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| RESEARCH ARTICLE

On the Angular Singularities of a Smooth Function

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| ABSTRACT
The paper studies angular singularities of a real smooth function of the 4th degree using real analysis and catastrophe theory. After that, we apply an ordinary differential equation (ODE) with its boundary conditions. We show that the real smooth function equivalent to the key function associated with the ODE's function by applying the Lyapunov-Schmidt local technique. The angular singularities have been used to study the bifurcation analysis of the real smooth function. We have discovered the (caustic) bifurcation set's parametric equation and geometric interpretation. Moreover, the critical spots' bifurcated spread has been identified.

| KEYWORDS

Bifurcation's solutions, Angular's Singularities, Caustic.

| ARTICLE INFORMATION

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1. Introduction

The method of Lyapunov-Schmidt (LS) can convert the dimensional equation from the infinite to the finite in Banach spaces. The method (LS) reduces most equations that appear in mathematics and physics of the form:

$$f(x,\lambda)=b, x\in O, b\in Y, \lambda\in\mathbb{R}^n \quad (1)$$

to the finite dimension of the form:

$$\Theta(\xi,\lambda)=\beta, \xi\in M, \beta\in N \quad (2)$$

in which X and Y are Banach's spaces, and f is a smooth Fredholm map with index zero., $O\subseteq X$ is open and, M and N are smooth finite dimensional manifolds. All of the topological and analytical characteristics of equation (1) such as multiplicity and the bifurcation diagram etc. are present in equation (2), (see [Loginov, 1985, Sapronov, 1973, Sapronov, 1996, Vainberg, 1975]. The study of the bifurcation solutions of BVPs heavily relies on the singularities of smooth maps [Golubitsky, 1985]. In the early years, the study of smooth map singularities and its applications to VPB piqued the interest of the Sapronov group in several of their studies [Arnol'd, 1994, Hussain, 2005, Hussain, 2010, Krasnosel'skii, 1964, Mizeal, 2012, Shveriova, 2002]. In recent years, works similar to those mentioned above have appeared [Ali, 2023, Hussain, 2024, Kadhim, 2020, Madhi, 2021, Madhi, 2022].

2. Literature Review

There are many works referred to in the aforementioned references about the study of singularities and their applications in most engineering and physical fields, etc. For example, the boundary singularities of the following function has been examined by Shveriova [2002], $\widetilde{W}(\eta,\gamma)=\eta_1^4+(c\eta_1+\eta_2)^2-2\varepsilon_1\eta_1^2+2\varepsilon_2\eta_1^2\eta_2+2\varepsilon_3\eta_1\eta_2+2\varepsilon_4\eta_1+2\varepsilon_5\eta_2$, where $\eta=(\eta_1,\eta_2)$, $\gamma=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5),\eta_1,\ \eta_2\geq 0$, such that she deemed the functional,

$$V(z,\alpha)=\int_0^{\pi}\left(\frac{(z')^2}{2}+\alpha(\cos(z(x))-1)\right)dx$$

with the additional setting, $\langle z,\omega\rangle = \int_0^{\pi} z(x)\omega(x)dx \geq 0$, as an applying of her outcomes. Also, another example, Kadhim [14] has investigated the following function's border singularities,

$W(s,\rho) = \frac{\eta_1^3}{3} + \frac{\eta_2^3}{3} + \eta_2\eta_1^2 + \eta_2^2\eta_1 + \lambda_1\eta_2\eta_1 + \lambda_2(\eta_1^2 + \eta_2^2)$ where, $s = (\eta_1, \eta_2), \eta_1, \eta_2 \geq 0, \rho = (\lambda_1, \lambda_2)$ with considering the functional,

$$V(z,\lambda) = \frac{1}{2}\int_0^1 (-\alpha(z')^2 + \beta z^2 + z^3 + z(z')^2)dx,$$

with the additional conditions, $\langle e_1, z\rangle + a\langle e_2, z\rangle \geq 0$ and $\langle e_1, z\rangle - a\langle e_2, z\rangle \geq 0$.

