Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Some Generalized Inequalities via (s, m) -Convexity and their Applications

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| ABSTRACT
The main objective of this study is to formulate novel inequalities of the trapezoidal, mid-point, and Simpson types in connection with an extension of the Bullen-type inequality using the concept of (s,m)-convexity. Additionally, the investigation includes the exploration of particular cases and the application of these derived inequalities to some special means.

| KEYWORDS
Convex functions; (s,m)-convex functions; Bullen type inequality; Simpson type inequality.

| ARTICLE INFORMATION

ACCEPTED: 01 March 2025 **PUBLISHED:** 06 April 2025 **DOI:** 10.32996/jmss.2025.6.2.1

1. Introduction

Convexity is one of the simple and natural concepts. It has a significant history that goes back at least to Archimedes and has a wide range of applications. The basic structures of this theory which play a crucial role in its advancement and applications of it in various branches of applied and pure mathematics, particularly in the areas of optimization and theory of inequalities are convex sets and convex functions. Following is the definition of convex function:
If a function $\vartheta: [u, v] \subset \mathbb{R} \to \mathbb{R}$ has the property that

$$\vartheta(\lambda\tau + (1-\lambda)\mu) \leq \lambda\vartheta(\tau) + (1-\lambda)\vartheta(\mu), \qquad (1,1)$$

Holds for all $\tau, \mu \in [u, v]$ and $\lambda \in [0,1]$,then ϑ is said to be convex function on $[u, v]$, see, [17]
Some researchers have generalized the notion of convex functions as follows:
Toader [21], gave the following definitions for the m- convex functions: a function $\vartheta: [0, v] \to \mathbb{R}$ is called m- convex if the inequality

$$\vartheta(\lambda\tau + m(1-\lambda)\mu) \leq \lambda\vartheta(\tau) + m(1-\lambda)\vartheta(\mu), \qquad (1.2)$$

is true for every τ , $\mu \in [0, v]$ and λ in [0,1], where $m \in (0,1]$.

Breckner [2], characterized the class of s- convex functions in the second sense as follows: a function $\vartheta: [0, \infty] \to \mathbb{R}$ is called s- convex in the second sense if the inequality

$$\vartheta(\lambda\tau + (1-\lambda)\mu) \leq \lambda^s\vartheta(\tau) + (1-\lambda)^s\vartheta(\mu), \qquad (1.3)$$

is true for every τ , $\mu \in [0, v]$ and $\lambda \in [0,1]$, and for some fixed $m \in (0,1]$.

Eftekhari [7], by combining the definitions of m- convex and s- convex functions, has provided the following definition for (s, m)- convex functions: A function $\vartheta: [0, v] \to \mathbb{R}$ is called (s, m)- convex on $[0, v]$ if

$$\vartheta(\lambda\tau + m(1-\lambda)\mu) \leq \lambda^s\vartheta(\tau) + m(1-\lambda)^s\vartheta(\mu) \qquad (1.4)$$

is true for every τ , $\mu \in [0, v]$ and $\lambda \in [0,1]$, where $s, m \in (0,1]^2$.
More details on numerous classes of convex functions can be found in [2, 9, 15, 21, 25] and in the references therein.
The Hermite-Hadamard inequality

$$\vartheta\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \vartheta(\xi)\, d\xi \leq \frac{\vartheta(u)+\vartheta(v)}{2}, \tag{1.5}$$

is one that derives from the use of the convexity concept and is a foundation of the modern theory of inequalities, see [5,11,18,17].

There are numerous improvements to the above inequality in literature. The following improvement to the inequality (1.5) has been proposed by Bullen [3].

$$\vartheta\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \vartheta(\xi)\, d\xi \leq \frac{1}{2}\left[\vartheta\left(\frac{u+v}{2}\right) + \frac{\vartheta(u)+\vartheta(v)}{2}\right]. \tag{1.6}$$

The second inequality in (1.6) is referred to as a Bullen-type inequality in the literature.
Bullen [3], also established the following inequality for 4-convex and integrable functions on [-1,1].

$$\frac{1}{2}\int_{-1}^{1} \vartheta(\xi)\, d\xi \leq \frac{1}{6}[\vartheta(-1) + 4\vartheta(0) + \vartheta(1)].$$

In general, if a function ϑ has a fourth derivative on (u,v) such that $\|\vartheta^{(4)}\|_\infty = sup_{\xi\in(u,v)}|\vartheta^{(4)}(\xi)| < \infty$, then

$$\left|\int_u^v \vartheta(\xi)\, d\xi - \frac{1}{6}\left[\vartheta(u) + 4\vartheta\left(\frac{u+v}{2}\right) + \vartheta(v)\right]\right| \leq \frac{(v-u)^5\|\vartheta^{(4)}\|_\infty}{2880}, \tag{1.7}$$

See [6].

The inequality (1.7), known as Simpson- type inequality been improved and generalized by using a variety of approaches; for more information, see[1,8,12,13,19,26].

Tseng et al. [22], derived the following Hermite-Hadamard-type inequality, which improves the in-equality (1.5).

$$\vartheta\left(\frac{u+v}{4}\right) \leq \frac{1}{2}\left[\vartheta\left(\frac{3u+v}{4}\right) + \vartheta\left(\frac{u+3v}{4}\right)\right]$$

$$\leq \left(\frac{1}{v-u}\right)\int_u^v \vartheta(\xi)\, d\xi$$

$$\leq \frac{1}{2}\left[\vartheta\left(\frac{u+v}{2}\right) + \left(\frac{\vartheta(u)+\vartheta(v)}{2}\right)\right] \leq \frac{\vartheta(u)+\vartheta(v)}{2}. \tag{1.8}$$

The reader can see [10, 16, 23, 24], for further details concerning the second and third inequalities in (1.8).

In [14], the following generalization of (1.8) and several associated inequalities are presented.

$$\vartheta\left(\frac{u+\omega}{2}\right) + \vartheta\left(\frac{\omega+u}{2}\right) \leq \frac{1}{\omega-u}\int_u^\omega \vartheta(\xi)\, d\xi + \frac{1}{u-\omega}\int_\omega^v \vartheta(\xi)\, d\xi$$

$$\leq \vartheta(\omega) + \frac{\vartheta(u)+\vartheta(v)}{2}, \tag{1.9}$$

where $\omega \in [u,v]$.

The inequality (1.9) has been recently proved for s-convex functions in [20], and inequalities associated with (1.9) have been obtained through various types of convexity.

The aim of this paper is to establish some new inequalities associated with (1.9) and (1.8), via (s,m)- convexity. Applying the obtained results in this work we derive new inequalities for some special functions.

We expect that the methods and ideas in this paper will encourage the reader to conduct further study in this field.

2. INEQUALITIES OF THE TRAPEZOID TYPE

To determine the error bounds via (s,m)- convexity for the inequalities of the trapezoidal, midpoint, Bullen, and Simpson types associated with (1.9) and (1.8), we first review the definition of the beta function as it is stated in [4].

