Let (χ, φ) is $\mathrm{DIT}_2 - \mathrm{S}$ and $(\text{Ɐ}, \varphi_{\text{Ɐ}})$ is a Double intuitionistic topological subspace of (χ, φ), to prove$(\text{Ɐ}, \varphi_{\text{Ɐ}})$ is$\mathrm{DIT}_2 - \mathrm{S}$. Let $(\tilde{p}, \tilde{p}), (\tilde{ł}, \tilde{ł}) \in \text{Ɐ}$, $(\tilde{p}, \tilde{p}) \neq (\tilde{ł}, \tilde{ł}) \rightarrow (\tilde{p}, \tilde{p}), (\tilde{ł}, \tilde{ł}) \in \chi$ (since $\text{Ɐ} \cdot \chi) \cdot\cdot$ Since$\cdot\chi$ is $\mathrm{DIT}_2 - \mathrm{S}$, so there exist (υ_1, V_1), $(\upsilon_2, V_2) \in \text{Ɐ}$; $(\upsilon_1, V_1) \cdot \cap\ (\upsilon_2, V_2) = (\tilde{\emptyset}, \tilde{\emptyset})$, $((\tilde{p}, \tilde{p}) \in (\upsilon_1, V_1) \cdot\cdot (\tilde{ł}, \tilde{ł}) \in (\upsilon_2, V_2)) \cdot \rightarrow (\upsilon_1, V_1) \cap \text{Ɐ} \cdot (\upsilon_2, V_2) \cap \text{Ɐ} \in \varphi_{\text{Ɐ}}$ (by definition 2.8); $((\upsilon_1, V_1) \cap \text{Ɐ}) \cap ((\upsilon_2, V_2) \cap \text{Ɐ}) = ((\upsilon_1, V_1) \cdot \cap (\upsilon_2, V_2) \cap \text{Ɐ}) = (\tilde{\emptyset}, \tilde{\emptyset}) \cap \text{Ɐ} = (\tilde{\emptyset}, \tilde{\emptyset})$.

Theorem 3.18 Let (χ, φ) & $(\text{Ɐ}, \varphi^*)$ be two DITS. Then the Double I-product space $\chi \times \text{Ɐ}$ is $\mathrm{DIT}_2 - \mathrm{S}$ if and only if each χ & Ɐ are $\mathrm{DIT}_2 - \mathrm{Space}$.

Proof $(\cdot)$ Let $(\tilde{p}, \tilde{p})_1, (\tilde{p}, \tilde{p})_2 \in \chi$, $(\tilde{p}, \tilde{p})_1 \neq (\tilde{p}, \tilde{p})_2$ and $(\tilde{ł}, \tilde{ł})_1, (\tilde{ł}, \tilde{ł})_2 \in \text{Ɐ}$, $(\tilde{ł}, \tilde{ł})_1 \neq (\tilde{ł}, \tilde{ł})_2 \rightarrow ((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1), ((\tilde{p}, \tilde{p})_2, (\tilde{ł}, \tilde{ł})_2) \in \chi \times \text{Ɐ}$, $((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1) \neq ((\tilde{p}, \tilde{p})_2, (\tilde{ł}, \tilde{ł})_2)$. Since $\chi \times \text{Ɐ}$ are $\mathrm{DIT}_2 - \mathrm{S} \rightarrow \exists\ (\upsilon_1, V_1) \cdot\cdot (\upsilon_2, V_2) \in \varphi_{\chi \times \text{Ɐ}}$; $((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1 \in (\upsilon_1, V_1) \cdot\cdot (\tilde{p}, \tilde{p})_2, (\tilde{ł}, \tilde{ł})_2 \in (\upsilon_2, V_2)) \cdot (\upsilon_1, V_1) \cdot \cap (\upsilon_2, V_2) = (\tilde{\emptyset}, \tilde{\emptyset})$ that means there exist Double$\cdot\cdot$-open sets $((\upsilon_1, V_1)_1 \times (\upsilon_2, V_2)_1)$, $((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2) \in \varphi_{\chi \times \text{Ɐ}}$, $(\upsilon_1, V_1)_1 \times (\upsilon_2, V_2)_1) \cdot \cap ((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2) = (\tilde{\emptyset}, \tilde{\emptyset}) \cdot\cdot\cdot ((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1) \in ((\upsilon_1, V_1)_1 \times (\upsilon_2, V_2)_1) \cdot\cdot\ ((\tilde{p}, \tilde{p})_2, (\tilde{ł}, \tilde{ł})_2 \in ((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2$, so there exist $\cdot\cdot ((\upsilon_1, V_1)_1, (\upsilon_2, V_2)_1 \in \varphi$; $\cdot ((\upsilon_1, V_1)_1 \cap (\upsilon_2, V_2)_2) = (\tilde{\emptyset}, \tilde{\emptyset})$; $((\tilde{p}, \tilde{p})_1 \in (\upsilon_1, V_1)_1 \cdot\cdot\cdot (\tilde{p}, \tilde{p})_2 \in ((\upsilon_1, V_1)_2$.So$\cdot\cdot\chi$ is $\mathrm{DIT}_2 - \mathrm{Space}$ and$\cdot \exists ((\upsilon_2, V_2)_1, (\upsilon_2, V_2)_2) \in \varphi^* \cdot\cdot ((\upsilon_2, V_2)_1 \cap (\upsilon_2, V_2)_2) = (\tilde{\emptyset}, \tilde{\emptyset})$. Hence, Ɐ is $\mathrm{DIT}_2 - \mathrm{Space}$.

Conversely$\cdot\cdot\cdot$) Let $\left((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1\right), ((\tilde{p}, \tilde{p})_2, (\tilde{ł}, \tilde{ł})_2) \in \chi \times \text{Ɐ}$; $\left((\tilde{p}, \tilde{p})_1, (\tilde{ł}, \tilde{ł})_1\right) \neq \left(\tilde{p}, \tilde{p}, (\tilde{ł}, \tilde{ł})_2\right) \rightarrow ((\tilde{p}, \tilde{p})_1), ((\tilde{p}, \tilde{p})_2 \in \chi \cdot\cdot\cdot ((\tilde{p}, \tilde{p})_1) \neq (\tilde{p}, \tilde{p})_2) \cdot\cdot (\tilde{ł}, \tilde{ł})_1, (\tilde{ł}, \tilde{ł})_2 \in \text{Ɐ} \cdot\cdot (\tilde{ł}, \tilde{ł})_1 \neq (\tilde{ł}, \tilde{ł})_2$. Since χ is $\mathrm{DIT}_2 - \mathrm{S}$, there exists $((\upsilon_1, V_1)_1, (\upsilon_1, V_1)_2 \in \varphi$; $((\upsilon_1, V_1)_1 \cap (\upsilon_2, V_2)_2) = (\tilde{\emptyset}, \tilde{\emptyset})$, $((\tilde{p}, \tilde{p})_1 \in (\upsilon_1, V_1)_1 \cdot\cdot (\tilde{p}, \tilde{p})_2 \in (\upsilon_1, V_1)_2) \cdot$. Since Ɐ is $\mathrm{DIT}_2 - \mathrm{S}$, then there exists $((\upsilon_2, \nu_2)_1, \cdot (\upsilon_2, \nu_2)_2) \cdot \in \varphi^*; ((\upsilon_2, V_2)_1 \cap (\upsilon_2, V_2)_2) = (\tilde{\emptyset}, \tilde{\emptyset})$, $((\tilde{ł}, \tilde{ł})_1 \in (\upsilon_2, V_2)_1 \cdot\cdot\cdot (\tilde{ł}, \tilde{ł})_2 \in (\upsilon_2, V_2)_2) \cdot$ so$\cdot\cdot$there exist Double$\cdot\cdot$-basis open sets $((\upsilon_1, V_1)_1 \times$

$(\upsilon_2, V_2)_1), ((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2); ((\upsilon_1, V_1)_1 \times (\upsilon_2, V_2)_1) \cap ((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2) \cdots ((\upsilon_1, V_1)_1 \cap ((\upsilon_1, V_1)_2) \times ((\upsilon_2, V_2)_1 \cdot \cap ((\upsilon_2, V_2)_2)$ $= (\widetilde{\emptyset}, \widetilde{\emptyset})$, $((\tilde{p}, \tilde{p})_1, (\tilde{I}, \tilde{I})_1) \in ((\upsilon_1, V_1)_1 \times (\upsilon_2, V_2)_1) \cdot\cdot ((\tilde{p}, \tilde{p})_2, (\tilde{I}, \tilde{I})_2) \in ((\upsilon_1, V_1)_2 \times (\upsilon_2, V_2)_2)$. Therefore χ ×∀ is DIT_2 −Space.

Theorem 3.19 Every Double I-compact space in $DIT_2 - S$ is Double I-closed.