The method (LS) assumes that $g: \Omega \subset E \to F$ is an index zero smooth nonlinear Fredholm map. If there is a functional V, then the map g has a variational feature if and only if $V : \Omega \subset E \to \mathbb{R}$ and $\frac{\partial V}{\partial y}(y,\alpha)k = \langle g(y,\alpha), h\rangle_H, \forall y \in \Omega, k \in E$, where Hilbert space H's scalar product is represented by $\langle .,.\rangle_H$ and $E \subset F \subset H$. The critical points of functional $V(y,\alpha)$ are the solutions to the equation $g(y,\alpha) = 0$. The method (LS) can reduce the problem,

$$V(y,\alpha) \to \text{extr},$$

$y \in E, \alpha \in \mathbb{R}^n$ to an identical problem,

$$W(\zeta,\alpha) \to \text{extr}, \zeta \in \mathbb{R}^n$$

where, $W(\xi,\alpha)$ is named key function. The topological and analytical characteristics of the functional V (multiplicity, bifurcation diagram, etc.) are possessed by the function W [Sapronov, 1973]. Investigating bifurcating solutions for functional V is analogous to examining bifurcating solutions for key function.

3. Methodology

This paper studies the boundary singularities of the following real smooth function,

$$W(z,\alpha) = \frac{z_1^4}{4} + \frac{z_2^4}{4} + {z_1}^2{z_2}^2 + {z_1}^3 + z_1{z_2}^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2 \quad (3)$$

where $\mathrm{z} = (z_1, z_2), \alpha = (\alpha_1, \alpha_2)$, and α_1 and α_2 are parameters with considering the functional,

$$V(r,\lambda) = \frac{1}{2}\int_0^1 (-\beta(r')^2 + \gamma r^2 + r^4 + r(r')^2)ds \quad (4)$$

Where $r = r(s)$ and $\lambda = (\beta,\gamma)$.

The bifurcating solution's sections of equation (18) (here equation (18) is an application of our work (see section 5)) are to be identified, where every bifurcate solution is equal to functional (4)'s critical point, and each functional (4)'s critical point matches with a crucial point of functional (4)'s key function [Darinskii, 2007]. Therefore, we shall show that the function (3) is tantamount to the functional (4)'s key function. That is, analyzing the bifurcating solutions of equation (18) is similar to researching function's bifurcating solutions (3). Hence, we are interested in researching function's bifurcating solutions (3).

3.1 Fredholm functional's angular singularities [Darinskii, 2007]

To examine the behavior of Fredholm's functional near an angular singularity point, the problem of reducing to equivalent extremes is employed:

$$W(x) \to \text{extr}$$

were, $x \in D, D = \{x = (x_1, x_2)^\top \in \mathbb{R}^2 : x_2 \geq 0\}$.

When a smooth function W has a point a in D, we say that it is conditional critical in $\mathbb{R}^2$, if the least face of D that contains a is orthogonal to $\mathrm{gradW}(a)$ (grad denotes gradient of W).The quotient algebra has dimension $\bar{\mu}$, which represents the conditional critical spot's multiplicity a, where the quotient algebra denotes by $\bar{Q} = \frac{\Gamma_a(\mathbb{R}^2)}{I}$, such that $\Gamma_a(\mathbb{R}^2)$ represents the smooth functions ring of germs on $\mathbb{R}^2$ at a, and the angular Jacobi ideal in $\Gamma_a(\mathbb{R}^2)$ is defined as $I = \left(\frac{\partial W}{\partial x_1}, x_2\frac{\partial W}{\partial x_2}\right)$.

The multiplicity $\bar{\mu}$ of a conditional critical point is equal to the summation of its multiplicities $\mu + \mu_0$, here μ is the (normal) multiplicity of W on $\mathbb{R}^2$ and μ_0 is the (normal) multiplicity of the restriction $W \mid \partial D$ (where ∂D is the border of the set).