Let $x > 0$ and $y > 0$, then the function beta is defined by

$$\beta(x,y) = \int_0^1 \xi^{x-1}(1-\xi)^{y-1}\, d\xi$$

and $(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, where Γ is the classical gamma function.

Lemma2.1. Assume that $\vartheta: [u,v] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function on (u,v).
If $\vartheta' \in [u,v]$,where $u < v$, then for every $\omega \in [u,v]$, the following equality holds:

$$\frac{1}{\omega-u}\int_u^\omega \vartheta(\xi)\, d\xi + \frac{1}{v-\omega}\int_\omega^v \vartheta(\xi)\, d\xi - \left[\vartheta(\omega) + \frac{\vartheta(u)+\vartheta(v)}{2}\right]$$

$$\frac{\omega-u}{4}\int_0^1 (1-\tau)\left[\vartheta'\left(\tau\frac{u+\omega}{2} + (1-\tau)u\right) - \vartheta'\left(\tau\frac{u+\omega}{2} + (1-\tau)\omega\right)\right] d\tau$$

$$+\frac{\upsilon-\omega}{4}\int_0^1(1-\tau)\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau\ . \quad (2.1)$$

Proof. Using integration by parts we have

$$\frac{\omega-u}{4}\int_0^1(1-\tau)\left[\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$

$$+\frac{\upsilon-\omega}{4}\int_0^1(1-\tau)\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau$$

$$=\frac{\omega-u}{4}\left\{\begin{array}{l}\frac{-2\vartheta(u)}{\omega-u}+\frac{2}{(\omega-u)}\int_0^1\vartheta\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)d\tau\\ \frac{-2\vartheta(\omega)}{\omega-u}+\frac{2}{(\omega-u)}\int_0^1\vartheta\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)d\tau\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{4}\left\{\begin{array}{l}\frac{-2\vartheta(\omega)}{\upsilon-\omega}+\frac{2}{(\upsilon-\omega)}\int_0^1\vartheta\left(\tau\frac{\upsilon+\omega}{2}+(1-\tau)\upsilon\right)d\tau\\ \frac{-2\vartheta(\upsilon)}{\upsilon-u}+\frac{2}{(\upsilon-u)}\int_0^1\vartheta\left(\tau\frac{\upsilon+\omega}{2}+(1-\tau)\upsilon\right)d\tau\end{array}\right\}$$

Applying the variable-changing rule, we get

$$\frac{\omega-u}{4}\int_0^1(1-\tau)\left[\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$

$$+\frac{\upsilon-\omega}{4}\int_0^1(1-\tau)\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau$$

$$=-\frac{\vartheta(u)+\vartheta(\omega)}{2}+\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)\,d\xi-\frac{\vartheta(\omega)+\vartheta(\upsilon)}{2}+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi\ .$$

This completes the proof. □

Theorem 2.1. Suppose that $0\le u<\upsilon$ and $\vartheta:\left[u,\frac{\upsilon}{m}\right]\to\mathbb{R}$ is a differentiable function on $\left(u,\frac{\upsilon}{m}\right)$, where $m\in(0,1]$. $If\,\vartheta'\in L\left[u,\frac{\upsilon}{m}\right]$ and $|\vartheta'|$ is$(s,m)-$ convex on$\left[u,\frac{\upsilon}{m}\right]$, with $s\in(0,1]$, then for every $\omega\in[u,\upsilon]$, the following inequality

$$\left|\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]\right|$$

$$\le\frac{1}{4}\left\{\begin{array}{l}(\omega-u)\left(2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\beta(s+1,2)+m\beta(1,s+2)\left(\left|\vartheta'\left(\frac{u}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\\ +(\upsilon-\omega)\left(2\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|\beta(s+1,2)+m\beta(1,s+2)\left(\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\end{array}\right\} \quad (2.2)$$

is fulfilled.

Proof. From Lemma 2.1 and using the (s,m) convexity of $|\vartheta'|$ on $\left[u,\frac{\upsilon}{m}\right]$, we get

$$\left|\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]\right|$$

$$\le\frac{\omega-u}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|+\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)\right|+\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right|\right]d\tau$$

$$=\frac{\omega-u}{4}\int_0^1(1-\tau)\left[\left|\vartheta'\left(\tau\frac{u+\omega}{2}+m(1-\tau)\frac{u}{m}\right)\right|+\left|\vartheta'\left(\tau\frac{u+\omega}{2}+m(1-\tau)\frac{\omega}{m}\right)\right|\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1(1-\tau)\left[\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+m(1-\tau)\frac{\omega}{m}\right)\right|+\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+m(1-\tau)\frac{\upsilon}{m}\right)\right|\right]d\tau$$

$$\le\frac{\omega-u}{4}\left\{\left(2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\int_0^1(1-\tau)\tau^s\,d\tau+m\left|\vartheta'\left(\frac{u}{m}\right)\right|\int_0^1(1-\tau)\tau^{s+1}\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\int_0^1(1-\tau)\tau^{s+1}\,d\tau\right)\right\}$$
$$+\frac{(\upsilon-\omega)}{4}\left(2\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|\int_0^1(1-\tau)\tau^s\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\int_0^1(1-\tau)\tau^{s+1}\,d\tau+m\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|\int_0^1(1-\tau)\tau^{s+1}\,d\tau\right)$$

$$=\frac{1}{4}\left\{\begin{array}{l}(\omega-u)\left(2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\beta(s+1,2)+m\beta(1,s+2)\left(\left|\vartheta'\left(\frac{u}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\\+(v-\omega)\left(2\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|\beta(s+1,2)+m\beta(1,s+2)\left(\left|\vartheta'\left(\frac{v}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\end{array}\right\}.$$

This ends the proof. □

Theorem 2.2. Suppose that $0 \le u < v$ and $\vartheta\colon \left[u,\frac{v}{m}\right] \to \mathbb{R}$, is a differentiable function on $\left(u,\frac{v}{m}\right)$, where $m \in (0,1]. If \vartheta' \in L\left[u,\frac{v}{m}\right]$ $and\ for\ any\ \rho > 1, |\vartheta'|^{\rho}$ is $(s,m)-$ convex on $\left[u,\frac{v}{m}\right]$, with $s \in (0,1]$, then for every $\omega \in [u,v]$, we have

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\le\frac{\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\\+(v-\omega)\left[\left(\frac{\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\},\quad(2.3)$$

where $\frac{1}{\rho}+\frac{1}{\mu}=1$.

