Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS

AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Decompositions of Hypercube Graphs into Diametral Paths and Cycle Decompositions

Anwar N. Jasim[1]✉ and Hayder R. Hashim[2]
[12]*Faculty of Computer Science and Mathematics, University of Kufa, P.O. Box 21, 54001 Al Najaf, Iraq*
Corresponding Author: Anwar N. Jasim , **E-mail**: anwern.jaseem@uokufa.edu.iq

| ABSTRACT

One of the most well-known and well-studied issues in graph theory is graph decomposition. Graph decomposition has been studied in great detail by extensive research. There are two main types of decomposition problems such as edge decompositions and vertex decompositions. It entails meeting certain requirements in order to divide an input graph into smaller segments (subgraphs). In this paper, an investigation into decomposition in hypercube graphs using diametral routes is studied. Additionally, we study the finding of the diametral path decomposition number, index, and hypercube graph's cycle decomposition.

| KEYWORDS

Diameter, Hypercube graph, Diametral path, Cycle decomposition.

| ARTICLE INFORMATION

ACCEPTED: 01 January 2025 **PUBLISHED:** 20 January 2025 **DOI:** 10.32996/jmss.2025.6.1.2

1. Introduction

Graphs are used in many applications that we encounter every day. In several scientific disciplines, including sociology, chemistry, physics, game theory, and many more, graphs are regarded as extremely effective modeling tools [7]. Graph theory is often used to describe challenges in the fields of distributed systems and computer networking. A strong framework for comprehending and resolving issues present in these systems is provided by graph theoretic techniques [4]. In many branches of mathematics, breaking down a complex item into smaller parts with a specific structure is a highly common activity. In reality, a lot of graph theory problems may be formulated as decomposition problems [11]. Numerous authors use for graph decomposition results include fault tolerance in network architecture, structure analysis, graph decomposition into particular patterns, graph compression and summarization, graph similarity and subgraph matching, property analysis in large graphs, and more [5]. Network theory, coding theory, geometry, and other significant fields are closely related to the study of graph decomposition [9].

This encouraged researchers to delve deeply into this field. Following Galecki's discovery of a Hamiltonian decomposition of a full network possessing an odd quantity of vertices, other unresolved issues have surfaced [5]. Since at least 1960, researchers have been actively studying graph decomposition and graph partitioning difficulties. Sethuraman and Murugan have put out a new conjecture to break down a whole graph into copies of two arbitrary trees [6]. Yamamoto et al. investigated the challenge of breaking down a complete graph into a union of line disjoint claws or stars [15]. Abueida and Daven looked at variations on the covering, factorization problems, and subgraph packing to determine an efficient method to decompose a big complete graph into generally well-behaved subgraphs [1]. Barát and Thomassen examined graph decompositions into trees [8]. Regarding the hypercube decomposition problems, they date back to Ringel, who demonstrated that when $2n$ and $n > 2$, the hypercube graph Q_n has a decomposition into Hamiltonian cycles. A Hamiltonian decomposition is the breakdown of a graph into Hamiltonian cycles. It was demonstrated by Stout, Horak, and other shown that Q_n can be decomposed into specific trees [13]. Fink, and others shown that Q_n can be decomposed into cycles of specific durations when n is even [14]. According to the findings of Aubert, Ringel, Axenovich, Tompkins, Schneider, and Offner, Q_n can be decomposed into comparatively lengthy cycles with lengths of a power of two [10]. It is demonstrated by Tapadia, Waphare, and Borse that Q_n can be broken down into very brief cycles with a power of two length [14].

Therefore, the need to divide networks into subsets with particular characteristics such as being acyclic or having a predetermined number of nodes or cycles of a given size motivates the graph decomposition problem. [5]. Studying peripheral vertices and diametral path is essential for network analysis and design because it provides important insights into a graph's structure and aids in the resolution of many problems. The idea of a diametral path is therefore very important to study the ideas of pathways, diameter, diametral paths, and their applications [5]. In a graph, the diameter provides important information about the distance between vertices, and the diametral path shows the critical path between these vertices [16].

In this paper, we examine the decomposition process that includes diametral paths and find the diametral path decomposition number of the hypercube graph. Also, the diametral path decomposition index of the hypercube graph.