Proof Let (χ , φ) be $DIT_2 - S$ and $(\mathcal{A}, \mathcal{K}) \cdot\cdot$ χ; $\cdot(\mathcal{A}, \mathcal{K})\cdot$ is Double I-compact in φ, to prove $(\mathcal{A}, \mathcal{K})$ is DIC in χ; $(\mathcal{A},\mathcal{K})^c$ is DIO(χ). Let $(\tilde{p}, \tilde{p}) \in (\mathcal{A},\mathcal{K})^c \to (\tilde{p}, \tilde{p}) \notin (\mathcal{A},\mathcal{K})$, $(\tilde{p}, \tilde{p}) \neq \alpha; \cdot\ \alpha \in \cdot(\mathcal{A},\mathcal{K})$. Since (χ , φ) is $DIT_2 - S$,so ∃ $((G_{1\alpha}, G_{1\alpha}), (G_{2\alpha}, G_{2\alpha}) \in \varphi$; $(G_{1\alpha}, G_{1\alpha}) \cap (G_{2\alpha}, G_{2\alpha}) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, $(\tilde{p}, \tilde{p}) \in (G_{1\alpha}, G_{1\alpha}) \cdot\cdot\ \alpha \in (G_{2\alpha}, G_{2\alpha})$;· $\alpha \in (\mathcal{A},\mathcal{K})$, $\{(G_{1\alpha}, G_{1\alpha})\}_{\alpha\in(\mathcal{A},\mathcal{K})}$ (all element in this family contains $(\tilde{p}, \tilde{p})$) and $\{(G_{2\alpha}, G_{2\alpha})\}_{\alpha\in(\mathcal{A},\mathcal{K})}$ (all element in this family contains one of the elements $(\tilde{p}, \tilde{p})$) and every $(G_{1\alpha}, G_{1\alpha})$ conforming $(G_{2\alpha}, G_{2\alpha})$s. t $(G_{1\alpha}, G_{1\alpha}) \cap (G_{2\alpha}, G_{2\alpha}) = (\widetilde{\emptyset}, \widetilde{\emptyset}) \to \{(G_{2\alpha}, G_{2\alpha})\}_{\alpha\in(\mathcal{A},\mathcal{C})}$ is DIO cover of $(\mathcal{A}, \mathcal{K})$.(i.e., $(\mathcal{A}, \mathcal{K}) \cdot \cup_{\alpha\in(\mathcal{A},\mathcal{K})} (G_{2\alpha}, G_{2\alpha})$. Since $(\mathcal{A}, \mathcal{K})$ is Double I-compact→ $\exists\alpha_1, \alpha_2, \dots, \alpha_n$; $(\mathcal{A},\mathcal{K}) \cdot \cup_{j=1}{}^{n} (G_{2\alpha j}, G_{2\alpha j})$. So, there is a Double limited family $\{(G_{1\alpha}, G_{1\alpha})\}_{\alpha\in(\mathcal{A},\mathcal{K})}$ conforming to the Double finite family) $\{(G_{1\alpha}, G_{1\alpha})\}_{\alpha\in(\mathcal{A},\mathcal{K})}\{(G_{2\alpha j}, G_{2\alpha j})\}^{n}{}_{j=1}$which is $(G_{1\alpha}, G_{1\alpha})$. Since every $(G_{1\alpha}, G_{1\alpha})$ contains $(\tilde{p}, \tilde{p})$, then $(\tilde{p}, \tilde{p}) \in \cap_{j=1}{}^{n} (G_{1\alpha j}, G_{1\alpha j})$. Since everyone $(G_{1\alpha j}, G_{1\alpha j})$ is DIO (χ), then $\cap_{j=1}{}^{n} (G_{1\alpha j}, G_{1\alpha j})$ is DIOS contains $(\tilde{p}, \tilde{p})$(second condition of the definition of DITS). Say, $(\gamma_1, \gamma_1) = \cap_{j=1}{}^{n} (G_{1\alpha j}, G_{1\alpha j}) \to (\tilde{p}, \tilde{p}) \in (G_{1\alpha}, G_{1\alpha}) \in \varphi$ on the other hand $\cup_{j=1}{}^{n} (G_{2\alpha j}, G_{2\alpha j})$ is DIOS (third condition of definition of DITS). Say, $(\gamma_2, \gamma_2) = \cup_{j=1}{}^{n} (G_{2\alpha j}, G_{2\alpha j}) \to (\mathcal{A}, \mathcal{K}) \cdot\cdot (G_{2\alpha}, G_{2\alpha}) \in \varphi$. Notes that, $(G_{1\alpha}, G_{1\alpha}) \cap (G_{2\alpha}, G_{2\alpha}) = (\widetilde{\emptyset}, \widetilde{\emptyset}) \to (G_{1\alpha}, G_{1\alpha}) \cdots\cdot (\mathcal{A}, \mathcal{K})^c \to (\tilde{p}, \tilde{p}) \in (G_{1\alpha}, G_{1\alpha}) \cdot\cdot (\mathcal{A}, \mathcal{K})^c \cdot\cdot (G_{1\alpha}, G_{1\alpha}) \in \varphi$, so $(\mathcal{A}, \mathcal{K})^c$ is DIOS (χ); $\cdot(\tilde{p}, \tilde{p}) \in (\mathcal{A},\mathcal{K})^c \to (\mathcal{A},\mathcal{K})$ is Double I-closed in χ.

Theorem 3.20 Let (χ, φ) be DITS, and let $(\mathcal{A},\mathcal{K})\cdot$be a Double I-closed set and $(\mathcal{O}, \mathcal{Z})$ be a Double I-compact set in $DIT_2 - S$, then $(\mathcal{A},\mathcal{K})\cdot\cap (\mathcal{O},\mathcal{Z})$ is Double I compact.

Proof Since χ is $DIT_2 - S$ and $(\mathcal{O}, \mathcal{Z})$ is Double I-compact in χ, then $(\mathcal{O}, \mathcal{Z})$ is DIC(χ). Since $(\mathcal{A}, \mathcal{K})$, $(\mathcal{O}, \mathcal{Z})$ are DICS(χ), then $(\mathcal{A},\mathcal{K})\cdot\cap (\mathcal{O},\mathcal{Z})$ is DICS(χ) (second condition of definition of DITS). Since $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})\cdot\cdot (\mathcal{O},\mathcal{Z})$, so $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})\cdot$be Double intuitionistic topological subspace of $(\mathcal{O}, \mathcal{Z})$ and $(\mathcal{O}, \mathcal{Z})$ is Double I compact. Therefore, $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})$ is Double I-compact (by Theorem 3.19).

Corollary 3.21 Let (χ, φ) be DITS, and let $(\mathcal{A}, \mathcal{K})\cdot\cdot (\mathcal{O},\mathcal{Z})$ ·be Double I-compact in $DIT_2 - S$, then $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})$ are Double I-compact.

Proof Let$(\mathcal{A},\mathcal{K})$ is Double I-compact in χ, and χ is $DIT_2 - S$, then $(\mathcal{A}, \mathcal{K})\cdot$is DIC(χ) (by Theorem 3.19). Since $(\mathcal{A}, \mathcal{K})$ is DIC(χ)··$(\mathcal{O}, \mathcal{Z})$ is Double I-compact in χ·and χ is $DIT_2 - S$, then $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})$ is Double I compact and χ be $DIT_2 - S$ and $(\mathcal{O}, \mathcal{Z})$ is Double I compact in χ, then $(\mathcal{O}, \mathcal{Z})$ is DIC(χ). Since $(\mathcal{A}, \mathcal{K})$, $(\mathcal{O}, \mathcal{Z})$ are DICS(χ), then $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})$ is DICS(χ) (second condition of definition of DITS). Since $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})\cdot\cdot (\mathcal{O}, \mathcal{Z})$, so $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})\cdot$is a Double intuitionistic topological subspace of $(\mathcal{O}, \mathcal{Z})$ and $(\mathcal{O}, \mathcal{Z})$ is Double I compact. Therefore, $(\mathcal{A}, \mathcal{K})\cdot\cap (\mathcal{O}, \mathcal{Z})$ is Double I compact (by Theorem 3.19).

4. Double Intuitionistic Regular − space, Double IntuitionisticT_3 − space, Double Intuitionistic Normal − space and Double IntuitionisticT_4 − space in Double Intuitionistic Topological Spaces

Definition 4.1 Let (χ, φ) is DITS. Then space (χ, φ) is said to be Double intuitionisticRegular − space (DIR − S, for short) if and only if for each $(\tilde{p}, \tilde{p}) \in \varphi, \cdot(\upsilon_1, V_2)^c \in \varphi^c$, $(\tilde{p}, \tilde{p}) \notin (\upsilon_1, V_2)^c$ is a Double I-closed set, there exists$\cdot(\upsilon_1, V_1)\cdot\cdot(\upsilon_2, V_2) \in \varphi\cdots(\upsilon_1, V_1)\cdot\cdot\cap (\upsilon_2, V_2) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, $(\tilde{p}, \tilde{p}) \in (\upsilon_1, V_1) \cdots (\upsilon_1, V_2)^c\cdots(\upsilon_2, V_2)$.

Example 4.2· Let χ = $\{\hbar, d, \jmath\}$; φ = φ^c = $\{(\widetilde{\emptyset}, \widetilde{\emptyset}), (\widetilde{\chi}, \widetilde{\chi}), (\mathcal{T}_1, \mathcal{T}_1), (\mathcal{T}_1^c, \mathcal{T}_1^c)\}$ where $(\mathcal{T}_1, \mathcal{T}_1) = (\langle\{\hbar\}, \{d, \jmath\}\rangle, \langle\{\hbar\}, \{d, \jmath\}\rangle)$, $(\mathcal{T}_1^c, \mathcal{T}_1^c) = (\langle\{d, \jmath\}, \{\hbar\}\rangle, \langle\{d, \jmath\}, \{\hbar\}\rangle)$. Take $(\widetilde{\chi}, \widetilde{\chi})$ but every element belongs to $(\widetilde{\chi}, \widetilde{\chi})$. Take $(\widetilde{\emptyset}, \widetilde{\emptyset})$ and $(\tilde{\hbar}, \tilde{\hbar}), (\tilde{d}, \tilde{d}), (\tilde{\jmath}, \tilde{\jmath}) \notin (\widetilde{\emptyset}, \widetilde{\emptyset})$, so ∃ $(\upsilon_1, \nu_1)\cdot = (\widetilde{\chi}, \widetilde{\chi})$ & $(\upsilon_2, \nu_2) = (\widetilde{\emptyset}, \widetilde{\emptyset})$ such that $(\tilde{\hbar}, \tilde{\hbar}), (\tilde{d}, \tilde{d}), (\tilde{\jmath}, \tilde{\jmath}) \in (\widetilde{\chi}, \widetilde{\chi})$, (υ_1, ν_1) and $(\widetilde{\emptyset}, \widetilde{\emptyset})\cdot\cdot (\widetilde{\emptyset}, \widetilde{\emptyset})\cdot\cdot (\upsilon_2, \nu_2)$ & $(\widetilde{\chi}, \widetilde{\chi}) \cap (\widetilde{\emptyset}, \widetilde{\emptyset}) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, so the definition satisfies. Take, $(\upsilon_1, \nu_2)^c = (\mathcal{T}_1, \mathcal{T}_1)$ is DICS (χ), $(\tilde{d}, \tilde{d}), (\tilde{\jmath}, \tilde{\jmath}) \notin (\upsilon_1, \nu_2)^c \to \exists\ (\upsilon_1, \nu_1) = (\mathcal{T}_1^c, \mathcal{T}_1^c)$ & $(\upsilon_2, \nu_2) = (\mathcal{T}_1, \mathcal{T}_1)$, $((\mathcal{T}_1, \mathcal{T}_1), (\mathcal{T}_1^c, \mathcal{T}_1^c)) \in \varphi$; $((\mathcal{T}_1, \mathcal{T}_1) \cap (\mathcal{T}_1^c, \mathcal{T}_1^c) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, $((\tilde{d}, \tilde{d}), (\tilde{\jmath}, \tilde{\jmath})) \in (\mathcal{T}_1^c, \mathcal{T}_1^c) \in \varphi \cdot (\upsilon_1, V_2)^c \cdot (\upsilon_2, V_2) = (\mathcal{T}_1, \mathcal{T}_1)$. Take $(\upsilon_1, V_2)^c = (\mathcal{T}_1^c, \mathcal{T}_1^c)$ is DICS(χ); $(\tilde{\hbar}, \tilde{\hbar}) \notin (\upsilon_1, V_2)^c$, so ∃ $(\upsilon_1, V_1) = (\mathcal{T}_1, \mathcal{T}_1)$ & $(\upsilon_2, V_2) = (\mathcal{T}_1^c, \mathcal{T}_1^c) = (\upsilon_1, V_2)^c$ then $(\upsilon_1, V_1)\cdot\cap (\upsilon_2, V_2) = (\widetilde{\emptyset}, \widetilde{\emptyset})$; $(\tilde{\hbar}, \tilde{\hbar}) \in (\upsilon_1, V_1) \cdot (\upsilon_1, V_2)^c = (\mathcal{T}_1^c, \mathcal{T}_1^c) \cdot (\upsilon_2, V_2)$. Hence (χ, φ)is DIR − S.