Proof. Using Lemma 2.1 and Holder's inequality, we get

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\le\frac{\omega-u}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\,u\right)\right|+\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|\right]d\tau$$
$$+\frac{v-\omega}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)\,\omega\right)\right|+\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)v\right)\right|\right]d\tau$$

$$\le\frac{(\omega-u)\left(\int_0^1(1-\tau)^{\mu}\,d\tau\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\int_0^1\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$
$$+\frac{(v-\omega)\left(\int_0^1(1-\tau)^{\mu}\,d\tau\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\int_0^1\left|\vartheta'\left(\tau\frac{v+\omega}{2}+(1-\tau)\omega\right)\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\\+\left(+\int_0^1\left|\vartheta'\left(\tau\frac{v+\omega}{2}+(1-\tau)v\right)\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Applying the $(s,m)-$ convexity of $[\vartheta']^{\rho}$ on $\left[u,\frac{v}{m}\right]$ yields

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\le\frac{\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}(\omega-u)}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\\+\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(v-\omega)\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\\\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\\+(v-\omega)\left[\left(\frac{\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\}.$$

This completes the proof. □

Theorem 2.3. Suppose that $0 \le u < v$ and $\vartheta: \left[u, \frac{v}{m}\right] \to \mathbb{R}$, is a differentiable function on $\left(u, \frac{v}{m}\right)$, where $m \in (0,1]$. $If\, \vartheta' \in L\left[u, \frac{v}{m}\right]$ and , $|\vartheta'|^{\rho}$ is $(s,m)-$ convex $for\ \ \rho \ge 1$ on $\left[u, \frac{v}{m}\right]$, with $s \in (0,1]$, then for every $\omega \in [u, v]$, the following inequality

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\le\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(\omega-u)}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\+\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(v-\omega)}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\+\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\end{array}\right\} \qquad (2.4)$$

is true

Proof. From Lemma 2.1 and using the power- mean inequality, we have

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\le\frac{\omega-u}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|+\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|\right]d\tau$$

$$+\frac{v-\omega}{4}\int_0^1|1-\tau|\left[\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)\omega\right)\right|+\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)v\right)\right|\right]d\tau$$

$$\le\frac{(\omega-u)\left(\int_0^1(1-\tau)\ d\tau\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\int_0^1(1-\tau)\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|^{\rho}\ d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1(1-\tau)\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|^{\rho}\ d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(\omega-u)\left(\int_0^1(1-\tau)\ d\tau\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\int_0^1(1-\tau)\left|\vartheta'\left(\tau\frac{v+\omega}{2}+(1-\tau)\,\omega\right)\right|^{\rho}\ d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1(1-\tau)\left|\vartheta'\left(\tau\frac{v+\omega}{2}+(1-\tau)\,v\right)\right|^{\rho}\ d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Applying the $(s,m)-$ convexity of $[\vartheta']^{\rho}$ on $\left[u, \frac{v}{m}\right]$ provides

$$\left|\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi-\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\leq \frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(\omega-u)}{4}\left\{\begin{array}{l}\left(\begin{array}{l}\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1(1-\tau)\tau^s\,d\tau\\+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^{s+1}\,d\tau\end{array}\right)^{\frac{1}{\rho}}\\+\left(\begin{array}{l}\int_0^1\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}(1-\tau)\tau^s\,d\tau\\+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^{s+1}\,d\tau\end{array}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(v-\omega)}{4}\left\{\begin{array}{l}\left(\begin{array}{l}\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1(1-\tau)\tau^s\,d\tau\\+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^{s+1}\,d\tau\end{array}\right)^{\frac{1}{\rho}}\\+\left(\begin{array}{l}\int_0^1\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}(1-\tau)\tau^s\,d\tau\\+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^{s+1}\,d\tau\end{array}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(\omega-u)}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\+\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(v-\omega)}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\+\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

This ends the proof. □

Corollary 2.1. If we select $\omega=\frac{u+v}{2}$ in Theorem 2.1. then we get the following Bullen-type inequality

$$\left|\frac{1}{v-u}\int_u^v\vartheta(\xi)\,d\xi+\frac{1}{2}\left[\vartheta\left(\frac{u+v}{2}\right)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\leq\frac{(v-u)}{16}\left\{\left(\begin{array}{c}4\left|\vartheta'\left(\frac{3u+v}{4}\right)\right|\beta(s+1,2)\\+m\beta(1,s+2)\left(\left|\vartheta'\left(\frac{u}{m}\right)\right|+2\left|\vartheta'\left(\frac{u+v}{2m}\right)\right|+\left|\vartheta'\left(\frac{v}{m}\right)\right|\right)\end{array}\right)\right\}.$$

Corollary 2.2. If we select $\omega=\frac{u+v}{2}$ in Theorem 2.2. then the inequality (2.3), leads the following Bullen-type inequality.

$$\left|\frac{1}{v-u}\int_u^v\vartheta(\xi)\,d\xi-\frac{1}{2}\left[\vartheta\left(\frac{u+v}{2}\right)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\leq\frac{(v-u)\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{8}\left\{\begin{array}{l}\left[\left(\frac{\left|\vartheta'\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{u+v}{2m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\\+\left[\left(\frac{\left|\vartheta'\left(\frac{3v+u}{4}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{u+v}{2m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\vartheta'\left(\frac{3v+u}{4}\right)\right|^{\rho}+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\}.$$

Corollary 2.3. The following Bullen-type inequality obtains from the inequality (2.3) if we choose $\omega=\frac{u+v}{2}$ in Theorem 2.3.

$$\left|\frac{1}{v-u}\int_u^v\vartheta(\xi)\,d\xi-\frac{1}{2}\left[\vartheta\left(\frac{u+v}{2}\right)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]\right|$$

$$\leq \frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(\upsilon-u)}{8}\left\{\begin{array}{c}\left(\left|\vartheta'\left(\frac{3u+\omega}{4}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\ +\left(\left|\vartheta'\left(\frac{3u+\upsilon}{4}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{u+\upsilon}{2m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\ +\left(\left|\vartheta'\left(\frac{3\upsilon+u}{4}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{u+\upsilon}{2m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\\ +\left(\left|\vartheta'\left(\frac{3\upsilon+u}{4}\right)\right|^{\rho}\beta(s+1,2)+m\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|^{\rho}\beta(1,s+2)\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Corollary 2.4. If $\rho=1$ in Theorem 2.4.then inequality (2.4), leads to (2.2).

3. INEQUALITIES OF THE MIDPOINT TYPE

Lemma 3.1. Assume that $\vartheta:[u,\upsilon]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable function on (u,υ) .If $\vartheta'\in L[u,\upsilon]$, where $u<\upsilon$, then for every $\omega\in[u,\upsilon]$, the following equality holds:

$$\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+\upsilon}{2}\right)-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_u^{\omega}\vartheta(\xi)\,d\xi\right]$$
$$=\frac{\omega-u}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau. \qquad (3.1)$$

Proof. Applying integration by parts, we obtain

$$=\frac{\omega-u}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau.$$
$$=\frac{\omega-u}{4}\left\{\begin{array}{c}\frac{2\vartheta\left(\frac{u+\omega}{2}\right)}{\omega-u}-\frac{2}{\omega-u}\int_0^1\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)d\tau\\ -\frac{2\vartheta\left(\frac{u+\omega}{2}\right)}{u-\omega}-\frac{2}{u-\omega}\int_0^1\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)d\tau\end{array}\right\}$$
$$+\frac{\upsilon-\omega}{4}\left\{\begin{array}{c}\frac{2\vartheta\left(\frac{\upsilon+\omega}{2}\right)}{\upsilon-\omega}-\frac{2}{\upsilon-\omega}\int_0^1\vartheta'\left(\tau\frac{\upsilon+\omega}{2}+(1-\tau)\omega\right)d\tau\\ -\frac{2\vartheta\left(\frac{\upsilon+\omega}{2}\right)}{\omega-\upsilon}+\frac{2}{\omega-\upsilon}\int_0^1\vartheta'\left(\tau\frac{\upsilon+\omega}{2}+(1-\tau)\upsilon\right)d\tau\end{array}\right\}.$$