2. Definitions and notation

A graph $G = (V(G), E(G))$ is a simple, connected, undirected, and finite graph. The degree of a vertex, denoted by $deg(v)$ where $v \in V(G)$, refers to the number of edges incident to v. A path in a graph G is denoted by P , $P = (v_1, v_2, \dots, v_n)$, $v_1, v_2, , \dots, v_n \in V$. If a path starts and ends at the same vertex, we say that cycle in a graph G is denoted by C, $C = (v_1, v_2, \dots, v_n, v_1)$ [12]. The number of edges in P refers to the length of path P. The length of the shortest path between any two vertices in a graph refers to the distance between these vertices, and denoted by $d(u, v)$ where $u, v \in V(G)$. The diameter of a graph G is the maximum distance in G and denoted by $diam(G)$, $diam(G) = max\{d(u, v) : u, v \in V\}$ [3]. A few typical outcomes for specific graph classes are shown below [2] [16].

- i- $diam(K_n) = 1$, K_n refers to complete graph with $n \geq 2$.
- ii- $diam(W_n) = 2$, W_n refers to wheel graph with $n \geq 5$.
- iii- $diam(K_{1,n}) = 2$, $K_{1,n}$ refers to star graph with $n \geq 2$.
- iv- $diam(K_{r,s}) = 2$, $K_{m,n}$ refers to complete bipartite graph with $m \geq 2$ and $n \geq 2$.
- v- $diam(P_n) = n - 1$, P_n refers to path graph with $n \geq 2$.
- vi- $diam(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$, C_n refers to cycle graph with $n \geq 3$.
- vii- $diam(Q_n) = n$, Q_n refers to hypercube graph with $n \geq 2$.

In a graph G, the diametral path is the shortest path that connects two vertices so that its length is equal to the graph's diameter [16].

3. Decomposition of graphs

In graph theory, one of the most well-known problems is the decomposition of graphs. It entails dividing an input graph into subgraphs that meet certain requirements. These problems can be broadly divided into two categories: The first decomposes the input graph into identically typed edge-disjoint subgraphs, which are known as simple decompositions. The second method decomposes the input graph into at least two different kinds of edge-disjoint subgraphs. Which is known as multiple decomposition [5].

Diametral path decomposition is a set of graph's edge-disjoint diametral paths so that for each graph edge to appear in precisely one diametral path. This collection's cardinality is called the diametral path decomposition number and denoted by $d_{ec}(G)$. The number of such decompositions is called the diametral path decomposition number and denoted by $d_{ec}(G)$. The number of such decompositions is called the diametral path decomposition index and denoted by $D_{ec}(G)$ [16]. A few typical outcomes to $d_{ec}(G)$ and $D_{ec}(G)$ for certain classes of graphs are shown below:

Lemma (1)

- i- $d_{ec}(K_n) = n$ and $D_{ec}(K_n) = 1$ where $n \geq 2$.
- ii- $d_{ec}(W_n) = n - 1$ where $n \geq 5$.
- iii- $d_{ec}(K_{1,n}) = \frac{n}{2}$ and $D_{ec}(K_{1,n}) = (n - 1)(n - 3)(n - 5) \dots .1$, where $n \geq 2$.
- iv- $d_{ec}(K_{m,n}) = \frac{mn}{2}$ where $(m \geq 2$ and $n \geq 2)$ m or n is even.
- v- $d_{ec}(P_n) = 1 = D_{ec}(P_n)$ where $n \geq 2$.
- vi- $d_{ec}(C_n) = 2$ and $D_{ec}(C_n) = \frac{n}{2}$ where n is even.

4. Decomposition of hypercube graph

The hypercube graph Q_n stands for the n −dimensional hypercube graph, which is the graph with 2^n vertices labeled $V(Q_n) = \{0,1\}^n$, and $n2^{n-1}$ edges labeled $E(Q_n) = \{uv: u, v \in V(Q_n)\}$, where u and u vary in a single coordinate [2]. The significant property of the hypercube Q_n is that it can be generated as the iterated cartesian product of K_2 from lower-dimensional cubes i.e.

$$Q_n = \begin{cases} K_2 & n = 1 \\ Q_{n-1} \times K_2 & n \geq 2 \end{cases}.$$

Theorem.1

For every $n \geq 2$, the hypercube graph Q_n has a diametral path decomposition with $de(Q_n) = n$ and $De(Q_n) = \frac{2^n}{2}$.

Proof. Suppose Q_n be a hypercube graph such that Q_2 denotes the hypercube graph with 4 vertices and 4 edges. Then we get Q_2 isomorphic to C_4.