In the event that a critical point is normal, the row (r_0, r_1, r_2), where r_i is the number of critical points of the Morse index i; $i = 0,1,2$, represents the spreading of bifurcating extremes (bifurcation spread). The following order 2×3 matrix represents bifurcation spread, if the crucial point we need to deal with is angular (or border):

$$\begin{pmatrix} r_0^1 & r_1^1 & r_2^1 \\ r_0 & r_1 & r_2 \end{pmatrix}$$

, here r_i^j denotes the number of the angular critical points of index i (for $j = 1$), and r_i is the number of normal (situated inside D) critical points of index $i (i = 0,1,2)$.

4. Results

In this section, we take into account the function (3), which is defined in the first section. The function (3) has codimension eight at the origin, hence it has multiplicity nine. The main objectives are to determine the geometrical description (bifurcation diagram) of the caustic of the function (3) and the distribution of its critical points. In order to stay away from certain challenges when examining the function (3), one makes the following assumptions, $z_1 = y, z_2{}^2 = z$. Therefore, researching function (3) is equivalent to investigating the function listed below:

$$W(x,\alpha) = \frac{y^4}{4} + \frac{z^2}{4} + y^2 z + y^3 + yz + \alpha_1 y^2 + \alpha_2 z \tag{5}$$

where $x = (y,z), \alpha = (\alpha_1, \alpha_2)$ and $z \geq 0$. Given that the function (5)'s germ (the principal part) $W_0 = \frac{y^4}{4} + \frac{z^2}{4}$, so from the 2nd section we get, $I = \left(\frac{\partial W_0}{\partial y}, z\frac{\partial W_0}{\partial z}\right) = \left(y^3, \frac{z^2}{2}\right) = (y^3, z^2)$ and $\bar{\mu} = 6$ such that $\mu = \mu_0 = 3$. Considering that multiplicity $\bar{\mu}$ = the number of crucial points [18] , as a result, function (5) has six critical points, three of which are on the border $z = 0$ and three of which are inside. Thus, the following union of three sets is the caustic of function (5):

$$\Pi = \Pi_{1,0}^{\text{int}} \cup \Pi_{1,0}^{\text{ext}} \cup \Pi_{1,1}.$$

The caustic sets (or components) that indicate the boundary singularity degeneration along the border and along the normal are denoted by $\Pi_{1,0}^{\text{int}}$ and $\Pi_{1,0}^{\text{ext}}$, respectively, while the component corresponding to the degeneration of the interior critical points (non-boundary) is represented by $\Pi_{1,1}$.

Lemma 1. (**Degeneration towards the border $z = 0$ (internal degeneration)**) The set $\Pi_{1,0}^{\text{int}}$ is expressed by the parametric equation of the following form: $\alpha_1(8\alpha_1 - 9) = 0$.

Proof: All points $(y, 0, \alpha_1, \alpha_2)$ that fulfill the following relations are represented by the set $\Pi_{1,0}^{\text{int}}: \frac{\partial W(y,0,\alpha_1,\alpha_2)}{\partial y} = \frac{\partial^2 W(y,0,\alpha_1,\alpha_2)}{\partial y^2} = 0$. From these relations, we have: $y^3 + 3y^2 + 2y\alpha_1 = 3y^2 + 6y + 2\alpha_1 = 0$. We may use the following equations system to express the prior relationships:

$$y^3 + 3y^2 + 2y\alpha_1 = 0 \tag{6a}$$
$$3y^2 + 6y + 2\alpha_1 = 0 \tag{6b}$$

Multiplying the equation (6b) by $-\frac{y}{3}$ and then adding the result to the equation (6a) gives us:
$\frac{1}{3}y(3y + 4\alpha_1) = 0$, and from this equation we obtain, $y = 0$ or $3y + 4\alpha_1 = 0$. Put $y = 0$ into the equation (6b) to get:

$$\alpha_1 = 0 \qquad (7).$$

Let $y \neq 0$ and $3y + 4\alpha_1 = 0$ this implies $y = -\frac{4\alpha_1}{3}$. Put $y = -\frac{4\alpha_1}{3}$ into the equation (6b) to get: $\frac{2}{3}\alpha_1(8\alpha_1 - 9) = 0$. Since, $y \neq 0$ and $y = -\frac{4\alpha_1}{3}$, so $\alpha_1 \neq 0$. Therefore,

$$(8\alpha_1 - 9) = 0 \qquad (8).$$

The result of multiplying the equations (7 and 8) is as follows: $\alpha_1(8\alpha_1 - 9) = 0$.