Using the variable-changing rule, we get

$$=\frac{\omega-u}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1\tau\left[\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)-\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right]d\tau.$$
$$=\frac{\omega-u}{4}\left\{\frac{4\vartheta\left(\frac{u+\omega}{2}\right)}{\omega-u}-\frac{4}{(\omega-u)^2}\int_u^{\frac{u+\omega}{2}}\vartheta(\xi)\,d\xi-\frac{4}{(\omega-u)^2}\int_{\frac{u+\omega}{2}}^{\omega}\vartheta(\xi)\,d\xi\right\}$$
$$+\frac{\upsilon-\omega}{4}\left\{\frac{4\vartheta\left(\frac{\upsilon+\omega}{2}\right)}{\upsilon-\omega}-\frac{4}{(\upsilon-\omega)^2}\int_{\omega}^{\frac{\upsilon+\omega}{2}}\vartheta(\xi)\,d\xi-\frac{4}{(\upsilon-\omega)^2}\int_{\frac{\upsilon+\omega}{2}}^{\upsilon}\vartheta(\xi)\,d\xi\right\}.$$

This completes the proof. □

Theorem 3.1. Suppose that $0\leq u<\upsilon$ and $\vartheta:\left[u,\frac{\upsilon}{m}\right]\to\mathbb{R}$, is differentiable function on $\left(u,\frac{\upsilon}{m}\right)$, where $m\in(0,1]$. $If\,\vartheta'\in L\left[u,\frac{\upsilon}{m}\right]\,and$, $|\vartheta'|$ is $(s,m)-$ convex on $\left[u,\frac{\upsilon}{m}\right]$, with $s\in(0,1]$, then for every $\omega\in[u,\upsilon]$, the following inequality:

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+\upsilon}{2}\right)-\left[\frac{1}{\omega-u}-\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_{\omega}^{\upsilon}\vartheta(\xi)\,d\xi\right]\right|$$

$$\leq \frac{1}{4}\left\{\begin{array}{l}(\omega-u)\left(2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\beta(s+2,1)+m\beta(2,s+1)\left(\left|\vartheta'\left(\frac{u}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\\(\upsilon-\omega)\left(2\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|\beta(s+2,1)+m\beta(2,s+1)\left(\left|\vartheta'\left(\frac{\omega}{m}\right)\right|+\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|\right)\right)\end{array}\right\} \quad (3.2)$$

is true.

Proof. From Lemma 3.1 and using the $(s,m)-$ convexity of $|\vartheta'|$ on $\left[u,\frac{\upsilon}{m}\right]$, we get

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+\upsilon}{2}\right)-\left[\frac{1}{\omega-u}-\int_u^\omega\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi\right]\right|$$

$$\leq\frac{\omega-u}{4}\int_0^1\tau\left[\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|+\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|\right]d\tau$$
$$+\frac{\upsilon-\omega}{4}\int_0^1\tau\left[\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\omega\right)\right|+\left|\vartheta'\left(\tau\frac{\omega+\upsilon}{2}+(1-\tau)\upsilon\right)\right|\right]d\tau$$
$$\leq\frac{\omega-u}{4}\left\{\begin{array}{c}2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\int_0^1\tau^{s+1}\,d\tau+m\left|\vartheta'\left(\frac{u}{m}\right)\right|\int_0^1\tau(1-\tau)^s\,d\tau\\+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\int_0^1\tau(1-\tau)^s\,d\tau\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{4}\left\{\begin{array}{c}2\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|\int_0^1\tau^{s+1}\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\int_0^1\tau(1-\tau)^s\,d\tau\\+m\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|\int_0^1\tau(1-\tau)^s\,d\tau\end{array}\right\}$$
$$=\frac{1}{4}\left\{\begin{array}{l}(\omega-u)\left(2\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|\beta(s+2,1)+m\beta(2,s+1)\left(\left|\vartheta'\left(\frac{u}{m}\right)\right|+\left|\vartheta'\left(\frac{\omega}{m}\right)\right|\right)\right)\\(\upsilon-\omega)\left(2\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|\beta(s+2,1)+m\beta(2,s+1)\left(\left|\vartheta'\left(\frac{\omega}{m}\right)\right|+\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|\right)\right)\end{array}\right\}$$

This ends the proof. □

Theorem 3.2. Suppose that $0\leq u<\upsilon$ and $\vartheta\colon\left[u,\frac{\upsilon}{m}\right]\to\mathbb{R}$, is a differentiable function on $\left(u,\frac{\upsilon}{m}\right)$, where $m\in(0,1]$. $If\,\vartheta'\in L\left[u,\frac{\upsilon}{m}\right]$ $and\ for\ any\ \rho>1$, $|\vartheta'|^\rho$ is $(s,m)-$ convex on $\left[u,\frac{\upsilon}{m}\right]$, with $s\in(0,1]$, then for every $\omega\in[u,\upsilon]$, we have

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+\upsilon}{2}\right)-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi\right]\right|$$

$$\leq\frac{\left(\frac{1}{\mu+1}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\begin{array}{l}\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^\rho+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^\rho}{s+1}\right)^{\frac{1}{\rho}}\\+\left(\frac{\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^\rho+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^\rho}{s+1}\right)^{\frac{1}{\rho}}\end{array}\right]\\+(\upsilon-\omega)\left[\begin{array}{l}\left(\frac{\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^\rho}{s+1}\right)^{\frac{1}{\rho}}\\+\left(\frac{\left|\vartheta'\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho+m\left|\vartheta'\left(\frac{\upsilon}{m}\right)\right|^\rho}{s+1}\right)^{\frac{1}{\rho}}\end{array}\right]\end{array}\right\}, \quad (3.3)$$

where $\frac{1}{\mu}+\frac{1}{\rho}=1$.