From Lemma (1) $de(C_4) = 2$ and $De(C_4) = 2$. Hence,

$$de(Q_2) = 2 \text{ and } De(Q_2) = 2. \tag{1}$$

Since, $deg(u_i) = n$, $u_i \in Q_n$ with $1 \leq i \leq 2^n$. Therefore, we can find n diametral path in Q_n. Hence,

$$de(Q_n) = n. \tag{2}$$

Now, by the structure of Q_n, we can find $De(Q_n)$, with $n \geq 2$ by the following cases:

If $n = 2$: from (1), we get $De(Q_2) = 2$ which can be expressed as follows:

$$De(Q_2) = \frac{2^2}{2}. \tag{3}$$

- **If $n = 3$:** There can be four decompositions as they shown in Figure (1b):

$\{(a_1, a_2, b_2, b_3), (a_1, b_1, b_4, b_3), (a_1, a_4, a_3, b_3)\}, \{(a_2, a_1, a_4, b_4), (a_2, b_2, b_1, b_4), (a_2, a_3, b_3, b_4)\}$,
$\{(a_3, a_4, a_1, b_1), (a_3, b_3, b_4, b_1), (a_3, a_2, b_2, b_1)\}$, and $\{(a_4, a_1, a_2, b_2), (a_4, b_4, b_1, b_2), (a_4, a_3, b_3, b_2)\}$.

So $De(Q_3) = 4$, which can be expressed as follows:

$$De(Q_3) = \frac{2^3}{2}. \tag{4}$$

- **If $n = 4$:** There can be eight decompositions as shown in Figure (1c):

$\{(a_1, a_2, c_2, d_2, d_3), (a_1, c_1, d_1, d_4, d_3), (a_1, a_4, c_4, c_3, d_3), (a_1, b_1, b_4, b_3, d_3)\}$,

$\{(a_2, a_1, c_1, c_4, d_4), (a_2, b_2, d_2, d_3, d_4), (a_2, c_2, c_1, b_1, d_4), (a_2, a_3, b_3, b_4, d_4)\}$,

$\{(a_3, c_3, d_3, d_4, d_1), (a_3, a_4, c_4, c_1, d_1), (a_3, b_3, b_2, b_1, d_1), (a_3, a_2, c_2, d_2, d_1)\}$,

$\{(a_4, c_4, c_3, c_2, d_2), (a_4, a_1, c_1, d_1, d_2), (a_4, a_3, c_3, d_3, d_2), (a_4, b_4, b_1, b_2, d_2)\}$,

$\{(b_1, b_4, d_4, d_3, c_3), (b_1, d_1, d_2, c_2, c_3), (b_1, b_2, b_3, a_3, c_3), (b_1, a_1, c_1, c_4, c_3)\}$,

$\{(b_2, b_1, b_4, a_4, c_4), (b_2, d_2, d_1, d_4, c_4), (b_2, b_3, a_3, d_3, c_4), (b_2, a_2, c_2, c_1, c_4)\}$,

$\{(b_3, a_3, a_4, c_4, c_1), (b_3, b_4, d_4, d_1, c_1), (b_3, b_2, a_2, a_1, c_1), (b_3, d_3, d_2, c_2, c_1)\}$,

$\{(b_4, b_1, a_1, c_1, c_2), (b_4, b_3, b_2, d_2, c_2), (b_4, a_4, a_1, a_2, c_2), (b_4, d_4, c_4, c_3, c_2)\}$.

So $De(Q_4) = 8$, which can be expressed as follows:

$$De(Q_4) = \frac{2^4}{2}. \tag{5}$$

- **If $n = 5$:** There can be sixteen decompositions as they shown in Figure (1c) in the same way when $n = 3$ and $n = 4$. So $De(Q_5) = 16$, which can be expressed as follows:

$$De(Q_5) = \frac{2^5}{2}. \tag{6}$$

Continuing in the same way in 3, 4, 5, and 6, we can get the general formula to $De(Q_n), n \geq 2$. Hence, $De(Q_n) = \frac{2^n}{2}$.

