Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Nilpotent Covers of Small Symmetric and Alternating Groups

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| ABSTRACT
We compute the size of a minimal nilpotent cover for small alternating and symmetric groups, An and Sn. We give precise values for values of n up to 8. For n=9 we give upper and lower bounds for the size of a minimal nilpotent cover of A9.

| KEYWORDS
Alternating group; nilpotent cover; symmetric group

| ARTICLE INFORMATION
ACCEPTED: 12 April 2025 **PUBLISHED:** 16 May 2025 **DOI:** 10.32996/jmss.2025.6.2.3

1. Introduction

In this paper we calculate the least number of nilpotent subgroups of the symmetric group on n letters S_n that are necessary to cover S_n, for values of n up to and including 9. We also perform similar calculations for the alternating group A_n.

To explain our results in more detail, consider a finite group G. A *nilpotent cover* of G is a finite family M of nilpotent subgroups of G for which

$$G = \bigcup_{H \in M} H .$$

Every finite group G has a nilpotent cover comprising the family of cyclic subgroups of G. A nilpotent cover M of G is said to be *minimal* if no other nilpotent cover of G has fewer members. Let $\Sigma_N(G)$ denote the size of a minimal nilpotent cover of G. Notice that if G is itself nilpotent, then $\{G\}$ is the unique minimal nilpotent cover of G and $\Sigma_N(G) = 1$.

When calculating the minimal size of a nilpotent cover, we can immediately restrict our attention to the maximal nilpotent subgroups of G. Here a *maximal nilpotent subgroup* of G is a subgroup of G that is maximal with respect to inclusion in the class of nilpotent subgroups of G.

This paper should be seen as a companion to the paper [GKS22] which, as we will see below, gave a formula for $\Sigma_N(S_n)$; on the other hand this same paper showed that the treatment given there did not apply neatly to the groups A_n.

2. Symmetric groups

Theorem 1.1 of [GKS22] asserts that there is unique minimal cover of S_n by maximal nilpotent subgroups. In order to give the size of this cover, we define a *distinct partition* of a positive integer n to be a set $T = \{t_1, t_2, \dots, t_k\}$, where $t_1, t_2, \dots, t_k$ are distinct positive integers and $n = t_1 + t_2 + \cdots + t_k$. Let us write DP($n$) for the set of all distinct partitions of n. We then have

$$\Sigma_N(S_n) = \sum_{T \in \mathrm{DP}(n)} \left(\frac{n!}{\prod_{t \in T} \prod_{i=1}^{\ell} (p_i - 1)^{a_i} p_i^{e_i}} \right).$$

Key words and phrases. alternating group; nilpotent cover; symmetric group.

1

In this expression $t = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell}$ is the prime factorisation of *t* (which depends on *t*) and $e_i = ({p_i}^{a_i} - 1)/(p^i - 1)$ for i = 1,2,...,ℓ. The product

$$\prod_{i=1}^{\ell} (p_i - 1)^{a_i} p_i^{e_i}$$

is considered to take the value 1 if t = 1. Table 2.1 displays the first few values of DP(n) and $\Sigma_N(S_n)$.

n	DP(n)	$\Sigma_N(S_n)$
2	$\{2\}$	1
3	$\{1,2\},\{3\}$	4
4	$\{1,3\},\{4\}$	7
5	$\{1,4\},\{2,3\},\{5\}$	31
6	$\{1,2,3\},\{1,5\},\{2,4\},\{6\}$	201
7	$\{1,2,4\},\{1,6\},\{2,5\},\{3,4\},\{7\}$	1086
8	$\{1,2,5\},\{1,3,4\},\{1,7\},\{2,6\},\{3,5\},\{8\}$	5139
9	$\{1,2,6\},\{1,3,5\},\{2,3,4\},\{1,8\},\{2,7\},\{3,6\},\{4,5\},\{9\}$	37507

Table 2.1. Values of DP(n) and $\Sigma_N(S_n)$, for n = 2,3,...,9.

3. Alternating groups

For the alternating groups, no formula is known for the minimal size of a nilpotent cover. Table 3.1 displays the first few values of $\Sigma_N(A_n)$; these were first calculated in the first author's PhD thesis. We justify the numbers given there below.

n	$\Sigma_N(A_n)$
2	1
3	1
4	5
5	21
6	91
7	666
8	3571
9	between 28120 and 30955

Table 3.1. Values of $\Sigma_N(A_n)$, for n = 2,3,...,9.

First, when n = 2 or 3, the group A_n is itself nilpotent, hence $\Sigma_N(A_n)$ = 1. When n = 4, the Sylow subgroups of A_n are maximal and, because they are *p*groups, they nilpotent. What is more, every element of A_4 lies in a unique Sylow subgroup. Thus we just need to count Sylow subgroups of A_4: there are five in total and the result follows.