Proof. Using Lemma 3.1 and Holder's inequality, we get

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+\upsilon}{2}\right)-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)\,d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)\,d\xi\right]\right|$$

$$\leq \frac{(\omega-u)\left(\int_0^1 \tau^{\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\int_0^1\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|^{\rho} d\tau\right)^{\frac{1}{\rho}} \\ +\left(\int_0^1\left|\vartheta'\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(v-\omega)\left(\int_0^1 \tau^{\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\int_0^1\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)\omega\right)\right|^{\rho} d\tau\right)^{\frac{1}{\rho}} \\ +\left(\int_0^1\left|\vartheta'\left(\tau\frac{\omega+v}{2}+(1-\tau)v\right)\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Applying the $(s,m)-$convexity of $|\vartheta'|^{\rho}$ on $\left[u,\frac{v}{m}\right]$ gives

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+v}{2}\right)-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)\,d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)\,d\xi\right]\right|$$

$$\leq \frac{(\omega-u)\left(\frac{1}{\mu+1}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{u}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}} \\ +\left(\left|\vartheta'\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(v-\omega)\left(\frac{1}{\mu+1}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}} \\ +\left(\left|\vartheta'\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\vartheta'\left(\frac{v}{m}\right)\right|^{\rho}\int_0^1(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\left(\frac{1}{\mu+1}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{ll}(\omega-u) & \left[\left(\frac{\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right] \\ +(v-\omega) & \left[\left(\frac{\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{p}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\}$$

This completes the proof. ■

Theorem 3.3. Suppose that $0 \leq u < v$ and $\vartheta:\left[u,\frac{v}{m}\right]\longrightarrow\mathbb{R}$, is a differentiable function on $\left(u,\frac{v}{m}\right)$, where $m\in(0,1]$. If $\acute{\vartheta}\in\left[u,\frac{v}{m}\right]$ and, $\left|\acute{\vartheta}\right|^{\rho}$ is $(s,m)-$convex for$\rho\geq 1$ on $\left[u,\frac{v}{m}\right]$, with $\in(0,1]$, then for every $\omega\in[u,v]$, the following inequality

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+v}{2}\right)-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)d\xi\ \right]\right|$$

$$\leq\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{ll}(\omega-u) & \left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\end{array}\right] \\ +(v-\omega) & \left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\end{array}\right]\end{array}\right\} \qquad (3.4)$$

is fulfilled.

Proof. From Lemma (3.1) and using the power-mean inequality, we have

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+v}{2}\right)-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)d\xi+\ \right]\right|$$

$$\leq \frac{(\omega-u)\left(\int_0^1 \tau\, d\tau\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\int_0^1 \tau\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)u\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1 \tau\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)\omega\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(v-\omega)\left(\int_0^1 \tau\, d\tau\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\int_0^1 \tau\left|\acute{\vartheta}\left(\tau\frac{v+\omega}{2}\right)+(1-\tau)\omega\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1 \tau\left|\acute{\vartheta}\left(\tau\frac{v+\omega}{2}\right)+(1-\tau)v\right|^{\rho} d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Applying the (s,m)-convexity of $|\acute{\vartheta}|^{\rho}$ on $\left[u,\frac{v}{m}\right]$ provides

$$\left|\vartheta\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+v}{2}\right)-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)d\xi\right]\right|$$

$$\leq \frac{(\omega-u)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1 \tau^{s+1}\, d\tau+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\int_0^1 \tau(1-\tau)^s\, d\tau\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1 \tau^{s+1}\, d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1 \tau(1-\tau)^s\, d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(v-\omega)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1 \tau^{s+1}\, d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1 \tau(1-\tau)^s\, d\tau\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\int_0^1 \tau^{s+1}\, d\tau+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\int_0^1 \tau(1-\tau)^s\, d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{ll}(\omega-u) & \left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\end{array}\right]\\+(v-\omega) & \left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\end{array}\right]\end{array}\right\}.$$

This ends the proof. ■

Corollary 3.1. If we select $\omega=\frac{u+v}{2}$ in Theorem (3.1) , then we get the following inequality related to (1.8).

$$\left|\frac{\vartheta\left(\frac{3u+v}{4}\right)+\vartheta\left(\frac{u+3v}{4}\right)}{2}-\frac{1}{v-u}\int_u^v \vartheta(\xi)d\xi\right|$$

$$\leq \frac{(v-u)}{8}\left\{\begin{array}{c}2\beta(s+2,1)\left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|+\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|\right)\\+m\beta(2,s+1)\left(2\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|+\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|\right)\end{array}\right\}.$$

Corollary 3.2. If we select $\omega=\frac{u+v}{2}$ in Theorem (3.2), then the inequality (3.3), leads to the following Inequality associated with (1.8)

$$\left|\frac{\vartheta\left(\frac{3u+v}{4}\right)+\vartheta\left(\frac{u+3v}{4}\right)}{2}-\frac{1}{v-u}\int_u^v \vartheta(\xi)d\xi\right|$$

$$\leq \frac{(v-u)\left(\frac{1}{\mu+1}\right)^{\frac{1}{\mu}}}{8}\left\{\begin{array}{l}\left(\frac{\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\\+\left(\frac{\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}}{s+1}\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Corollary 3.3. The following inequality associated with (1.8), obtains from the inequality (3.3), if we Choose $\omega=\frac{u+v}{2}$ in Theorem (3.3).

$$\left|\frac{\vartheta\left(\frac{3u+v}{4}\right)+\vartheta\left(\frac{u+3v}{4}\right)}{2}-\frac{1}{v-u}\int_{u}^{v}\vartheta(\xi)d\xi\right|$$

$$\leq \frac{(v-u)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{8}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}\beta(s+2,1)+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\beta(2,s+1)\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Corollary 3.4. If $\rho=1$ in Theorem (3.3), then the inequality (3,4), reduces to (3.2).

4. INEQUALITIES OF THE SIMPSON TYPE

In this section, in order to obtain Simpson-type inequalities related to (1.9). We need to prove the following Lemma.

Lemma 4.1. Let $\vartheta:[u,v]\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on (u,v). If $\acute{\vartheta}\epsilon L[u,v]$, where $u<v$, then for every $\omega\epsilon[u,v]$, the following equality holds:

$$\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+v}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]-\left[\frac{1}{\omega-u}\int_{u}^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-w}\int_{\omega}^{v}\vartheta(\xi)d\xi\right]$$

$$=\frac{\omega-u}{12}\left\{\int_{0}^{1}(3\tau-1)\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)d\tau+\int_{0}^{1}(1-3\tau)\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)d\tau\right\}$$
$$+\frac{v-\omega}{12}\left\{\int_{0}^{1}(3\tau-1)\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)\omega\right)d\tau+\int_{0}^{1}(1-3\tau)\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)v\right)d\tau\right\}. \qquad (4.1)$$

Proof. From Lemma (2.1) and (3.1) respectively, we have

$$\frac{1}{3}\left[\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]-\frac{1}{3}\left[\frac{1}{\omega-u}\int_{u}^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)d\xi\right]$$
$$=\frac{\omega-u}{12}\int_{0}^{1}(1-\tau)\left[\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)-\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right]d\tau$$
$$+\frac{v-\omega}{12}\int_{0}^{1}(1-\tau)\left[\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)v\right)-\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)\omega\right)\right]d\tau \cdot \qquad (4.2)$$

and

$$\frac{2}{3}\left[\left(\frac{u+\omega}{2}\right)+\vartheta\left(\frac{\omega+v}{2}\right)\right]-\frac{2}{3}\left[\frac{1}{\omega-u}\int_{u}^{\omega}\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_{\omega}^{v}\vartheta(\xi)d\xi\right]$$
$$=\frac{\omega-u}{12}\int_{0}^{1}2\tau\left[\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)-\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right]d\tau$$

$$+\frac{v-u}{12}\int_0^1 2\tau\left[\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)\omega\right)-\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)v\right)\right]d\tau\cdot \tag{4.3}$$