Lemma 2. (Degeneration towards the normal of the border $\mathbf{z = 0}$ (external degeneration)) The structure of parametrical formula describing the parametric set $\Pi_{1,0}^{\text{ext}}$ is as follows:

$$\alpha_2(4\alpha_1{}^2 - 4\alpha_1\alpha_2 + \alpha_2{}^2 - 4\alpha_1 + 6\alpha_2) = 0.$$

Proof: All points $(y, 0, \alpha_1, \alpha_2)$ that fulfill the following relations are represented by the set $\Pi_{1,0}^{ext}$: $\frac{\partial W(y,0,\alpha_1,\alpha_2)}{\partial y} = \frac{\partial W(y,0,\alpha_1,\alpha_2)}{\partial z} = 0$, this implies $y^3 + 3y^2 + 2y\alpha_1 = y^2 + y + \alpha_2 = 0$.

These relations are equivalent to the following equations system:

$$y^3 + 3y^2 + 2y\alpha_1 = 0 \quad (9a)$$
$$y^2 + y + \alpha_2 = 0 \quad (9b)$$

Multiplying the equation (9b) by $-y$ and then adding the result to the equation (9a) gives us: $y(2y - \alpha_2 + 2\alpha_1) = 0$, and from this equation, we obtain $y = 0$ or $2y - \alpha_2 + 2\alpha_1 = 0$. Put $y = 0$ into the equation (9b) to get:

$$\alpha_2 = 0 \quad (10).$$

Let $y \neq 0$ and $2y - \alpha_2 + 2\alpha_1 = 0$, this implies $y = \frac{1}{2}\alpha_2 - \alpha_1$. Put $y = \frac{1}{2}\alpha_2 - \alpha_1$ into the equation (9b) to get:

$$4\alpha_1^2 - 4\alpha_1\alpha_2 + \alpha_2{}^2 - 4\alpha_1 + 6\alpha_2 = 0 \quad (11).$$

The result of multiplying the equations (10 and 11) is as follows: $\alpha_2(4\alpha_1{}^2 - 4\alpha_1\alpha_2 + \alpha_2{}^2 - 4\alpha_1 + 6\alpha_2) = 0$.

Lemma 3. (**Interior (non-boundary) degradation**)The parametrical equation:

$$-8\alpha_1{}^3 + 48\alpha_1{}^2\alpha_2 - 96\alpha_1\alpha_2^2 + 64\alpha_2{}^3 + 21\alpha_1{}^2 - 30\alpha_1\alpha_2 + 57\alpha_2{}^2 - 18\alpha_1 + 5 = 0$$

is what creating the parametric set $\Pi_{1,1}$.

Proof: We take into account the function (5)'s critical points that the following structure defines in order to find the set $\Pi_{1,1}$: $\frac{\partial W(y,z,\alpha_1,\alpha_2)}{\partial y} = \frac{\partial W(y,z,\alpha_1,\alpha_2)}{\partial z} = 0, z > 0$, this implies:

$$y^3 + 3y^2 + 2yz + 2y\alpha_1 + z = \frac{z}{2} + y^2 + y + \alpha_2 = 0 \quad (12)$$

Then, to obtain the following equation, put the function (5)'s Hessian matrix's determinate equal to zero:

$$-\frac{5}{2}y^2 - y + z + \alpha_1 - 1 = 0 \quad (13)$$

The relations (12 and 13) can be rewritten as follows:

$$y^3 + 3y^2 + 2yz + 2y\alpha_1 + z = 0 \quad (14a)$$
$$\frac{z}{2} + y^2 + y + \alpha_2 = 0 \quad (14b)$$
$$-\frac{5}{2}y^2 - y + z + \alpha_1 - 1 = 0 \quad (14c)$$