Remark 4.3 In the preceding example 4.2, notice that χ it is not $DIT_0 - S$**,** not $DIT_1 - S$ and not $DIT_2 - S$, so the not DIR − S is not necessarily not $DIT_0 - S$ or not $DIT_1 - S$ or not $DIT_2 - S$. (i.e., DIR $\nrightarrow$ DIT_0 ··DIR $\nrightarrow$ DIT_1 ··DIR $\nrightarrow$ DIT_2.

Theorem 4.4 Let (χ, φ) be DIR-S if and only if for each $(\tilde{p}, \tilde{p}) \in \chi$ and every Double I open set (υ_3, ν_3) containing $(\tilde{p}, \tilde{p})$, there exists a Double I open set (υ_1, ν_1) s.t $(\tilde{p}, \tilde{p}) \in (\upsilon_1, \nu_1) \cdot$ cl $(\upsilon_1, \nu_1) \cdot (\upsilon_3, \nu_3)$.

Proof (·) Let $(\tilde{p}, \tilde{p}) \in \chi$, $(\upsilon_3, \nu_3) \in \varphi$; $(\tilde{p}, \tilde{p}) \in (\upsilon_3, \nu_3) \to (\tilde{p}, \tilde{p}) \notin (\upsilon_3, \nu_3)^c$ and $(\upsilon_3, \nu_3)^c \in (\upsilon_1, \nu_1)^c$. Since χ is DIR-S, then there exist $(\upsilon_1, \nu_1), (\upsilon_2, \nu_2) \in \varphi$; $(\upsilon_1, \nu_1) \cap (\upsilon_2, \nu_2) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, $((\tilde{p}, \tilde{p}) \in (\upsilon_1, \nu_1) \cdot\cdot(\upsilon_3, \nu_3)^c \cdot (\upsilon_2, \nu_2))$. Since $(\upsilon_1, \nu_1) \cap (\upsilon_2, \nu_2) = (\widetilde{\emptyset}, \widetilde{\emptyset})$, so $(\upsilon_1, \nu_1) \cdot (\upsilon_2, \nu_2)^c$ and $(\upsilon_2, \nu_2)^c \cdot (\upsilon_3, \nu_3) \to cl(\upsilon_1, \nu_1) \cdot cl(\upsilon_2, \nu_2)^c$ (by theorem 2.12 (3)) $\to cl(\upsilon_1, \nu_1) \cdot (\upsilon_2, \nu_2)^c$ (by theorem 2.12(2)) $\to cl(\upsilon_1, \nu_1) \cdot (\upsilon_2, \nu_2)^c\cdots\cdot(\upsilon_2, \nu_2)^c \cdot (\upsilon_3, \nu_3) \to$ cl $(\upsilon_1, \nu_1) \cdot (\upsilon_3, \nu_3)$, so $(\tilde{p}, \tilde{p}) \in (\upsilon_1, \nu_1) \cdot cl(\upsilon_1, \nu_1) \cdot (\upsilon_3, \nu_3)$ (by theorem 2.12(1)).

Conversely⋅⋅⋅) Let $(\tilde{p},\tilde{p}) \in \chi$ and $(\upsilon_1,\nu_2)^c$ be DICS(χ); $(\tilde{p},\tilde{p}) \notin (\upsilon_1,\nu_2)^c \rightarrow (\tilde{p},\tilde{p}) \in ((\upsilon_1,\nu_2)^c)^c \in \varphi$ (since $(\upsilon_1, \nu_2)^c$ is DIC(χ)), there exists $(\upsilon_1,\nu_1) \in \varphi$; $(\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1)$ ▪ cl (υ_1,ν_1) ▪ $((\upsilon_1, \nu_2)^c)^c$, then cl$(\upsilon_1,\nu_1)$ ▪ $((\upsilon_1,\nu_2)^c)^c \rightarrow (\upsilon_1,\nu_2)^c$ ▪ $(\mathrm{cl}(\upsilon_1,\nu_1))^c$. But, $(\mathrm{cl}(\upsilon_1,\nu_1))^c$ is DIOS(χ), since cl(υ_1,ν_1) is DIC(χ), say $(\mathrm{cl}(\upsilon_1,\nu_1))^c = (\upsilon_2,\nu_2) \rightarrow (\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1)$ ▪ $(\upsilon_1,\nu_2)^c$▪▪ (υ_2,ν_2) ▪$(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$ (since (υ_1, ν_1) ▪ cl(υ_1, ν_1) and cl$(\upsilon_1,\nu_1) \cap (\mathrm{cl}(\upsilon_1,\nu_1))^c = (\tilde{\emptyset}, \tilde{\emptyset})) \rightarrow (\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$. Hence, χ is DIR-S.

Theorem 4.5 The property of being $\mathrm{DIR}-\mathrm{S}$ is a hereditary property.

Proof Let (χ,φ) is $\mathrm{DIR}-\mathrm{S}$ and $(\text{Ɏ},\varphi_{\text{Ɏ}})$ is Double intuitionistic topological subspace of χ, to prove $(\text{Ɏ},\varphi_{\text{Ɏ}})$ is $\mathrm{DIR}-\mathrm{S}$. Let $(\tilde{p},\tilde{p}) \in \text{Ɏ}$ and $(\mathcal{O},\mathcal{Z})$ is DICS $\in \varphi$; $(\tilde{p},\tilde{p}) \notin (\mathcal{O},\mathcal{Z}) \rightarrow (\tilde{p},\tilde{p}) \in \chi$▪▪ since Ɏ▪▪ χ) there exists $(\mathcal{A}, \mathcal{K}) \in (\upsilon_1,\nu_2)^c$, $(\mathcal{O}, \mathcal{Z}) = (\mathcal{A}, \mathcal{K}) \cap \text{Ɏ}$ (i.e., $(\mathcal{O}, \mathcal{Z})$ is DIC(χ). Since χ is $\mathrm{DIR}-\mathrm{S}$, there exist $(\upsilon_1,\nu_1), (\upsilon_2,\nu_2) \in \varphi$; $(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$, $((\tilde{\emptyset},\tilde{\emptyset}) \in (\upsilon_1,\nu_1)$▪▪ $(\mathcal{A}, \mathcal{K})$▪▪ $(\upsilon_2,\nu_2))$▪$\rightarrow$ $(\upsilon_1,\nu_1)\cap \text{Ɏ}$ ▪▪$(\upsilon_2,\nu_2) \cap \text{Ɏ} \in \varphi_{\text{Ɏ}}$ (by definition 2.8); $((\upsilon_1,\nu_1) \cap \text{Ɏ})\cap ((u_2,\nu_2) \cap \text{Ɏ}) = ((\upsilon_1,\nu_1) \cap (u_2,\nu_2) \cap \text{Ɏ}) = (\tilde{\emptyset}, \tilde{\emptyset}) \cap \text{Ɏ} = (\tilde{\emptyset}, \tilde{\emptyset})$ and $((\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1) \cap \text{Ɏ})$ (since $(\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1)$▪▪$(\tilde{p},\tilde{p}) \in \text{Ɏ})$ ▪▪($\mathcal{O},\mathcal{Z})$ ▪ $(\upsilon_2,\nu_2) \cap \text{Ɏ})$ (since $(\mathcal{O}, \mathcal{Z})$ ▪ $(\mathcal{A},\mathcal{K}) \cap \text{Ɏ}) \rightarrow (\mathcal{O}, \mathcal{Z})$ ▪ $(\mathcal{A}, \mathcal{K})$▪▪ $(\mathcal{O}, \mathcal{Z})$ ▪ Ɏ $\rightarrow$($\mathcal{O}, \mathcal{Z})$ ▪ (υ_2,ν_2)▪▪ $(\mathcal{O}, \mathcal{Z})$ ▪ Ɏ $\rightarrow$($\mathcal{O},\mathcal{Z})$ ▪ $(\upsilon_2,\nu_2) \cap \text{Ɏ}$. Therefore, $(\text{Ɏ},\varphi_{\text{Ɏ}})$ is $\mathrm{DIR}-\mathrm{S}$.

Remark 4.6 Notes that $\mathrm{DIT}_0 - \mathrm{S} \nrightarrow \mathrm{DIR}-\mathrm{S}$ ▪▪$\mathrm{DIT}_1 - \mathrm{S} \nrightarrow \mathrm{DIR}-\mathrm{S}$ ▪▪$\mathrm{DIT}_2 - \mathrm{S} \nrightarrow \mathrm{DIR}-\mathrm{S}$.