Adding (4.2) and (4.3) and ordering the resulting equality yields the desired identity. ∎

Theorem 4.1. Suppose that $0\le u<v$ and $\vartheta:\left[u,\frac{v}{m}\right]\longrightarrow\mathbb{R}$ is a differentiable function on $\left(u,\frac{v}{m}\right)$, where $m\in(0,1]$. If $\acute{\vartheta}\in L\left[u,\frac{v}{m}\right]$ and $|\acute{\vartheta}|$ is (s,m) -convex on $\left[u,\frac{v}{m}\right]$, with $\in(0,1]$, then for every $\omega\in[u,v]$, the following inequality

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+v}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_\omega^v\vartheta(\xi)d\xi\right]\right|$$

$$\le\frac{1}{12(s^2+3s+2)}\left\{\begin{array}{l}(\omega-u)\left[\begin{array}{c}(4\times3^{-s-1}+4s+2)\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|\\+m(s+8\times3^{-s-1}-1)\left(\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|\right)\end{array}\right]\\+(v-\omega)\left[\begin{array}{c}(4\times3^{-s-1}+4s+2)\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|\\+m(s+8\times3^{-s-1}-1)\left(\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|\right)\end{array}\right]\end{array}\right\} \tag{4.4}$$

is true.

Proof. From Lemma (4.1) and using the (s,m) –convexity of $|\acute{\vartheta}|$ on $\left[u,\frac{v}{m}\right]$, we get

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+v}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(v)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)d\xi+\frac{1}{v-\omega}\int_\omega^v\vartheta(\xi)d\xi\right]\right|$$

$$\le\frac{\omega-u}{12}\left\{\int_0^1|3\tau-1|\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)u\right)\right|d\tau+\int_0^1|3\tau-1|\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}+(1-\tau)\omega\right)\right|d\tau\right\}$$

$$+\frac{v-\omega}{12}\left\{\int_0^1|3\tau-1|\left|\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)\omega\right)\right|d\tau+\int_0^1|3\tau-1|\left|\acute{\vartheta}\left(\tau\frac{v+\omega}{2}+(1-\tau)v\right)\right|d\tau\right\}$$

$$\le\frac{\omega-u}{12}\left\{\begin{array}{l}2\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)t^sd\tau+\int_{\frac{1}{3}}^1(3t-1)t^sd\tau\right)\\+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)(1-t)^sd\tau+\int_{\frac{1}{3}}^1(3t-1)(1-t)^sd\tau\right)\\+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)(1-t)^sd\tau+\int_{\frac{1}{3}}^1(3t-1)(1-t)^sd\tau\right)\end{array}\right\}$$

$$+\frac{v-\omega}{12}\left\{\begin{array}{l}2\left|\acute{\vartheta}\left(\frac{v+\omega}{2}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)t^sd\tau+\int_{\frac{1}{3}}^1(3t-1)t^sd\tau\right)\\+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)(1-t)^sd\tau+\int_{\frac{1}{3}}^1(3t-1)(1-t)^sd\tau\right)\\+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|\left(\int_0^{\frac{1}{3}}(1-3t)(1-t)^sd\tau+\int_{\frac{1}{3}}^1(3t-1)(1-t)^sd\tau\right)\end{array}\right\}$$

$$=\frac{\omega-u}{12}\left\{\begin{array}{l}2\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}\\+m\frac{s+8\times3^{-s-1}-1}{(s+1)(s+2)}\left(\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|\right)\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left\{\begin{array}{c} 2\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|\frac{2\times 3^{-s-1}+2s+1}{(s+1)(s+2)} \\ +m\frac{s+8\times 3^{-s-1}-1}{(s+1)(s+2)}\left(\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|\right)\end{array}\right\}.$$

This ends the proof. ∎

Theorem 4.2. Suppose that $0\le u<\upsilon$ and $\vartheta:\left[u,\frac{\upsilon}{m}\right]\longrightarrow\mathbb{R}$ is a differentiable function on $\left(u,\frac{\upsilon}{m}\right)$, where $m\in(0,1]$. If $\acute{\vartheta}\in L\left[u,\frac{\upsilon}{m}\right]$ and for any $\rho>1$, $\left|\acute{\vartheta}\right|^{\rho}$ is (s,m)-convex on $\left[u,\frac{\upsilon}{m}\right]$, with $\in(0,1]$, then for every $\omega\in[u,\upsilon]$, we have

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_{\omega}^{\upsilon}\vartheta(\xi)d\xi\right]\right|$$

$$\le\frac{\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)(\beta(s+1,1))^{\frac{1}{\rho}}}{12}\left\{\begin{array}{l}(\omega-u)\left[\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}\right] \\ +(\upsilon-\omega)\left[\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}+\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}\right]\end{array}\right\}\quad(4.5)$$

Where $\frac{1}{\mu}+\frac{1}{\rho}=1$.

Proof. Using Lemma (4.1) and Holders inequality, we obtain

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_{\omega}^{\upsilon}\vartheta(\xi)d\xi\right]\right|$$

$$\le\frac{\omega-u}{12}\left(\int_0^1|3\tau-1|^{\mu}\,d\tau\right)^{\frac{1}{\mu}}\left\{\begin{array}{l}\left(\int_0^1\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)u\right|^{\rho}d\tau\right)^{\frac{1}{\rho}} \\ +\left(\int_0^1\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)\omega\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\int_0^1|3\tau-1|^{\mu}\,d\tau\right)^{\frac{1}{\mu}}\left\{\begin{array}{l}\left(\int_0^1\left|\acute{\vartheta}\left(\tau\frac{\upsilon+\omega}{2}\right)+(1-\tau)\omega\right|^{\rho}d\tau\right)^{\frac{1}{\rho}} \\ +\left(\int_0^1\left|\acute{\vartheta}\left(\tau\frac{\upsilon+\omega}{2}\right)+(1-\tau)\upsilon\right|^{\rho}d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}.$$

Applying the (s,m) –convexity of $\left|\acute{\vartheta}\right|^{\rho}$ on $\left[u,\frac{\upsilon}{m}\right]$ provides

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^{\omega}\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_{\omega}^{\upsilon}\vartheta(\xi)d\xi\right]\right|$$

$$\le\frac{\omega-u}{12}\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)^{\frac{1}{\mu}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\int_0^1(1-\tau^s)\,d\tau\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau^s)\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)^{\frac{1}{\mu}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\int_0^1(1-\tau^s)\,d\tau\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}\int_0^1\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^{\rho}\int_0^1(1-\tau^s)\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\omega-u}{12}\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)^{\frac{1}{\mu}}(\beta(s+1,1))^{\frac{1}{\rho}}\left\{\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)^{\frac{1}{\mu}}(\beta(s+1,1))^{\frac{1}{\rho}}\left\{\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}+\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}\right\}.$$