Multiplying the equation (14b) by $\frac{5}{2}$ and then adding the result to the equation (14c) gives us: $\frac{3}{2}y + \frac{9}{4}z + \alpha_1 - 1 + \frac{5}{2}\alpha_2 = 0$, its solution for y gives:

$$y = -\frac{3}{2}z - \frac{2}{3}\alpha_1 + \frac{2}{3} - \frac{5}{3}\alpha_2 \quad (15).$$

Put the equation (15) in the equation (14b), to get:

$$z = -\frac{4}{9}\alpha_1 - \frac{10\alpha_2}{9} + \frac{2}{3} \pm \frac{2}{9}\sqrt{2\alpha_1 - 4\alpha_2 - 1} \quad (16).$$

Substituting the equations (15 and 16) in the equation(14a), and simplifying the result we obtain: $\frac{\frac{2}{3}\alpha_1 + \frac{2}{3}\alpha_2 - \frac{4}{9}}{\frac{4}{9}\alpha_1 - \frac{8\alpha_2}{9} - \frac{2}{9}} = \pm\sqrt{2\alpha_1 - 4\alpha_2 - 1}$, thus, after squaring and simplifying both sides of this equation, one finds

$$8\alpha_1{}^3 - 48\alpha_1{}^2\alpha_2 + 96\alpha_1\alpha_2{}^2 - 64\alpha_2{}^3 - 12\alpha_1{}^2 + 48\alpha_1\alpha_2 - 48\alpha_2{}^2 + 6\alpha_1 - 12\alpha_2 - 1 = 9\alpha_1^2 + 18\alpha_1\alpha_2 + 9\alpha_2^2 - 12\alpha_1 - 12\alpha_2 + 4$$

Finally, moving all the terms of the last equation to the right, we find:

$$-8\alpha_1{}^3 + 48\alpha_1{}^2\alpha_2 - 96\alpha_1\alpha_2{}^2 + 64\alpha_2{}^3 + 21\alpha_1{}^2 - 30\alpha_1\alpha_2 + 57\alpha_2{}^2 - 18\alpha_1 + 5 = 0.$$

Theorem 1. The formula that follows represents the function (5)'s parametric equation of its bifurcation set(the caustic):

$$\alpha_2\alpha_1(8\alpha_1 - 9)(4\alpha_1{}^2 - 4\alpha_1\alpha_2 + \alpha_2{}^2 - 4\alpha_1 + 6\alpha_2)(-8\alpha_1{}^3 + 48\alpha_1{}^2\alpha_2 - 96\alpha_1\alpha_2{}^2 + 64\alpha_2{}^3 + 21\alpha_1{}^2 - 30\alpha_1\alpha_2 + 57\alpha_2{}^2 - 18\alpha_1 + 5) = 0.$$

Proof: Considering that the union of the following three sets constitutes the caustic of function (5):

$$\Pi = \Pi_{1,0}^{int} \bigcup \Pi_{1,0}^{\text{ext}} \bigcup \Pi_{1,1}$$

Therefore, the parametrical equation for the caustic will be made up of the result of multiplying each left portion of the caustic components formulas together with setting the result to be zero. We know the equations of the caustic components have been found in the lemmas (1), (2) and (3), thus, $\alpha_2\alpha_1(8\alpha_1 - 9)(4\alpha_1{}^2 - 4\alpha_1\alpha_2 + \alpha_2{}^2 - 4\alpha_1 + 6\alpha_2)(-8\alpha_1{}^3 + 48\alpha_1{}^2\alpha_2 - 96\alpha_1\alpha_2{}^2 + 64\alpha_2{}^3 + 21\alpha_1{}^2 - 30\alpha_1\alpha_2 + 57\alpha_2{}^2 - 18\alpha_1 + 5) = 0$, will serve as the parametric equation for the bifurcating set (caustic) of the function (5).