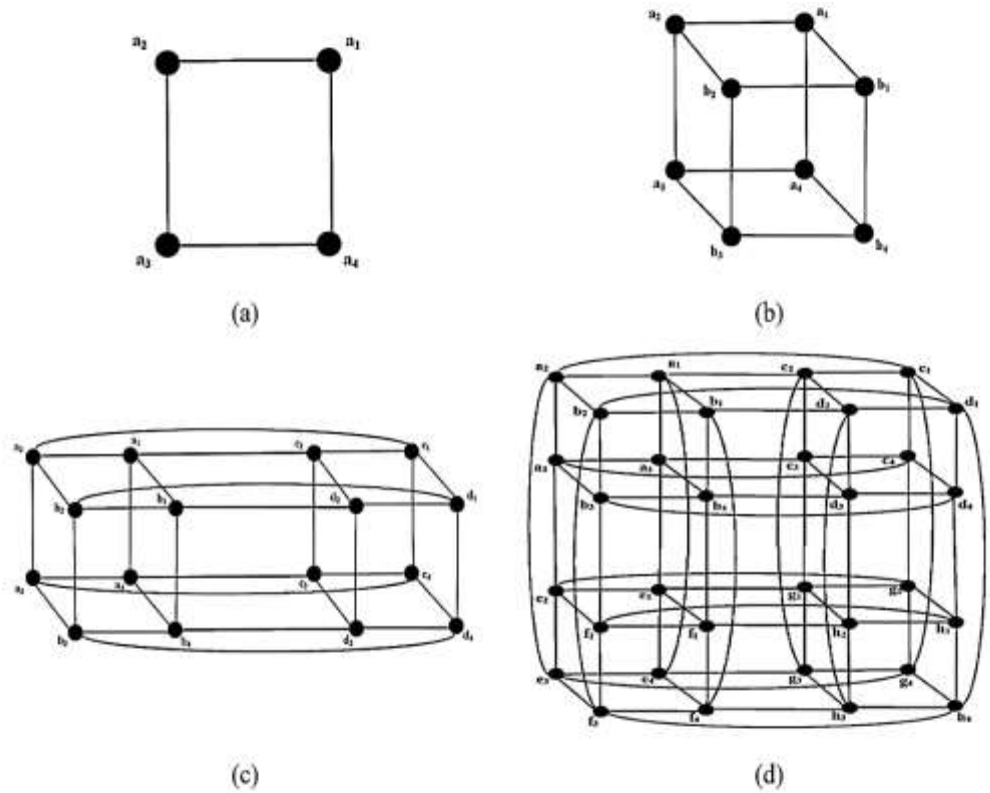

Figure 1: Hypercube graph with $\boldsymbol{n} = \{2, 3, 4, 5\}$.

Theorem. 2
For every $n \geq 2$, the hypercube graph Q_n has a cycle decomposition $CD(Q_n) = 2\, CD(Q_{n-1})$

Proof. Since, the significant property of the hypercube Q_n is that it can be generated as the iterated cartesian product of K_2 from lower-dimensional cubes.

By the structure of Q_n, we can find cycle decomposition by the following cases:

- **If $\boldsymbol{n} = 5$:** Q_2 contains one-cycle decomposition denoted by $CD(Q_2)$, which has four vertices (a_1, a_2, a_3, a_4) and four edges, see figure (1a). Hence,
$$CD(Q_2) = 1 \qquad (7)$$
- **If $\boldsymbol{n} = 3$:** Q_3 contains two-cycle decomposition $C_1D(Q_3)$ and $C_2D(Q_3)$, every cycle has four vertices $C_1D(Q_3) = (a_1, a_2, a_3, a_4)$ and $C_2D(Q_3) = (b_1, b_2, b_3, b_4)$, such that the vertices $a_i \in C_1D(Q_3), i \in \{1,2,3,4\}$ adjacent to $b_j \in C_2D(Q_3), j \in \{1,2,3,4\}$, see figure (1b). Hence $CD(Q_3) = 2$, which can be expressed as follows:
$$CD(Q_3) = 2\, CD(Q_2) \qquad (8)$$
- **If $\boldsymbol{n} = 4$:** Q_4 contains four-cycle decomposition $C_1D(Q_4)$, $C_2D(Q_4)$, $C_3D(Q_4)$ and $C_4D(Q_4)$, every cycle has four vertices, $C_1D(Q_4) = (a_1, a_2, a_3, a_4)$, $C_2D(Q_4) = (b_1, b_2, b_3, b_4)$, $C_3D(Q_4) = (c_1, c_2, c_3, c_4)$ and $C_4D(Q_4) = (d_1, d_2, d_3, d_4)$, such that the vertices $a_i \in C_1D(Q_4), i \in \{1,2,3,4\}$ adjacent to $b_j \in C_2D(Q_4)$ and $c_j \in C_3D(Q_4)$ with $j \in \{1,2,3,4\}$, the vertices $d_i \in C_4D(Q_4), i \in \{1,2,3,4\}$ adjacent to $b_j \in C_2D(Q_4)$ and $c_j \in C_3D(Q_4)$ with $j \in \{1,2,3,4\}$, see Figure (1c). Hence $CD(Q_4) = 4$, which can be expressed as follows:
$$CD(Q_4) = 2\, CD(Q_3) \qquad (9)$$