This completes the proof. ∎

Theorem 4.3. Suppose that $0\le u<\upsilon$ and $\vartheta:\left[u,\frac{\upsilon}{m}\right]\longrightarrow\mathbb{R}$ is a differentiable function on $\left(u,\frac{\upsilon}{m}\right)$, where $m\in(0,1]$. If $\acute{\vartheta}\in L\left[u,\frac{\upsilon}{m}\right]$ and, $\left|\acute{\vartheta}\right|^{\rho}$ is (s,m)-convex for $\rho\ge 1$ on $\left[u,\frac{\upsilon}{m}\right]$, with $\in(0,1]$, then for every $\omega\in[u,\upsilon]$, the following inequality

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)d\xi\right]\right|$$

$$\leq\frac{\left(\frac{5}{6}\right)^{1-\frac{1}{\rho}}}{12}\left\{\begin{array}{l}(\omega-u)\left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^\rho\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}2^s-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^\rho\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}2^s-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\end{array}\right]\\(\upsilon-\omega)\left[\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}2^s-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}2^s-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\end{array}\right]\end{array}\right\}\quad(4.6)$$

Is fulfilled.

Proof. From Lemma (4.1) and using the power-mean inequality, we gain

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)d\xi\right]\right|$$

$$\leq\frac{\omega-u}{12}\left(\int_0^1|1-3\tau|\,d\tau\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\int_0^1|1-3\tau|\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)u\right|^\rho d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1|1-3\tau|\left|\acute{\vartheta}\left(\tau\frac{u+\omega}{2}\right)+(1-\tau)\omega\right|^\rho d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\int_0^1|1-3\tau|\,d\tau\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\int_0^1|1-3\tau|\left|\acute{\vartheta}\left(\tau\frac{\upsilon+\omega}{2}\right)+(1-\tau)\omega\right|^\rho d\tau\right)^{\frac{1}{\rho}}\\+\left(\int_0^1|1-3\tau|\left|\acute{\vartheta}\left(\tau\frac{\upsilon+\omega}{2}\right)+(1-\tau)\upsilon\right|^\rho d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

Applying the (s,m) –convexity of $|\acute{\vartheta}|^\rho$ on $\left[u,\frac{\upsilon}{m}\right]$ yields

$$\left|\frac{1}{3}\left[2\vartheta\left(\frac{u+\omega}{2}\right)+2\vartheta\left(\frac{\omega+\upsilon}{2}\right)+\vartheta(\omega)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right]-\left[\frac{1}{\omega-u}\int_u^\omega\vartheta(\xi)d\xi+\frac{1}{\upsilon-\omega}\int_\omega^\upsilon\vartheta(\xi)d\xi\right]\right|$$

$$\leq\frac{\omega-u}{12}\left(\int_0^1|1-3\tau|\,d\tau\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^\rho\int_0^1|(1-3\tau)|\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^\rho\int_0^1|(1-3\tau)|(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|^\rho\int_0^1|(1-3\tau)|\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\int_0^1|(1-3\tau)|(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\int_0^1|1-3\tau|\,d\tau\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho\int_0^1|(1-3\tau)|\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\int_0^1|(1-3\tau)|(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|^\rho\int_0^1|(1-3\tau)|\tau^s\,d\tau+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^\rho\int_0^1|(1-3\tau)|(1-\tau)^s\,d\tau\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$=\frac{\omega-u}{12}\left(\frac{5}{6}\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{u+\omega}{2}\right)\right|\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{\upsilon-\omega}{12}\left(\frac{5}{6}\right)^{1-\frac{1}{\rho}}\left\{\begin{array}{l}\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\omega}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\\+\left(\left|\acute{\vartheta}\left(\frac{\upsilon+\omega}{2}\right)\right|\frac{2\times3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{\upsilon}{m}\right)\right|^\rho\frac{s+8\times3^{-s-1}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

This end the proof. ■

Corollary 4.1. The following Simpson type inequality related to (1.8), obtains from the inequality (4.3), if we choose $\omega=\frac{u+\upsilon}{2}$ in Theorem (4.1).

$$\left|\frac{1}{3}\left[2\left(\frac{\vartheta\left(\frac{3u+\upsilon}{4}\right)+\vartheta\left(\frac{u+3\upsilon}{4}\right)}{2}\right)+\frac{1}{2}\left(\vartheta\left(\frac{u+\upsilon}{2}\right)+\frac{\vartheta(u)+\vartheta(\upsilon)}{2}\right)\right]-\left[\frac{1}{\upsilon-u}\int_u^\upsilon\vartheta(\xi)d\xi\right]\right|$$

$$\leq \frac{(v-u)}{48(s^2+3s+2)}\begin{Bmatrix} (4\times 3^{-s-1}+4s+2)\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right| \\ +m(s+8\times 3^{-s-1}-1)\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|+\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right| \\ +(4\times 3^{-s-1}+4s+2)\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right| \\ +m(s+8\times 3^{-s-1}-1)\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|+\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right| \end{Bmatrix}.$$

Corollary 4.2. If we select $\omega=\frac{u+v}{2}$ in Theorem (4.2) , we get the following Simpson type inequality associated with (1.8).

$$\left|\frac{1}{3}\left[2\left(\frac{\vartheta\left(\frac{3u+v}{4}\right)+\vartheta\left(\frac{u+3v}{4}\right)}{2}\right)+\frac{1}{2}\left(\vartheta\left(\frac{u+v}{2}\right)+\frac{\vartheta(u)+\vartheta(v)}{2}\right)\right]-\left[\frac{1}{v-u}\int_u^v\vartheta(\xi)d\xi\right]\right|$$

$$\leq \frac{\left(\frac{1+2^{1+\mu}}{3(1+\mu)}\right)^{\frac{1}{\mu}}(v-u)\left(\beta(s+1,1)\right)^{\frac{1}{\rho}}}{48}\begin{Bmatrix} \left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} \end{Bmatrix}.$$

Corollary 4.3. If we take $\omega=\frac{u+v}{2}$ in Theorem (4.3) , then the inequality (4.6), reduces to the following following Simpson type inequality associated with (1.8).

$$\left|\frac{1}{3}\left[2\left(\frac{\vartheta\left(\frac{3u+v}{4}\right)+\vartheta\left(\frac{u+3v}{4}\right)}{2}\right)+\frac{1}{2}\left(\vartheta\left(\frac{u+v}{2}\right)+\frac{\vartheta(u)+\vartheta(v)}{2}\right)\right]-\left[\frac{1}{v-u}\int_u^v\vartheta(\xi)d\xi\right]\right|$$

$$\leq \frac{(v-u)\left(\frac{5}{6}\right)^{1-\frac{1}{\rho}}}{48}\begin{Bmatrix} \left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}\frac{2\times 3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{u}{m}\right)\right|^{\rho}\frac{s+8\times 3^{-s-1}2^{s}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{3u+v}{4}\right)\right|^{\rho}\frac{2\times 3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\frac{s+8\times 3^{-s-1}2^{s}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}\frac{2\times 3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{u+v}{2m}\right)\right|^{\rho}\frac{s+8\times 3^{-s-1}2^{s}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}} \\ +\left(\left|\acute{\vartheta}\left(\frac{u+3v}{4}\right)\right|^{\rho}\frac{2\times 3^{-s-1}+2s+1}{(s+1)(s+2)}+m\left|\acute{\vartheta}\left(\frac{v}{m}\right)\right|^{\rho}\frac{s+8\times 3^{-s-1}2^{s}-1}{(s+1)(s+2)}\right)^{\frac{1}{\rho}} \end{Bmatrix}.$$

Corollary 4.4. If $\rho=1$ in Theorem (4.3) , then the inequality (4.6), leads to (4.4).