Example 4.7 Let $\chi = \{s,r\}$; $\varphi = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\chi},\tilde{\chi}), (\omega_1,\omega_2), (\omega_3,\omega_3), (\omega_4,\omega_4), (\omega_3^c,\omega_5)\}$ where $(\omega_1,\omega_2) = (\langle\{s\},\{r\}\rangle, \langle\{s\},\emptyset\rangle)$, $(\omega_3,\omega_3) = (\langle\{r\},\emptyset\rangle, \langle\{r\},\emptyset\rangle)$, $(\omega_4,\omega_4) = (\langle\{s,r\},\emptyset\rangle, \langle\{s,r\},\emptyset\rangle)$, $(\omega_3^c,\omega_5) = (\langle\emptyset,\{r\}\rangle, \langle\emptyset,\emptyset\rangle)$. $\varphi^c = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\chi},\tilde{\chi}), (\omega_2^c,\omega_1^c), (\omega_3^c,\omega_3^c), (\omega_4^c,\omega_4^c), (\omega_5^c,\omega_3)\}$ where $(\omega_2^c,\omega_1^c) = (\langle\emptyset,\{s\}\rangle, \langle\{r\},\{s\}\rangle)$, $(\omega_3,\omega_3) = (\langle\emptyset,\{r\}\rangle, \langle\emptyset,\{r\}\rangle)$, $(\omega_4^c,\omega_4^c) = (\langle\emptyset,\{s,r\}\rangle, \langle\emptyset,\{s,r\}\rangle)$ and $(\omega_5^c,\omega_3) = (\langle\emptyset,\emptyset\rangle, \langle\{r\},\emptyset\rangle)$, since$(\tilde{s},\tilde{s}) \neq (\tilde{r},\tilde{r})$, then there exist DIO($\chi$) such that $(\tilde{s},\tilde{s}) \in (\omega_1,\omega_2)$, (ω_4,ω_4)▪▪▪$(\tilde{r},\tilde{r}) \notin (\omega_1,\omega_2)$. Clear (χ,φ) is $\mathrm{DIT}_0 - \mathrm{S}$. Take $(\tilde{\chi},\tilde{\chi})$ but every element belongs to $(\tilde{\chi},\tilde{\chi})$ and take $(\tilde{\emptyset},\tilde{\emptyset})$ and $(\tilde{s},\tilde{s})$, $(\tilde{r},\tilde{r}) \notin (\tilde{\emptyset},\tilde{\emptyset})$, so $\exists$ (υ_1,ν_1)▪$= (\tilde{\chi},\tilde{\chi})$ & $(\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$, so the definition satisfies. Take $(\upsilon_1,\nu_2)^c = (\omega_3^c,\omega_3^c) \in \varphi^c$, $(\tilde{r},\tilde{r}) \notin (\upsilon_1,\nu_2)^c$. Let $(\upsilon_1,\nu_1) = (\omega_3,\omega_3)$ & $(\upsilon_2,\nu_2) = (\omega_1,\omega_2)$; $(\omega_3,\omega_3) \cap (\omega_1, \omega_2) = (\omega_3^c,\omega_5) \neq (\tilde{\emptyset},\tilde{\emptyset})$, $(\tilde{r},\tilde{r}) \in (\upsilon_1,\nu_1)$. Take $(\upsilon_2,\nu_2)^c = (\omega_4^c, \omega_4^c)$, $(\tilde{s},\tilde{s}) \notin (\upsilon_2,\nu_2)^c$. Let $(\upsilon_1,\nu_1) = (\omega_1,\omega_2)$, $(\upsilon_2,\nu_2) = (\omega_4,\omega_4)$; $(\tilde{s},\tilde{s}) \in (\upsilon_1,\nu_1)$; $(\omega_1,\omega_2) \cap (\omega_4, \omega_4) = (\omega_1, \omega_2) \neq (\tilde{\emptyset},\tilde{\emptyset})$. Hence (χ,φ) not $\mathrm{DIR}-\mathrm{S}$.

Example 4.8 From example 3.7. Clear (χ,φ) is $\mathrm{DIT}_1 - \mathrm{S}$, but not $\mathrm{DIR}-\mathrm{S}$. First find the family of DICS(χ). $\varphi^c = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\chi},\tilde{\chi}), (\mathcal{S}_1,\tilde{\chi}), (\mathcal{S}_1,\mathcal{S}_1), (\tilde{\emptyset},\mathcal{S}_1^c)\}$ where $(\mathcal{S}_1,\mathcal{S}_1) = (\langle\{a\},\{b\}\rangle, \langle\{a\},\{b\}\rangle)$ and $(\tilde{\emptyset},\tilde{\emptyset}),(\tilde{\chi},\tilde{\chi})$ in the preceding example content the definition in general $(\tilde{\emptyset},\tilde{\emptyset})$, $(\tilde{\chi},\tilde{\chi})$ satisfy the definition in all examples. Take $(\upsilon_1,\nu_2)^c = (\mathcal{S}_1,\mathcal{S}_1)$; $(\tilde{b},\tilde{b}) \notin (\upsilon_1,\nu_2)^c$. Let $(\mathcal{S}_1^c,\mathcal{S}_1^c)$, $(\tilde{\emptyset},\mathcal{S}_1^c)$, $(\tilde{b},\tilde{b}) \in (\mathcal{S}_1^c,\mathcal{S}_1^c)$ and $(\mathcal{S}_1^c,\mathcal{S}_1^c) \cap (\tilde{\emptyset},\mathcal{S}_1^c) = (\tilde{\emptyset},\mathcal{S}_1^c) \neq (\tilde{\emptyset},\tilde{\emptyset})$. Take $(\upsilon_1,\nu_2)^c = (\tilde{\emptyset},\mathcal{S}_1^c)$; $(\tilde{a},\tilde{a}) \notin (\tilde{\emptyset},\mathcal{S}_1^c)$. Let $(\upsilon_1,\nu_1) = (\mathcal{S}_1,\tilde{\chi})$, $(\upsilon_2,\nu_2) = (\tilde{\emptyset},\mathcal{S}_1^c)$, $(\upsilon_1,\nu_2)^c$▪▪(υ_2,ν_2) and $(\mathcal{S}_1,\tilde{\chi}) \cap (\tilde{\emptyset},\mathcal{S}_1^c) = (\tilde{\emptyset},\mathcal{S}_1^c) \neq (\tilde{\emptyset},\tilde{\emptyset})$. So, is not $\mathrm{DIR}-\mathrm{S}$.

Example 4.9 Let $\chi = \{1,2\}$; $\varphi = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\chi},\tilde{\chi}), (\lambda_1, \lambda_1), (\lambda_1^c, \lambda_1^c), (\lambda_2,\tilde{\chi}), (\lambda_2^c,\lambda_1)\}$ where $(\lambda_1,\lambda_1) = (\langle\{1\},\{2\}\rangle, \langle\{1\},\{2\}\rangle)$, $(\lambda_1^c, \lambda_1^c) = (\langle\{2\},\{1\}\rangle, \langle\{2\},\{1\}\rangle)$, $(\lambda_2,\tilde{\chi}) = (\langle\{2\},\emptyset\rangle, \langle\chi,\emptyset\rangle)$ and $(\lambda_2^c,\lambda_1) = (\langle\emptyset,\{2\}\rangle, \langle\{1\},\{2\}\rangle)$. $\varphi^c = \{(\tilde{\emptyset},\tilde{\emptyset}), (\tilde{\chi}, \tilde{\chi}), (\lambda_1^c, \lambda_1^c), (\lambda_1,\lambda_1), (\tilde{\emptyset}, \lambda_2^c), (\lambda_1^c,\lambda_2)\}$ where $(\tilde{\emptyset}, \lambda_2^c) = (\langle\emptyset,\chi\rangle, \langle\emptyset,\{2\}\rangle)$, $(\lambda_1^c, \lambda_2) = (\langle\{2\},\{1\}\rangle, \langle\{2\},\emptyset\rangle)$. Let $(\tilde{1},\tilde{1}),(\tilde{2},\tilde{2}) \in \chi$, $(\tilde{1},\tilde{1}) \neq (\tilde{2},\tilde{2})$ then $(\tilde{1},\tilde{1}) \in (\lambda_1,\lambda_1)$▪▪▪$(\tilde{2},\tilde{2}) \in (\lambda_1^c, \lambda_1^c)$ such that (λ_1,λ_1)▪$\cap (\lambda_1^c, \lambda_1^c) = (\tilde{\emptyset},\tilde{\emptyset})$. So (χ,φ) is $\mathrm{DIT}_2 - \mathrm{S}$, but not $\mathrm{DIR}-\mathrm{S}$. Take $(\upsilon_1,\nu_2)^c = (\lambda_1^c,\lambda_1^c)$, $(\tilde{1},\tilde{1}) \notin (\upsilon_1,\nu_2)^c$. Let $(\upsilon_1,\nu_1) = (\lambda_1,\lambda_1)$, $(\upsilon_2,\nu_2) = (\lambda_2,\tilde{\chi})$, (υ_1,ν_2)▪▪(u_2,ν_2) and $(\tilde{1},\tilde{1}) \in (\lambda_1,\lambda_1)$; $(\lambda_1,\lambda_1) \cap (\lambda_2,\tilde{\chi}) = (\lambda_2^c,\lambda_1) \neq (\tilde{\emptyset},\tilde{\emptyset})$. Hence (χ,φ) is not $\mathrm{DIR}-\mathrm{S}$.

Definition 4.10 Let (χ,φ) be a DITS, then the space (χ,φ) is said to be $\mathrm{DIT}_3 - \mathrm{space}$ if and only if its $\mathrm{DIR}-\mathrm{S}$ and $\mathrm{DIT}_1 - \mathrm{S}$.

Example 4.11 See example 3.12, the space (χ,φ) is $\mathrm{DIT}_3 - \mathrm{S}$, since its $\mathrm{DIT}_1 - \mathrm{S}$ and DIR-S.

Theorem 4.12 The property of being $\mathrm{DIT}_3 - \mathrm{S}$ is a hereditary property.