continuing in the same way we can find $CD(Q_5)$, $CD(Q_6)$, as follows:

$$CD(Q_5) = 2\, CD(Q_4) \qquad (10)$$

$$CD(Q_6) = 2\, CD(Q_5) \qquad (11)$$

continuing in the same way in (7)-(11). Hence, we get the general formula of cycle decomposition of Hypercube graph

$CD(Q_n) = 2\,CD(Q_{n-1})$.

5. Conclusion

The decomposition of the hypercube graph, by identifying diametral paths and calculating the hypercube graph's diametral path decomposition number $de(Q_n) = n$, are the outcomes of the work of this paper. We also calculate the hypercube graph's diametral path decomposition index $De(Q_n) = \frac{2^n}{2}$. Lastly, figuring out the hypercube graph's cycle decomposition $CD(Q_n)$ for every $n > 2$.

Funding: This research received no external funding.
ORCID: https://orcid.org/0000-0002-1095-062X [1], https://orcid.org/0000-0001-5408-7496 [2]
Publisher's Note: All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers.

References

[1] A. A. Abueida and M. Daven, Multidecompositions of the complete graph, Ars Combinatoria, 72: 17-22, 2004.
[2] A. N Jasim and A.A Najim, "Edges deletion problem of hypercube graphs for some n", Discrete Mathematics, Algorithms and Applications, 2450045, 2024.
[3] A. N. Jasim and A. A Najim, "Solving edges deletion problem of complete graphs", Baghdad Science Journal, 21(12): 4073-4082, 2024.
[4] A. Sinha, S. Kumar Saha, and P. Basuchowdhuri, An Algorithm for the Decomposition of Complete Graph into Minimum Number of Edge-disjoint Trees, arXiv preprint arXiv:2405.18506.
[5] Beggas, F., Decomposition and Domination of Some Graphs (Doctoral dissertation, Université Claude Bernard Lyon 1), 2017.
[6] G. Sethuraman and V. Murugan. Decomposition of complete graphs into arbitrary trees. Graphs and Combinatorics, 37(4):1191–1203, 2021.
[7] H. R. Hashim, F. Luca, H. B. Shelash, and A. A. Shukur, Generalized lucas graphs, Afrika Matematika, 34(1), 2023.
[8] J. Barát and C. Thomassen. Claw-decompositions and Tutte orientations. J. Graph Theory, 52:135–146, 2006.
[9] J. Kottarathil, S. Naduvath, and J. V. Kureethara, Graph Theory and Decomposition. CRC Press, 2024.
[10] M. Axenovich, D. Offner, and C. Tompkins, Long path and cycle decompositions of even hypercubes. European J. Combin, 95, 103320, 2021.
[11] M. Merker, Graph Decompositions, 2016.
[12] S. Arumugam, I.S. Hamid, and V. M. Abraham, Decomposition of graphs into paths and cycles, Journal of Discrete Mathematics, Vol.1: 1-6, 2013.
[13] S. Gibson, and D. Offner, Decompositions of even hypercubes into cycles whose length is a power of 2, 2024.
[14] S. Tapadia, B. Waphare, and Y. Borse, Cycle decompositions of the Cartesian product of cycles. Australas. J. Combin, 74, : 443–459, 2019.
[15] S. Yamamoto, H. Ikeda, S. sei Shige-Eda, K. Ushio, and N. Hamada, On claw-decomposition of complete graphs and complete bigraphs. Hiroshima Mathematical Journal, 5(1):33–42, 1975.
[16] T. M. Agnes, L. S. Reddy , J. Varghese, and J. S. Mangam, Decomposition of Graph into diametral paths, 6: 97-104, 2016.