3.1. **Cases** $n \geq 5$**.** From here on, the situation is more complicated. We seek to construct a minimal cover, $\mathcal{C}$, of $G = A_n$ by maximal nilpotent subgroups. Our strategy will be as follows:

(i) First, we list all conjugacy classses of the group $G = A_n$.
(ii) We enumerate those conjugacy classes, C_1, for which there exists another class, C_2, such that elements of C_1 are powers of elements in C_2. When we are constructing a nilpotent cover it will therefore be sufficient to ensure that all elements of C_2 are contained in an element of the cover, as this will automatically

imply that elements of C_1 are similarly contained.

(iii) Now let g be an element of a conjugacy class that does **not** contain powers of another class. We compute those maximal nilpotent subgroups of G that contain g. In many cases we find that there is only one such maximal nilpotent subgroup, N_g, and in that case N_g must be contained in the nilpotent cover, $\mathcal{C}$.
(iv) Where there is more than one such maximal nilpotent subgroup, further calculations are required.

Tables 3.2 to 3.6 summarise the situation. The notation we use is as follows:

(i) *Label* gives a name for each conjugacy class of $G = A_n$; recall that a conjugacy class in A_n is determined by cycle type unless that cycle type consists of distinct odd numbers, in which case the class splits in two (so we use labels "*a*" and "*b*") and we list the two corresponding conjugacy classes on the same row.
(ii) *element* gives an element from the conjugacy class(es) on the given row.
(iii) *power of* gives a conjugacy class of which the current class is a power, if such exists.
(iv) *subgroup* gives the unique maximal nilpotent subgroup N of G that contains a given element from this current class, should such a group exist. Note that we only need to list such groups for classes that do not have an entry in the *power of* column. Our notation here is largely standard; note that $P_{p,n}$ denotes a Sylow p-subgroup of A_n. In some cases, it turns out that the same group N can be the unique maximal nilpotent subgroup containing elements from more than one conjugacy class (e.g. for class Cl_3 and Cl_4 in A_6). In this case we write the subgroup N in parenthesis for one of these classes, to ensure that it is not counted twice; entries after the parenthetic entry will then be empty.
(v) *number* gives the number of conjugates of the group N.
(vi) *structure* gives the isomorphism class of the group N.

3.2. **Cases** n = 5,6,7**.** In this case, every single conjugacy class either has an entry in the *power of* column or in the *subgroup* column. As a consequence, the set of all conjugates of groups in the *subgroup* column is a minimal cover of maximal nilpotent subgroups. Hence the size of this cover is given by summing the entries in the *number* column. This yields the values given in Table 3.1.

Note that, in this case, as for n = 3,4,5, there is a unique cover of $G = A_n$ by maximal nilpotent subgroups. What is more, this cover is a union of conjugacy classes of subgroups (and so is known in the literature as a *normal cover*).

3.3. **Case** n = 8**.** In this case there is one conjugacy class that has no entry in the *power of* column and in the *subgroup* column, namely the class, Cl_8, of elements with cycle type 4 − 4. However, in this case, the class, Cl_7, of elements with cycle type 4 − 2 contains elements which lie in a unique maximal nilpotent subgroup, namely a Sylow 2-subgroup of $G = A_8$. Since the union of all the Sylow 2-subgroups contains all elements of cycle type 4 − 4, we find that the listed subgroups already yield a minimal cover of maximal nilpotent subgroups.

Thus, as before, the size of this cover is given by summing the entries in the *number* column. This yields the value 3571 given in Table 3.1. As before, this cover is unique and normal.

3.4 **Case** n = 9**.** In this case, once again, there is one conjugacy class that has no entry in the *power of* column and in the *subgroup* column, namely the class, Cl_{10}, of elements with cycle type 4 − 4. This time, though, this class is not contained in the union of subgroups given in the *subgroup* column.

Since the centralizer of an element in Cl_{10} is a 2-group, a maximal nilpotent subgroup that contains an element of this class must be a Sylow 2-subgroup. The difficulty is that, for any given element g in this class, there are 7 Sylow 2-subgroups containing g.

So a minimal cover of A_9 by maximal nilpotent subgroups must contain the 28120 listed subgroups – the union of these groups contains all conjugacy classes in A_9 apart from Cl_{10} – as well as some of the 2835 Sylow 2-subgroups of A_9. We obtain the bounds given in Table 3.1.