Proof Let the property $\mathrm{DIT}_1 - \mathrm{S}$ & $\mathrm{DIR}-\mathrm{S}$ be a hereditary property. Then $\mathrm{DIT}_3 - \mathrm{S}$ it is a hereditary property.

Theorem 4.13 Let (χ,φ) & $(\text{Ɏ},\varphi^*)$ be two DITS. Then the Double I-product space $\chi \times \text{Ɏ}$ is $\mathrm{DIT}_3 - \mathrm{S}$ if and only if each χ & Ɏ is $\mathrm{DIT}_3 - \mathrm{S}$.

Proof In proving in preceding theorems: that Double I-product space $\chi \times \text{Ɏ}$is $\mathrm{DIT}_1 - \mathrm{S}$ and DIR-S if and only if each χ & Ɏ are $\mathrm{DIT}_1 - \mathrm{S}$ (by theorem 3.10) and DIR-S. Hence, the Double I-product space χ & Ɏis $\mathrm{DIT}_3 - \mathrm{S}$ if and only if each χ & Ɏ is $\mathrm{DIT}_3 - \mathrm{S}$.

Theorem 4.14 Let (χ,φ) be $\mathrm{DIT}_3 - \mathrm{S}$, then χ is $\mathrm{DIT}_2 - \mathrm{S}$.

Proof Suppose that χ $\mathrm{DIT}_3 - \mathrm{S}$ (i.e., $\chi$is $\mathrm{DIT}_1 - \mathrm{S}$ and $\mathrm{DIR}-\mathrm{S}$), to proveχ is $\mathrm{DIT}_2 - \mathrm{S}$. Let $(\tilde{p},\tilde{p}), (\tilde{q},\tilde{q}) \in \chi$; $(\tilde{p},\tilde{p}) \neq (\tilde{q},\tilde{q})$, since χ is $\mathrm{DIT}_1 - \mathrm{Space} \rightarrow \{(\tilde{p},\tilde{p}), (\tilde{q},\tilde{q})\} \in (\upsilon_1,\nu_1)^c$ (by theorem 3.8), then $(\tilde{p},\tilde{p}) \notin \{(\tilde{q},\tilde{q})\}$ (since $(\tilde{p},\tilde{p}) \neq (\tilde{q},\tilde{q})$). Since χ is $\mathrm{DIR}-\mathrm{S}$, then there exists $(\upsilon_1,\nu_1), (\upsilon_2,\nu_2) \in \varphi$; $(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$, $((\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1)$ ▪▪$\{(\tilde{q},\tilde{q})\}$▪▪ $(\upsilon_2,\nu_2)) \rightarrow (\tilde{p},\tilde{p}) \in (\upsilon_1,\nu_1)$ ▪ $(\tilde{q},\tilde{q})$▪▪ $(\upsilon_2,\nu_2))$. Therefore (χ,φ) it is $\mathrm{DIT}_2 - \mathrm{S}$. In the previous theorem, take $(\tilde{p},\tilde{p}) \notin \{(\tilde{q},\tilde{q})\}$ and in a similar way. I can take $(\tilde{q},\tilde{q}) \notin \{(\tilde{p},\tilde{p})\}$ and have the same outcome.

Remark 4.15 From a above theorem, I have $DIT_3 - S \rightarrow DIT_2 - S \rightarrow\cdot DIT_1 - S \rightarrow DIT_0 - S$.

$\nleftarrow \quad \nleftarrow \quad \nleftarrow$

Definition 4.16 Let (χ,φ) be DITS. Then space (χ,φ) is said to be Double intuitionistic Normal − space(DIN − S, for short) if and only if for each $(\upsilon_1,V_2)^c$, $(\upsilon_3,V_4)^c \in \varphi^c$; $(\upsilon_1,V_2)^c \cap (\upsilon_3,V_4)^c = (\widetilde{\emptyset},\widetilde{\emptyset})$, there exist $(\upsilon_1,V_1)\cdot(\upsilon_2,V_2) \in \varphi$; $(\upsilon_1,V_1)\cdot\cap (\upsilon_2,V_2) = (\widetilde{\emptyset},\widetilde{\emptyset})$, $(\upsilon_1,V_2)^c\cdot\cdot (\upsilon_1,V_1)\cdot\cdot (\upsilon_3,V_4)^c\cdot\cdot\cdot(\upsilon_2,V_2)$.

Example 4.17 See example 4.7, Clear (χ,φ) is DIN − S. Take $((\upsilon_1,V_2)^c = (\omega_2^c, \omega_1^c) \cdot\cdot (\upsilon_3,V_4)^c = (\widetilde{\emptyset},\widetilde{\emptyset}) \in \varphi^c$, $(\omega_2^c,\omega_1^c) \cap (\widetilde{\emptyset},\widetilde{\emptyset}) = (\widetilde{\emptyset},\widetilde{\emptyset})$, there exist; $(\upsilon_1,V_1) = (\omega_3, \omega_3)\cdot\cdot(\upsilon_2,V_2) = (\widetilde{\emptyset},\widetilde{\emptyset}) \in \varphi$, $(\omega_3,\omega_3) \cap (\widetilde{\emptyset},\widetilde{\emptyset}) = (\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\upsilon_1,V_2)^c\cdot\cdot (\upsilon_1,V_1)\cdot\cdot (\upsilon_3,V_4)^c\cdot\cdot\cdot(\upsilon_2,V_2)$).

Remark 4.18 In the former example 4.17, remark that (χ,φ) not DIR-S and its DIN-S, accordingly that DIN-S $\nrightarrow$ DIR-S.

Remark 4.19 DIN-S $\nrightarrow DIT_1 - S \cdot$ DIN-S $\nrightarrow DIT_2 - S$.

See example 4.2, the space (χ,φ) is DIN-S. Take, $(\mathcal{T}_1, \mathcal{T}_1), (\mathcal{T}_1^c, \mathcal{T}_1^c) \in \varphi = \varphi^c$; $(\mathcal{T}_1,\mathcal{T}_1) \cap (\mathcal{T}_1^c, \mathcal{T}_1^c) = (\widetilde{\emptyset},\widetilde{\emptyset})$, but (χ,φ) is not $DIT_1 - S$ and not $DIT_2 - S$.

Remark 4.20 DIR-S $\nrightarrow$ DIN − S $\cdot$ $DIT_1 - S \nrightarrow$ DIN − S.

Example 4.21 See example 3.7, $\varphi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi},\widetilde{\chi}), (\mathcal{S}_1,\widetilde{\chi}), (\mathcal{S}_1,\mathcal{S}_1), (\widetilde{\emptyset},\mathcal{S}_1^c)\}$ where $(\mathcal{S}_1,\mathcal{S}_1) = (\langle\{a\},\{b\}\rangle, \langle\{a\},\{b\}\rangle)$. The space (χ,φ) is $DIT_1 - S$ and not DIN − S. Take $(\upsilon_1,V_2)^c = (\widetilde{\emptyset}, \mathcal{S}_1^c)\cdot\cdot (\upsilon_3,V_4)^c = (\mathcal{S}_1, \mathcal{S}_1) \in \varphi^c$, $(\upsilon_1,V_2)^c \cap (\upsilon_3,V_4)^c = (\widetilde{\emptyset},\widetilde{\emptyset})$, there exist; $(\upsilon_1,V_1) = (\mathcal{S}_1,\widetilde{\chi})$, $(\upsilon_2,V_2) = (\mathcal{S}_1^c,\mathcal{S}_1^c) \in \varphi$, $(\mathcal{S}_1, \widetilde{\chi}) \cap (\mathcal{S}_1^c,\mathcal{S}_1^c) = (\widetilde{\emptyset}, \mathcal{S}_1^c) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\upsilon_1,V_2)^c\cdot\cdot (\upsilon_1,V_1) \cdot\cdot\cdot (\upsilon_3,V_4)^c\cdot\cdot\cdot(\upsilon_2,V_2)$).

Example 4.22 Let $\chi = \{10, 20\}$; $\varphi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi}, \widetilde{\chi}), (\mathcal{H}_1,\mathcal{H}_1), (\mathcal{H}_2,\mathcal{H}_2), (\mathcal{H}_3, \mathcal{H}_3), (\mathcal{H}_3^c, \mathcal{H}_3^c), (\mathcal{H}_1^c, \mathcal{H}_1^c)\}$ where $(\mathcal{H}_1,\mathcal{H}_1) = (\langle\{10\},\emptyset\rangle, \langle\{10\},\emptyset\rangle)$, $(\mathcal{H}_2,\mathcal{H}_2) = (\langle\{20\},\{10\}\rangle, \langle\{20\},\{10\}\rangle)$, $(\mathcal{H}_3,\mathcal{H}_3) = (\langle\{20\},\emptyset\rangle, \langle\{20\},\emptyset\rangle)$, $(\mathcal{H}_3^c,\mathcal{H}_3^c) = (\langle\emptyset,\{20\}\rangle, \langle\emptyset,\{20\}\rangle)$ and $(\mathcal{H}_1^c,\mathcal{H}_1^c) = (\langle\emptyset,\{10\}\rangle, \langle\emptyset,\{10\}\rangle)$. $\varphi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi},\widetilde{\chi}), (\mathcal{H}_1^c,\mathcal{H}_1^c), (\mathcal{H}_2^c,\mathcal{H}_2^c), (\mathcal{H}_3^c,\mathcal{H}_3^c), (\mathcal{H}_1,\mathcal{H}_1), (\mathcal{H}_3,\mathcal{H}_3)\}$ where $(\mathcal{H}_2^c,\mathcal{H}_2^c) = (\langle\{10\},\{20\}\rangle, \langle\{10\},\{20\}\rangle)$. The space (χ,φ) is DIR − S, since $(u_1,v_2)^c = (\mathcal{H}_3^c,\mathcal{H}_3^c) \in \varphi^c$, $(\widetilde{20}, \widetilde{20}) \in (u_1,v_2)^c$, there exists $(\mathcal{H}_2, \mathcal{H}_2)$, $(\mathcal{H}_3^c,\mathcal{H}_3^c) \in \varphi$ then $(\mathcal{H}_2,\mathcal{H}_2) \cap (\mathcal{H}_3^c,\mathcal{H}_3^c) = (\widetilde{\emptyset},\widetilde{\emptyset})$.(since $(\tilde{2},\tilde{2}) \in (\mathcal{H}_2, \mathcal{H}_2)\cdot\cdot (u_1,v_2)^c \cdot (\mathcal{H}_3^c,\mathcal{H}_3^c)$). But not DIN − S, since $(\mathcal{H}_1^c, \mathcal{H}_1^c), (\mathcal{H}_2^c, \mathcal{H}_2^c) \in \varphi^c = \varphi$, $(\mathcal{H}_1^c,\mathcal{H}_1^c) \cap (\mathcal{H}_2^c,\mathcal{H}_2^c) = (\widetilde{\emptyset},\widetilde{\emptyset})$. $((\mathcal{H}_1^c,\mathcal{H}_1^c) \cdot (\mathcal{H}_2,\mathcal{H}_2) \cdot\cdot\cdot (\mathcal{H}_2^c,\mathcal{H}_2^c) \subsetneq (\mathcal{H}_3^c,\mathcal{H}_3^c))$.