Theorem 2. The following are the bifurcation spread matrices of the function (5)'s critical points:

$$\begin{pmatrix}0 & 3 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 3 & 0\\ 1 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 0\end{pmatrix} \quad (17)$$

Proof: From the caustic equation that it has been found by Theorem (1), the geometric representation of this equation can be found by Figure 1. The parameters' plane can be divided into eight regions (S_i, $\mathrm{i} = 1,2,3,4,5,6,7,8,9,10$) by this figure. Every zone has a certain number of real crucial spots that are not degenerate. Both internal and boundary points are included in this set of points. The second derivative test can be used to figure out the quality of the boundary and interior points. As a result, the crucial points are distributed as follows:

1) If the pair of parameters (α_1, α_2) is a member of S_1 or S_2 or S_5, then, there are four crucial points: one saddle point in the interior and three saddle points on the boundary $z = 0$.
2) if the parameters pair (α_1, α_2) belong to S_3 or S_4 or S_6 or S_7, then the boundary $z = 0$ has three crucial points that are saddle.
3) if the parameters pair (α_1, α_2) belong to S_8, then we have five critical points: three saddle points on the boundary $z = 0$, and two interior points: a minimum point, and a saddle point.
4) if the parameters pair (α_1, α_2) belong to S_9, then there are four crucial spots: two saddle spots on the border $z = 0$, one minimal spot on the border also, and one saddle spot inside.
5) if the parameters pair (α_1, α_2) belong to S_{10}, then we have three critical points (two saddle points and one minimum point on boundary $z = 0$).

The matrices are obtained from the five items above as are described in (17).

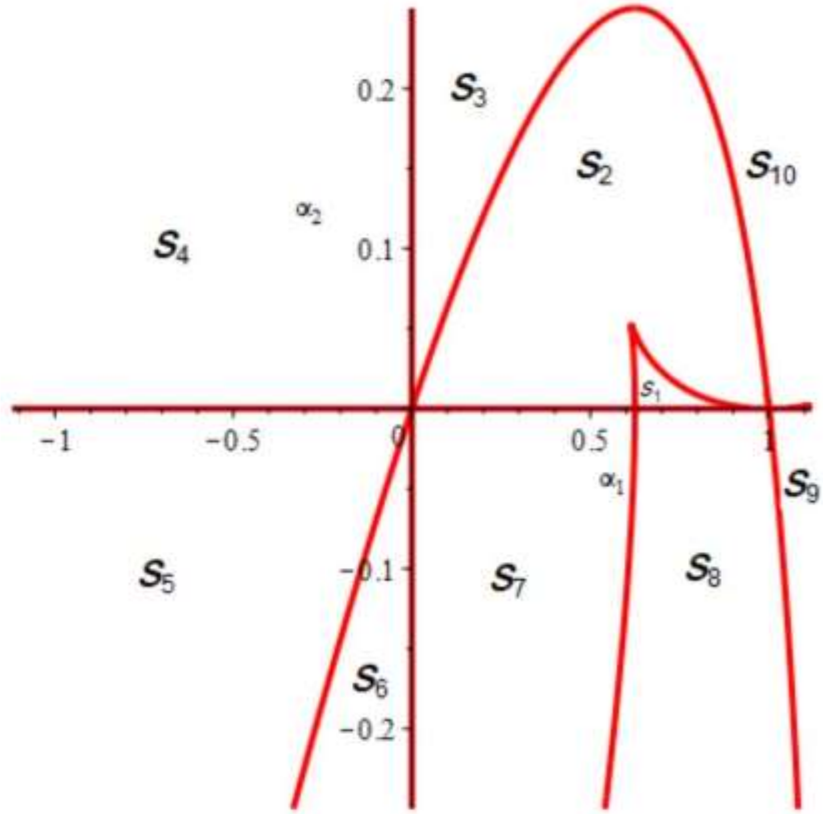
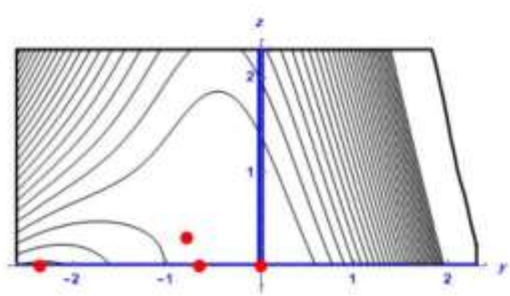
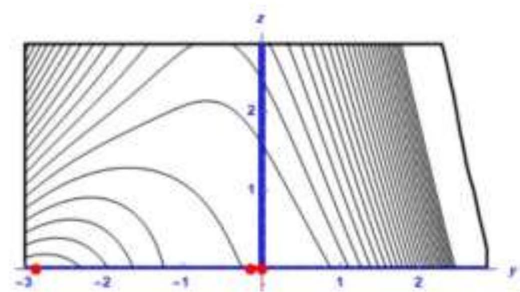