Label	element	power of	subgroup	number	structure
Cl_1	(1)	All	-	-	
Cl_2	$(1,2)(3,4)$	-	$P_{2,4}$	5	$C_2 \times C_2$
Cl_3	$(1,2,3)$	-	$P_{3,3}$	10	C_3
$Cl_{4a,4b}$	$(1,2,3,4,5)$	-	$P_{5,5}$	6	C_5
Total				21	

Table 3.2. Constructing a minimal nilpotent cover of A_5

Label	element	power of	subgroup	number	structure
Cl_1	(1)	All	-	-	-
Cl_2	$(1,2)(3,4)$	Cl_5	-	-	
Cl_3	$(1,2,3)$	-	$(P_{3,6})$	-	-
Cl_4	$(1,2,3)(4,5,6)$	-	$P_{3,6}$	10	$C_3 \times C_3$
Cl_5	$(1,2,3,4)(5,6)$	-	$P_{2,6}$	45	D_8
$Cl_{6a,6b}$	$(1,2,3,4,5)$	-	$P_{5,5}$	36	C_5
Total				91	

Table 3.3. Constructing a minimal nilpotent cover of A^6

Label	element	power of	subgroup	number	structure
Cl_1	(1)	All	-	-	-
Cl_2	$(1,2)(3,4)$	Cl_6	-	-	-
Cl_3	$(1,2,3)$	Cl_5	-	-	-
Cl_4	$(1,2,3)(4,5,6)$	-	$P_{3,6}$	70	$C_3 \times C_3$
Cl_5	$(1,2,3)(4,5)(6,7)$	-	$P_{2,4} \times P_{3,3}$	35	$C_2 \times C_2 \times C_3$
Cl_6	$(1,2,3,4)(5,6)$	-	$P_{2,6}$	315	D_8
Cl_7	$(1,2,3,4,5)$	-	$P_{5,5}$	126	C_5
$Cl_{8a,8b}$	$(1,2,3,4,5,6,7)$	-	$P_{7,7}$	120	C_7
Total				666	

Table 3.4. Constructing a minimal nilpotent cover of A_7

Label	element	power of	subgroup	number	structure
Cl_1	(1)	All	-	-	-
Cl_2	$(1,2)(3,4)$	Cl_5	-	-	-
Cl_3	$(1,2)(3,4)(5,6)(7,8)$	Cl_8	-	-	-
Cl_4	$(1,2,3)$	Cl_{10}	-	-	-
Cl_5	$(1,2,3)(4,5)(6,7)$	-	$P_{2,4} \times P_{3,3}$	280	$C_3 \times C_2 \times C_2$
Cl_6	$(1,2,3)(4,5,6)$	Cl_{11}	-	-	-
Cl_7	$(1,2,3,4)(5,6)$	-	$P_{2,8}$	315	$(C_2 \wr C_2) \wr C_2$
Cl_8	$(1,2,3,4)(5,6,7,8)$	-	-	-	-
Cl_9	$(1,2,3,4,5)$	Cl_{10}	-	-	-
$Cl_{10a,10b}$	$(1,2,3,4,5)(6,7,8)$	-	$P_{5,5} \times P_{3,3}$	336	$C_5 \times C_3$
Cl_{11}	$(1,2,3,4,5,6)(7,8)$	-	C_6	1680	C_6
$Cl_{12a,12b}$	$(1,2,3,4,5,6,7)$	-	$P_{7,7}$	960	C_7
Total				3571	

Table 3.5. Constructing a minimal nilpotent cover of A_8

Label	element	power of	subgroup	number	structure
Cl_1	(1)	*All*	-	-	-
Cl_2	(1,2)(3,4)	Cl_8	-	-	-
Cl_3	(1,2)(3,4)(5,6)(7,8)	Cl_{10}	-	-	-
Cl_4	(1,2,3)	Cl_5	-	-	-
Cl_5	(1,2,3)(4,5)(6,7)	Cl_9	-	-	-
Cl_6	(1,2,3)(4,5,6)	Cl_{14}	-	-	-
Cl_7	(1,2,3)(4,5,6)(7,8,9)	Cl_{16}	-	-	
Cl_8	(1,2,3,4)(5,6)	Cl_9	-	-	-
Cl_9	(1,2,3,4)(5,6,7)(8,9)	-	P_2,6 × P_3,3	3780	$D8 \times C3$
Cl_{10}	(1,2,3,4)(5,6,7,8)	-	-	?	-
Cl_{11}	(1,2,3,4,5)	Cl_{12}	-	-	-
Cl_{12}	(1,2,3,4,5)(6,7)(8,9)	-	P_5,5 × P_2,4	756	$C5 \times C2 \times C2$
$Cl13a,13b$	(1,2,3,4,5)(6,7,8)	-	P_5,5 × P_3,3	3024	$C5 \times C3$
Cl_{14}	(1,2,3,4,5,6)(7,8)	-	C_6	15120	C_6
Cl_{15}	(1,2,3,4,5,6,7)	-	$P7$,7	4320	C_7
$Cl16a,16b$	(1,2,3,4,5,6,7,8,9)	-	$P3$,9	1120	$C3 \wr C3$
Total				28120+?	

Table 3.6. Constructing a minimal nilpotent cover of A_9

4. Final remark

It is interesting to compare the values given in Table 3.1 with Sequence A218964 of [OEI25] which enumerates the total number of maximal nilpotent subgroups of the alternating groups A_n. Up to $n = 7$, the two sequences coincide.

Note that the just-cited sequence from [OEI25] is a corrected version of a sequence first appearing in [NP13].

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