Remark 4.23 $DIT_0 - S \nrightarrow$ DIN-S $\cdot$ DIN − S $\nrightarrow DIT_0 - S$.

Example 4.24 Recall example 3.2, $\varphi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi},\widetilde{\chi}), (\mho_2^c,\mho_1^c), (\mho_4^c,\mho_3^c), (\widetilde{\emptyset},\mho_5^c), (\mho_7^c,\mho_6^c)\}$ where $(\mho_2^c,\mho_1^c) = (\langle\{j\},\{i,h\}\rangle, \langle\{j\},\{i\}\rangle)$, $(\mho_4^c,\mho_3^c) = (\langle\{h\},\{i,j\}\rangle, \langle\{h\},\{j\}\rangle)$, $(\widetilde{\emptyset},\mho_5^c) = (\langle\emptyset,\chi\rangle, \langle\emptyset,\{i,j\}\rangle)$ and $(\mho_7^c,\mho_6^c) = (\langle\{j,h\},\{i\}\rangle, \langle\{j,h\},\emptyset\rangle)$. The space$(\chi,\varphi)$ is not DIN − S, the DICS (χ) in φ^c, then $(\mho_2^c,\mho_1^c) \cap (\mho_4^c,\mho_3^c) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$ there exist $(\mho_5,\widetilde{\chi}), (\mho_1,\mho_2) \in \varphi$, $(\mho_5,\widetilde{\chi}) \cap (\mho_1,\mho_2) \neq (\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\mho_2^c,\mho_1^c)\cdot\cdot (\mho_5,\widetilde{\chi})\cdot\cdot (\mho_4^c,\mho_3^c) \subsetneq (\mho_1,\mho_2)$. But (χ,φ) is $DIT_0 - S$.

Example 4.25 Recall example 4.2, the space (χ,φ) is not DIN − S. But is not $DIT_0 - S$, since $(\tilde{d},\tilde{d}) \neq (\tilde{j},\tilde{j})$, then $(\widetilde{\chi},\widetilde{\chi}), (\mathcal{T}_1^c, \mathcal{T}_1^c) \in \varphi$, $(\tilde{d},\tilde{d}) \in (\mathcal{T}_1^c,\mathcal{T}_1^c)\cdot\cdot (\tilde{j},\tilde{j}) \in (\widetilde{\chi},\widetilde{\chi})$.

Remark 4.26 Let (χ,φ) is DITS and $\varphi = \varphi^c$ (i.e., every DICS(χ) is $\in (\chi)$ or $(u_1,v_1) \in \varphi \leftrightarrow (u_1,v_1) \in \varphi^c$, then (χ,φ) is DIN −S.

The space $(\chi, \Im(\chi))$ and $(\chi, Đ(\chi))$ is the special case from these spaces.

Remark 4.27 The property of Bing a DIN −S is not hereditary and the following example displays that:

Example 4.28 Let $\chi = \{q, p, k\}$; $\varphi = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi},\widetilde{\chi}), (\mathcal{B}_1,\mathcal{B}_1), (\mathcal{B}_2,\mathcal{B}_3), (\mathcal{B}_4,\mathcal{B}_5), (\mathcal{B}_1^c,\mathcal{B}_1^c)\}$ where $(\mathcal{B}_1,\mathcal{B}_1) = (\langle\{q,k\},\{p\}\rangle, \langle\{q,k\},\{p\}\rangle)$, $(\mathcal{B}_2,\mathcal{B}_3) = (\langle\{p\},\{k\}\rangle, \langle\{q,p\},\emptyset\rangle)$, $(\mathcal{B}_4, \mathcal{B}_5) = (\langle\emptyset,\{p,k\}\rangle, \langle\{q\},\{p\}\rangle)$ and $(\mathcal{B}_1^c,\mathcal{B}_1^c) = (\langle\{p\},\{q,k\}\rangle, \langle\{p\},\{q,k\}\rangle)$. $\varphi^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (\widetilde{\chi},\widetilde{\chi}), (\mathcal{B}_1^c,\mathcal{B}_1^c), (\mathcal{B}_3^c,\mathcal{B}_2^c), (\mathcal{B}_5^c,\mathcal{B}_4^c), (\mathcal{B}_1,\mathcal{B}_1)\}$ where $(\mathcal{B}_1^c,\mathcal{B}_1^c) = (\langle\emptyset,\{q,p\}\rangle, \langle\{k\},\{p\}\rangle)$, $(\mathcal{B}_3^c,\mathcal{B}_2^c) = (\langle\emptyset,\{q,p\}\rangle, \langle\{k\},\{p\}\rangle)$ and $(\mathcal{B}_5^c,\mathcal{B}_4^c) = (\langle\{p\},\{q\}\rangle, \langle\{p,k\},\emptyset\rangle)$. Let $((\upsilon_1,\nu_2)^c = (\mathcal{B}_1,\mathcal{B}_1), (\upsilon_3,\nu_4)^c = (\mathcal{B}_1^c,\mathcal{B}_1^c)) \in \varphi = \varphi^c$, $(\mathcal{B}_1,\mathcal{B}_1) \cap (\mathcal{B}_1^c, \mathcal{B}_1^c) = (\widetilde{\emptyset},\widetilde{\emptyset})$, (since $(\upsilon_1,\nu_2)^c\cdot\cdot\cdot(\upsilon_1,\nu_1)\cdot\cdot (\upsilon_3,\nu_4)^c\cdot\cdot\cdot(\upsilon_2,\nu_2)$).$\cdot$Hence (χ,φ) is DIN − S. Now, take the Double intuitionistic topological subspace $¥ \cdot \chi ;\cdot¥\cdot\cdot\cdot\cdot q, p\cdot\cdot\cdot$and$\cdot\cdot¥,¥) \in \varphi\cdot\cdot$then $(¥,¥) \cdot\cdot(\langle\{q\},\{p\}\rangle, \langle\{q\},\{p\}\rangle)$. $\varphi_¥ = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (¥,¥), (\mathcal{B}_4,\mathcal{B}_5)\}$ and $\varphi_¥{}^c = \{(\widetilde{\emptyset},\widetilde{\emptyset}), (¥,¥), (\mathcal{B}_3^c,\mathcal{B}_6)\}$ where $(\mathcal{B}_3^c, \mathcal{B}_6) = (\langle\emptyset,\{q,p\}\rangle, \langle\emptyset,\{p\}\rangle)$, notes that $(\mathcal{B}_3^c, \mathcal{B}_6), (\widetilde{\emptyset},\widetilde{\emptyset}) \in \varphi_¥{}^c$ and $(\mathcal{B}_3^c, \mathcal{B}_6) \cap (\widetilde{\emptyset},\widetilde{\emptyset}) = (\widetilde{\emptyset},\widetilde{\emptyset})$. But there exist $(\mathcal{B}_3^c, \mathcal{B}_6) \in (\chi)$ in $\varphi_¥$, $(\mathcal{B}_4,\mathcal{B}_5) \cap (\widetilde{\emptyset},\widetilde{\emptyset}) = (\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\mathcal{B}_3^c,\mathcal{B}_6) \subsetneq (\mathcal{B}_4,\mathcal{B}_5)$). So that ¥ is not DIN-S, while χ is DIN-S.

Theorem 4.29 The space (χ,φ) is DIN − S if and only if for each Double I closed subset $(\upsilon_1,\nu_2)^c \cdot \chi$ and DI open in ¥ containing $(\upsilon_1,\nu_2)^c$, there exists an DI open set (υ_1,ν_1) such that $(\upsilon_1,\nu_2)^c\cdot\cdot\cdot(\upsilon_1,\nu_1)\cdot\cdot\ cl(\upsilon_1,\nu_1) \cdot\cdot¥$.