Figure 1: Depicts the function (5)'s caustic in $\alpha_1\alpha_2$- plane

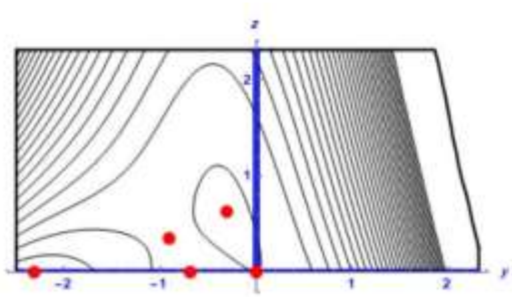
Note 1. Studying graphing by contour lines with critical points: Figure 2 sections (a_1), (a_2), (a_3), (a_4), and (a_5) depict the contour lines' placements in relation to the function's domain borders (5), as well as the number and kind of critical points that correspond to each region in the function's caustic regions (5), where depicting of (a_1) represents the regions S_1 or S_2 or S_5,(a_2) represents the regions S_3 or S_4 or S_6 or S_7, (a_3) represents the region S_8, (a_4) represents the region S_9 and (a_5) represents the region S_{10}.



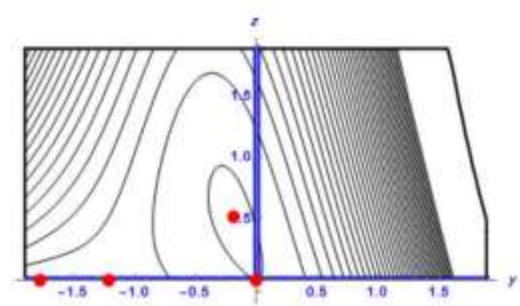
(a_1) contour lines' placements and kind of critical points in the regions S_1 or S_2 or S_5



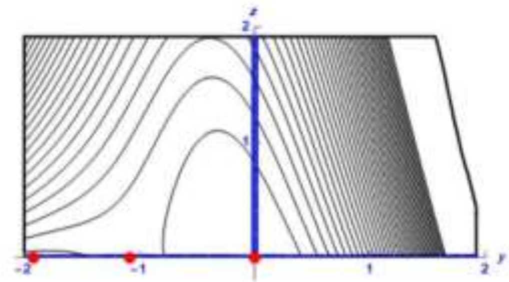
(a_2) contour lines' placements and kind of critical points in the regions S_3 *or* S_4 *or* S_6 *or* S_7



(a_3) contour lines' placements and kind of critical points in the region S_8



(a_4) contour lines' placements and kind of critical points in the region S_9



(a_5) contour lines' placements and kind of critical points in the region S_{10}

Figure 2. contour lines' placements and kind of critical points in the function's caustic regions (5)

5. Applications

The following ordinary differential equation (ODE) is an application of our study:

$$\beta r'' + \gamma\, r + 2r^3 - \left(\tfrac{1}{2}(r')^2 + rr''\right) = 0,$$

with the coming boundary conditions $r(0) = r(1) = 0$, where $r = r(s), s \in [0,1]$ and β and γ are parameters. The ODE has been converted from the Camassa Holm equation of the type:

$$v_t + 2cv_y - v_{yyt} + 6v^2 v_y = 2v_y v_{yy} + v v_{yyy}, \text{ [Li, 2013]}$$

where v is the fluid velocity in the y direction and the constant c is associated with the critical wave speed in shallow water, by the transformation, $s = y - \beta t$, $v(y,t) = r(s)$.