5. APPLICATIONS TO MEANS

Let u and v bet two positive real numbers. The arithmetic, logarithmic, and generalized logarithmic means are, respectively, given below for every pair of positive real numbers, v and v.

$$A(u,v)=\frac{u+v}{2},$$

$$\bar{L}(u,v)=\frac{v-u}{\ln v-\ln u}\quad,\quad u\neq v\ ,$$

$$\bar{L}_r(u,v)=\left(\frac{v^{r+1}-u^{r+1}}{(r+1)(v+u)}\right)^{\frac{1}{r}}\quad,\quad r\neq -1,0\cdot$$

See, (26).

Hudzik and Maligranda provided the following example in (9).

Let $p,q,r\in\mathbb{R}$ and $s\in(0,1)$, then the function

$$\vartheta(\xi):=\begin{cases} p & if \quad \xi=0,\\ q\xi^s+r & if \quad \xi>0,\end{cases}$$

is $(s,1)$-convex if $q\geq 0$, and $0\leq r\leq p$.

Applying the Theorems(2,2), (3,2), (2,3), and (3,3), to $\vartheta(\xi)=\frac{\xi^{s+1}}{s+1}$, where $s\in(0,1)$, respectively, yields the following inequalities for special means.

Proposition 5.1. Suppose that $0\leq u<v$ and $s\in(0,1)$. If $\rho>1$, with $\rho s\in(0,1)$, then for every $\omega\in[u,v]$, we have

$$|[L_{s+1}^{s+1}(u,\omega)+L_{s+1}^{s+1}(\omega,v)-(A(u^{s+1},\omega^{s+1})+A(\omega^{s+1},v^{s+1}))]|$$

$$\leq \frac{(s+1)\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\left(\frac{A^{\rho s}(u,\omega)+u^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{A^{\rho s}(u,\omega)+\omega^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}\right]\\+(v-\omega)\left[\left(\frac{A^{\rho s}(\omega,v)+\omega^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{A^{\rho s}(\omega,v)+v^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\}$$

and

$$\left|\left(A^{s+1}(u,\omega)+A^{s+1}(\omega,v)\right)-\left(L_{s+1}^{s+1}(u,\omega)+L_{s+1}^{s+1}(\omega,v)\right)\right|$$

$$\leq \frac{(s+1)\left(\frac{1}{1+\mu}\right)^{\frac{1}{\mu}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\left(\frac{A^{\rho s}(u,\omega)+u^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{A^{\rho s}(u,\omega)+\omega^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}\right]\\+(v-\omega)\left[\left(\frac{A^{\rho s}(\omega,v)+\omega^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}+\left(\frac{A^{\rho s}(\omega,v)+v^{\rho s}}{s+1}\right)^{\frac{1}{\rho}}\right]\end{array}\right\},$$

Where $\frac{1}{\mu}+\frac{1}{\rho}=1$.

Proposition 5.2. Suppose that $0 \leq u < v$ and $s \in (0,1)$. If $\rho \geq 1$, with $\rho s \in (0,1)$, then for every $\omega \in [u,v]$, the Following inequality holds:

$$\left|\left[L_{s+1}^{s+1}(u,\omega)+L_{s+1}^{s+1}(\omega,v)-\left(A(u^{s+1},\omega^{s+1})+A(\omega^{s+1},v^{s+1})\right)\right]\right|$$

$$\leq \frac{(s+1)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(\omega-u)}{4}\left\{\begin{array}{l}\left(\beta(s+1,2)A^{\rho s}(u,\omega)+\beta(1,s+2)u^{\rho s}\right)^{\frac{1}{\rho}}\\\left(\beta(s+1,2)A^{\rho s}(u,\omega)+\beta(1,s+2)\omega^{\rho s}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

$$+\frac{(s+1)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}(v-\omega)}{4}\left\{\begin{array}{l}\left(\beta(s+1,2)A^{\rho s}(\omega,v)+\beta(1,s+2)\omega^{\rho s}\right)^{\frac{1}{\rho}}\\\left(\beta(s+1,2)A^{\rho s}(\omega,v)+\beta(1,s+2)v^{\rho s}\right)^{\frac{1}{\rho}}\end{array}\right\}$$

and

$$\left|A^{s+1}(u,\omega)+A^{s+1}(\omega,v)-\left(L_{s+1}^{s+1}(u,\omega)+L_{s+1}^{s+1}(\omega,v)\right)\right|$$

$$\leq \frac{(s+1)\left(\frac{1}{2}\right)^{1-\frac{1}{\rho}}}{4}\left\{\begin{array}{l}(\omega-u)\left[\begin{array}{l}\left(\left(\beta(s+2,1)A^{\rho s}(u,\omega)+\beta(2,s+1)\right)^{\frac{1}{\rho}}u^{\rho s}\right)^{\frac{1}{\rho}}\\+\left(\left(\beta(s+2,1)A^{\rho s}(u,\omega)+\beta(2,s+1)\right)^{\frac{1}{\rho}}\omega^{\rho s}\right)^{\frac{1}{\rho}}\end{array}\right]\\+(v-\omega)\left[\begin{array}{l}\left(\left(\beta(s+2,1)A^{\rho s}(\omega,v)+\beta(2,s+1)\right)^{\frac{1}{\rho}}\omega^{\rho s}\right)^{\frac{1}{\rho}}\\+\left(\left(\beta(s+2,1)A^{\rho s}(\omega,v)+\beta(2,s+1)\right)^{\frac{1}{\rho}}v^{\rho s}\right)^{\frac{1}{\rho}}\end{array}\right]\end{array}\right\}.$$

5. Conclusion

In this study, we have established new trapezoidal, midpoint, Bullen, and Simpson types of inequalities related to (1.9) , using the existing identities in the literature in the literature and (s,m)-convexity. We have derived some inequalities associated with (1.8), by putting $\omega=\frac{u+v}{2}$ in the resulting inequalities. Applying the derived inequalities, we have investigated some inequalities for special means. We hope that the ideas and approaches presented in this study will motivate interested readers who are working in this area.

Funding: This research received no external funding

Conflicts of Interest: The authors declare no conflict of interest.

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