Proof $(\cdot)$ Let $¥ \in \varphi$, $(\upsilon_1,\nu_2)^c\cdot\cdot\cdot¥$, so $(\upsilon_1,\nu_2)^c\cdot\cap ¥^c = (\widetilde{\emptyset},\widetilde{\emptyset}) \cdot ¥^c \in \varphi^c$ (since $¥ \in \varphi$). Since χ is DIN-S, there exist $(\upsilon_1,\nu_1), (\upsilon_2,\nu_2) \in \varphi$; $(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\widetilde{\emptyset},\widetilde{\emptyset})$ (since$(\upsilon_1,\nu_2)^c \cdot\cdot\cdot(\upsilon_1,\nu_1) \cdot\cdot ¥\cdot{}^c\cdot\cdot\cdot(\upsilon_1,\nu_1)$), so $(\upsilon_1,\nu_2)^c\cdot\cdot\cdot¥$, since $(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\widetilde{\emptyset},\widetilde{\emptyset}) \rightarrow (\upsilon_1,\nu_1) \cdot (\upsilon_2,\nu_2)^c\cdot\rightarrow cl(\upsilon_1,\nu_1)\cdot\cdot\cdot cl(\upsilon_2,\nu_2)^c$ (by theorem 2.12(3))$\cdot\rightarrow cl(\upsilon_1,\nu_1)\cdot\cdot (\upsilon_2,\nu_2)^c$ (by theorem 2.12 (2))$\rightarrow (\upsilon_1,\nu_1)\cdot\cdot\cdot cl(\upsilon_1,\nu_1)\cdot\cdot\cdot(\upsilon_2,\nu_2)^c$ (by theorem 2.12 (1)), so$(\upsilon_1,\nu_2) \cdot\cdot (\upsilon_1,\nu_1) \cdot\cdot(\upsilon_1,\nu_1)\cdot\cdot\ cl(\upsilon_1,\nu_1)\cdot\cdot (\upsilon_2,\nu_2)^c\cdot\cdot (\upsilon_2,\nu_2)^c\cdot\cdot\cdot¥ \rightarrow (\upsilon_2,\nu_2)^c\cdot\cdot(\upsilon_1,\nu_1)\cdot\cdot\ cl(\upsilon_1,\nu_1) \cdot\cdot\cdot¥$**.**

Conversely $\cdot\cdot$) Let $(\upsilon_1,\nu_2)^c\cdot(\upsilon_3,\nu_4)^c \in \varphi^c$; $(\upsilon_1,\nu_2)^c \cap (\upsilon_3,\nu_4)^c\cdot = (\widetilde{\emptyset},\widetilde{\emptyset})$, $(\upsilon_1,\nu_2)^c\cdot\cdot (\upsilon_3,\nu_4)^c)^c \in \varphi$ (since $(\upsilon_3,\nu_4) \in \varphi^c$), there exist $(\upsilon_1,\nu_1) \in \varphi$; $(\upsilon_1,\nu_2)^c\cdot\cdot (\upsilon_1,\nu_1) \cdot\cdot\cdot cl(\upsilon_1,\nu_1)\cdot\cdot (\upsilon_3,\nu_4)^c)^c \rightarrow cl(\upsilon_1,\nu_1)\cdot\cdot (\upsilon_3,\nu_4)^c)^c \rightarrow (\upsilon_3,\nu_4)^c\cdot\cdot\cdot\cdot (cl(\upsilon_1,\nu_1))^c$. But $(cl(\upsilon_1,\nu_1))^c$ is DIO (χ), $cl(\upsilon_1,\nu_1)$ is DIC(χ), say $(cl(\upsilon_1,\nu_1))^c = (\upsilon_2,\nu_2) \rightarrow (\upsilon_3,\nu_4)^c\cdot\cdot (\upsilon_2,\nu_2)\cdot\cdot\cdot(cl(\upsilon_1,\nu_1))^c \cdot\cdot(u_1,\nu_2)^c\cdot\cdot\cdot(\upsilon_1,\nu_1) \cdot (\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\upsilon_1,\nu_1) \cdot\cdot cl(\upsilon_1,\nu_1)$ and $cl(\upsilon_1,\nu_2) \cap (cl(\upsilon_1,\nu_1))^c = (\widetilde{\emptyset},\widetilde{\emptyset}) \rightarrow (\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\widetilde{\emptyset},\widetilde{\emptyset})$). Therefore χ is DIN − S.

Theorem 4.30 A Double I-closed subspace of $\mathrm{DIN}-\mathrm{S}$ is $\mathrm{DIN}-\mathrm{S}$.

Proof Let (χ,φ) be $\mathrm{DIN}-\mathrm{S}$ and $(Ұ,\varphi_{Ұ})$ is Double I-closed subspace of χ, to prove $(Ұ,\varphi_{Ұ})$ is $\mathrm{DIN}-\mathrm{S}$. Let $(\upsilon_1,\nu_2)_{Ұ}{}^{c}$, $(\upsilon_3,\nu_4)_{Ұ}{}^{c}$ be DIC(χ) in ($Ұ$); $(\upsilon_1,\nu_2)_{Ұ}{}^{c}\cap(\upsilon_3,\nu_4)_{Ұ}{}^{c}=(\widetilde{\emptyset},\widetilde{\emptyset})$, there exist $(\upsilon_1,\nu_2)^{c}$, $(\upsilon_3,\nu_4)^{c}\in\varphi^{c}$ • • $(\upsilon_1,\nu_2)_{Ұ}{}^{c}=(\upsilon_1,\nu_2)^{c}\cap Ұ$ • • $(u_3,\nu_4)_{Ұ}{}^{c}=(u_3,\nu_4)^{c}\cap Ұ$; $(\upsilon_1,\nu_2)^{c}\cap(u_3,\nu_4)^{c}=(\widetilde{\emptyset},\widetilde{\emptyset})$. Since χ is $\mathrm{DIN}-\mathrm{S}$, there exist (υ_1,ν_1) • • $(\upsilon_2,\nu_2)\in$, (υ_1,ν_1) • $\cap$ $(\upsilon_2,\nu_2)=(\widetilde{\emptyset},\widetilde{\emptyset})$, • (since $(\upsilon_1,\nu_2)^{c}$ • (u_1,ν_1) • • $(u_3,\nu_4)^{c}$ • • (u_2,ν_2)) $\rightarrow(\upsilon_1,\nu_1)\cap Ұ$ • • $(u_2,\nu_2)\cap Ұ\in\varphi_{Ұ}$ (by definition $\varphi_{Ұ}$). $((\upsilon_1,\nu_1)\cap Ұ$ • • $(\upsilon_2,\nu_2)\cap Ұ)$ = $((\upsilon_1,\nu_1)\cap$ • $(\upsilon_2,\nu_2))\cap Ұ=(\widetilde{\emptyset},\widetilde{\emptyset})\cap Ұ=(\widetilde{\emptyset},\widetilde{\emptyset})$, since $(\upsilon_1,\nu_2)_{Ұ}{}^{c}=(\upsilon_1,\nu_2)^{c}\cap Ұ\rightarrow(\upsilon_1,\nu_2)_{Ұ}{}^{c}$ • • $(\upsilon_1,\nu_2)^{c}$ • • $(\upsilon_1,\nu_2)_{Ұ}{}^{c}$ • • • $Ұ\rightarrow(\upsilon_1,\nu_1)_{Ұ}{}^{c}$ • (υ_1,ν_1) • $(\upsilon_1,\nu_2)_{Ұ}{}^{c}$ • • $Ұ\rightarrow(\upsilon_1,\nu_2)_{Ұ}{}^{c}$ • • $(\upsilon_1,\nu_1)\cap Ұ$, • since $(\upsilon_3,\nu_4)_{Ұ}{}^{c}=(\upsilon_3,\nu_4)^{c}\cap Ұ\rightarrow(\upsilon_3,\nu_4)_{Ұ}{}^{c}$ • • $(\upsilon_3,\nu_4)^{c}$ • • $(\upsilon_3,\nu_4)_{Ұ}{}^{c}$ • $Ұ\rightarrow$ $(\upsilon_3,\nu_4)_{Ұ}$ • ^{c} • • (υ_1,ν_2) • • $(\upsilon_3,\nu_4)^{c}$ • • $Ұ\rightarrow(\upsilon_3,\nu_4)_{Ұ}{}^{c}$ • $(\upsilon_1,\nu_2)\cap Ұ$. Therefore, $(Ұ,\varphi_{Ұ})$ is $\mathrm{DIN}-\mathrm{S}$.

Definition 4.31 Let (χ,φ) be DITS. Then space (χ,φ) is said to be Double intuitionistic T_4 –S ($\mathrm{DIT}_4-\mathrm{S}$, for short) if and only if its $\mathrm{DIN}-\mathrm{S}$ and DIT_1 –S. (i.e., $\mathrm{DIT}_4-\mathrm{S}=\mathrm{DIT}_1-\mathrm{S}+\mathrm{DIN}{-}\mathrm{S}$).
Example 4.32 From example 3.12. Then the space (χ,φ) is $\mathrm{DIT}_4-\mathrm{S}$, since it $\mathrm{DIT}_1-\mathrm{S}$ and $\mathrm{DIN}{-}\mathrm{S}$.
Remark 4.33 The property of being a $\mathrm{DIT}_4-\mathrm{S}$ is not a hereditary property, since normality is not a hereditary property.

Example 4.34 See example 3.7. Then space (χ,φ) is not $\mathrm{DIT}_4-\mathrm{S}$, since its $\mathrm{DIT}_1-\mathrm{S}$, but not $\mathrm{DIN}{-}\mathrm{S}$.
Theorem 4.35 A Double I-closed subspace of $\mathrm{DIT}_4-\mathrm{S}$ is $\mathrm{DIT}_4-\mathrm{S}$.

Proof Let (χ,φ) be $\mathrm{DIT}_4-\mathrm{S}$ and $Ұ$ be Double I-closed subspace of χ, to prove $Ұ$ is $\mathrm{DIT}_4-\mathrm{S}$, since χ it is $\mathrm{DIT}_1-\mathrm{S}\rightarrow Ұ$ is $\mathrm{DIT}_1-\mathrm{S}$, since $Ұ$ is DICS (χ) and χ it is $\mathrm{DIN}-\mathrm{S}\rightarrow Ұ$ is $\mathrm{DIN}-\mathrm{S}$ (by theorem 4.30). Therefore, $Ұ$ is $\mathrm{DIT}_4-\mathrm{S}$.

Theorem 4.36 Every $\mathrm{DIT}_4-\mathrm{S}$ is $\mathrm{DIR}-\mathrm{S}$.

Proof Let (χ,φ) is $\mathrm{DIT}_4-\mathrm{S}\rightarrow\chi$ is $\mathrm{DIT}_1-\mathrm{S}$ and $\mathrm{DIN}-\mathrm{S}$. Let $(\tilde{p},\tilde{p})\in\chi$ and $(\upsilon_1,\nu_1)^{c}$ is DICS(χ); $(\tilde{p},\tilde{p})\notin(\upsilon_1,\nu_1)^{c}\rightarrow\{(\tilde{p},\tilde{p})\}\in(\upsilon_1,\nu_1)^{c}$ (by theorem 3.8) $\rightarrow\{(\tilde{p},\tilde{p})\}\cap(\upsilon_2,\nu_2)^{c}=(\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\tilde{p},\tilde{p})\notin(\upsilon_1,\nu_1)^{c}$). Since χ is $\mathrm{DIN}-\mathrm{S}$, then there exists $(\upsilon_1,\nu_1),(\upsilon_2,\nu_2)\in\varphi$; $(\upsilon_1,\nu_1)\cap(\upsilon_2,\nu_2)=(\widetilde{\emptyset},\widetilde{\emptyset})$, $\{(\tilde{p},\tilde{p})\}$ • • (υ_1,ν_1) • • $(u_1,\nu_1)^{c}$ • • (υ_2,ν_2)), so $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_1)$ • • • $(\upsilon_1,\nu_1)^{c}$ • • • (υ_2,ν_2). Hence χ is $\mathrm{DIR}-\mathrm{S}$.

Corollary 4.37 Every $\mathrm{DIT}_4-\mathrm{S}$ is $\mathrm{DIT}_3-\mathrm{S}$.
Proof By the above theorem and by definition of $\mathrm{DIT}_4-\mathrm{S}$. So χ is $\mathrm{DIT}_1-\mathrm{S}$ and $\mathrm{DIR}-\mathrm{S}$. Therefore χ is $\mathrm{DIT}_3-\mathrm{S}$.
Remark 4.38 Every $\mathrm{DIT}_4-\mathrm{S}$ is $\mathrm{DIT}_2-\mathrm{S}$, since every $\mathrm{DIT}_4-\mathrm{S}$ is $\mathrm{DIT}_3-\mathrm{S}$ and every $\mathrm{DIT}_3-\mathrm{S}$ is $\mathrm{DIT}_2-\mathrm{S}$, so that

$$\mathrm{DIT}_4-\mathrm{S}\underset{\nleftarrow}{\rightarrow}\mathrm{DIT}_3-\mathrm{S}\underset{\nleftarrow}{\rightarrow}\mathrm{DIT}_2-\mathrm{S}\underset{\nleftarrow}{\rightarrow}\mathrm{DIT}_1-\mathrm{S}\underset{\nleftarrow}{\rightarrow}\mathrm{DIT}_0-\mathrm{S}.$$

Theorem 4.39 Every Double I-compact $\mathrm{T}_2-\mathrm{S}$ is $\mathrm{DIR}-\mathrm{S}$.

Proof Let (χ,φ) be $\mathrm{DIT}_2-\mathrm{S}$ and DI-compact, to prove χ is $\mathrm{DIR}-\mathrm{S}$. Let $(\tilde{p},\tilde{p})\in\chi$ and $(\upsilon_1,\nu_2)^{c}\in\varphi^{c}$; $(\tilde{p},\tilde{p})\notin(\upsilon_1,\nu_2)^{c}\rightarrow(\tilde{p},\tilde{p})\neq(\tilde{q},\tilde{q})$ for each $(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)^{c}$. Since χ is $\mathrm{DIT}_2-\mathrm{S}$, then there exist $(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$, $(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}\in\varphi$;$(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\cap(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}=(\widetilde{\emptyset},\widetilde{\emptyset})$, (since $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$ • • • $(\tilde{q},\tilde{q})\in(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}$. I have two families of DIOS(χ) are $\{(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\}_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}$ and $\{(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}\}_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}$ such that all element in $(\upsilon_1,\nu_2)^{c}$ exists in one element of the family $\{(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}\}_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}$ and every element in the family $\{(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\}_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}$ contains the element $(\tilde{p},\tilde{p})$ and every $(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$ corresponding $(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}$ such that $(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\cap(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}=(\widetilde{\emptyset},\widetilde{\emptyset})$. Therefore, $\{(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}\}_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}$ it is a Double I-open cover for $(\upsilon_1,\nu_2)^{c}$. i.e., $(\upsilon_1,\nu_2)^{c}$ • $\cup_{(\tilde{q},\tilde{q})\in(\upsilon_1,\nu_2)c}(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}$. • Since $(\upsilon_1,\nu_2)^{c}$ is DIC(χ) in the Double I-compact space (by hypothesis), so that $(\upsilon_1,\nu_2)^{c}$ is a Double I-compact space and there exists $(\tilde{q},\tilde{q})_1,(\tilde{q},\tilde{q})_2,\ldots(\tilde{q},\tilde{q})_n$, $(\upsilon_1,\nu_2)^{c}$ • $\cup^{n}{}_{i=1}(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}$. Therefore $\{(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}\}^{n}{}_{i=1}$ is a limited family of DIOS(χ) covers $(\upsilon_1,\nu_2)^{c}$, (let $(\upsilon_1,\nu_2)=\cup^{n}{}_{i=1}(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}$) on the other then hand, $\{(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\}^{n}{}_{i=1}$ it is a finite family of DIOS(χ) and every element in this family contains $(\tilde{p},\tilde{p})$. (let $(\upsilon_1,\nu_1)=\cap^{n}{}_{i=1}(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$), then (υ_1,ν_1) and (υ_2,ν_2) are DIOS(χ) (by the second &third conditions of the definition in DITS) s.t $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_1)$ & $(\upsilon_1,\nu_2)^{c}$ • (υ_2,ν_2). Notes that $(\upsilon_1,\nu_1)\cap(\upsilon_2,\nu_2)=(\widetilde{\emptyset},\widetilde{\emptyset})$ (since $(\upsilon_1,\nu_1)=\cap^{n}{}_{i=1}(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$), (υ_1,ν_1) • $(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}$, $\forall\,\mathrm{I}$ and $(\upsilon_1,\nu_1)_{(\tilde{q},\tilde{q})}\cap(\upsilon_2,\nu_2)_{(\tilde{q},\tilde{q})}=(\widetilde{\emptyset},\widetilde{\emptyset})$. Hence χ is $\mathrm{DIR}-\mathrm{S}$.

Corollary 4.40 Every Double I-compact $\mathrm{T}_2-\mathrm{S}$ is $\mathrm{DIT}_3-\mathrm{S}$.
Proof by the above theorem, by remark 3.13, and by theorem 4.39. Hence χ is $\mathrm{DIT}_1-\mathrm{S}$ and $\mathrm{DIR}-\mathrm{S}$. Therefore χ is $\mathrm{DIT}_3-\mathrm{S}$.
Theorem 4.41 Every Double I-compact $\mathrm{T}_2-\mathrm{S}$ is $\mathrm{DIN}-\mathrm{S}$.

Proof Let $(\upsilon_1,\nu_2)^{c}$, $(u_3,\nu_4)^{c}\in\varphi^{c}$; $(\upsilon_1,\nu_2)^{c}\cap(\upsilon_3,\nu_4)^{c}=(\widetilde{\emptyset},\widetilde{\emptyset})$, so $(\upsilon_1,\nu_2)^{c}$, $(\upsilon_3,\nu_4)^{c}$ are Double I-compact (by proposition 2.13). Choose $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_2)^{c}\rightarrow(\tilde{p},\tilde{p})\notin(\upsilon_3,\nu_4)^{c}$, then there exist $(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})},(\upsilon_2,\nu_2)_{(u_3,\nu_4)c}\in\varphi$; $(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}\cap(\upsilon_3,\nu_4)c_{(\tilde{p},\tilde{p})}=(\widetilde{\emptyset},\widetilde{\emptyset})$, since $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}$ • • $(\upsilon_3,\nu_4)^{c}$ • • $(\upsilon_2,\nu_2)_{(\upsilon_3,\nu_4)c}$ • • • Now • • repeat this method on every element in $(\upsilon_1,\nu_2)^{c}$, I have a family of DIOS(χ) cover $\{(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}$; $(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_2)^{c}$ • • $(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}\in\varphi\}\rightarrow(\upsilon_1,\nu_2)^{c}$ • • $\cup_{(\tilde{p},\tilde{p})\in(\upsilon_1,\nu_2)c}(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}$. • • Since $(\upsilon_1,\nu_2)^{c}$ is Double I-compact, so there exist $(\tilde{p},\tilde{p})_1,(\tilde{p},\tilde{p})_2,\ldots(\tilde{p},\tilde{p})_n$; $(\upsilon_1,\nu_2)^{c}$ • • • $\cup^{n}{}_{i=1}(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}$. Also, I need a family of DIOS every element in this family include $(\upsilon_3,\nu_4)^{c}$,$\{(\upsilon_2,\nu_2)_i$; $\forall$ i = 1,2,…,n • $(\upsilon_2,\nu_2)_i\in\varphi\}$, $(\upsilon_2,\nu_2)_i\cap(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})}=(\widetilde{\emptyset},\widetilde{\emptyset})$ $\forall$ i =1, 2,…,n. Say (υ_1,ν_1) =

$\cup^{n}{}_{i=1}(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})} \to (u_1,\nu_2)^{c} \cdot\cdot\cdot\cdot\cdot (\upsilon_1,\nu_1) \in \varphi \cdot$(by a third of the definition DITS). Say $(\upsilon_2,\nu_2) = \cap^{n}{}_{i=1}(\upsilon_2,\nu_2)_i \to (\upsilon_3,\nu_4)^{c} \cdot\cdot\cdot\cdot (\upsilon_2,\nu_2) \in \varphi$(by second of the definition DITS). Notes that, $(\upsilon_1,\nu_1) \cap (\upsilon_2,\nu_2) = (\tilde{\emptyset},\tilde{\emptyset})$ (since $(\upsilon_1,\nu_1) \cdot\cdot \cup^{n}{}_{i=1}(\upsilon_1,\nu_1)_{(\tilde{p},\tilde{p})} \cap (\upsilon_2,\nu_2) = \cap^{n}{}_{i=1}(\upsilon_2,\nu_2)_i \cdot = (\tilde{\emptyset},\tilde{\emptyset})$. Therefore χ is $\mathrm{DIN} - \mathrm{S}$.

5. Conclusions

In this article, we received the following outcomes: we have presented an original set of the following notions: Double intuitionisticT_0 $-$S (resp., Double intuitionisticT_1 $-$S, Double intuitionisticT_2 $-$S, Double intuitionisticT_3 $-$S, Double intuitionisticT_4 $-$S, Double intuitionistic R $-$S, and Double intuitionistic N $-$S on DITS. Also, we investigate some of their properties and give relations among them.

Funding: This research received no external funding.
Conflicts of Interest: The authors declare no conflict of interest.
ORCID : https://orcid.org/0009-0002-3812-4167 .

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