We decided to $g: E \to F$ has an index of zero and is a nonlinear Fredholm operator, the space, $E = D^2([0,1], \mathbb{R})$ contains all continuous real functions whose derivatives have no more than two orders, $F = D^0([0,1], \mathbb{R})$ is the space of all continuous real functions and the following operator expression defines the function g as :

$$g(r,\alpha) = \beta r'' + \gamma r + 2r^3 - \left(\tfrac{1}{2}(r')^2 + rr''\right) = 0 \qquad (18).$$

In the following theorem, we show that the key function of the functional (4) and function (3) are interchangeable.

Theorem 3: The key function's normal form W_1, which corresponds to the functional (4), is

$$W_1(z,\alpha) = \frac{z_1^4}{4} + \frac{z_2^4}{4} + z_1^2 z_2^2 + z_1^3 + z_1 z_2^2 + \alpha_1 z_1^2 + \alpha_2 z_2^2$$

$z = (z_1, z_2), \alpha = (\alpha_1, \alpha_2)$ and α_1, α_2 are parameters.

Proof: The linearized expression that matches to equation (18) at $(0,\alpha)$ can be obtained by applying the Lyapunov-Schmidt technique. It has the following form:

$$Lk = 0, k \in E$$
$$k(0) = k(1) = 0,$$

where, $L = \beta \frac{d^2}{dx^2} + \gamma$.

The linearized expression's solution that meets the initial conditions is provided by $e_p(s) = c_p \sin(p\pi s), p = 1,2,\ldots$ and the characteristic formula corresponding to this solution is $-\beta(p\pi)^2 + \gamma = 0$.This formula provides characteristic lines ℓ_p in the $\beta\gamma -$ plane. The points (β,γ) for which there are non-zero solutions to the linearized equation make up the characteristic lines ℓ_p [Sapronov, 1996]. The bifurcation point for equation (18) can be found at $(\beta,\gamma) = (0,0)$, which is the place where the characteristic lines in the $\beta\gamma -$ plane intersect.

The bifurcation along the modes is caused by parameters β, γ that are localized as follows: $\beta = 0 + \delta_1, \gamma = 0 + \delta_2, \delta_1, \delta_2$. The modes are $e_1(s) = c_1 \sin(\pi s), e_2(s) = c_2 \sin(2\pi s)$, where $\|e_1\| = \|e_2\| = 1$. Then we get $c_1 = c_2 = \sqrt{2}$.

Let $N = \mathrm{Ker}(L) = \mathrm{span}\{e_1,\ e_2\}$, then both N and its orthogonal supplement where they represent direct summation that can be used to decompose the space E:

$$E = N \oplus N^{\perp}, N^{\perp} = \left\{v \in E: \int_0^1 v e_s dx = 0, s = 1,2\right\}.$$

Two projections are in existence $P: E \to N$ and $I - P: E \to N^{\perp}$ in which $Pn = \tau$ and $(I-P)n = w$, (I serves as the operator for identity). Therefore, any vector n in E may be expressed in the form,

$$n = \tau + w \qquad (19),$$

where $\tau = z_1 e_1 + z_2 e_2 \in N, w \in N^{\perp}, z_i = \langle n, e_i \rangle; i = 1,2$.

Consequently, a smooth map Θ exists according to the implicit function theorem, such that $\Theta: N \to N^{\perp}$ and $\widetilde{W}(\zeta, \eta) = V(\Theta(w, \eta), \eta),\ \zeta = (z_1, z_2), \eta = (\delta_1, \delta_2)$.

The key function $\widetilde{W}$ can therefore be expressed as follows:

$$\begin{aligned}\tilde{W}(\zeta, \eta) &= V(z_1 e_1 + z_2 e_2 + \Theta(z_1 e_1 + z_2 e_2, \eta), \eta) \\ &= W_2(\zeta, \eta) + o(|\zeta|^4) + O(|\zeta|^4) O(\eta),\end{aligned}$$

where,

$$W_2(\zeta, \eta) = \frac{3}{4} z_1{}^4 + \frac{2}{3}\pi\sqrt{2} z_1{}^3 + \left(-\frac{1}{2}\beta\pi^2 + \frac{\gamma}{2}\right) z_1{}^2 + 3 z_1{}^2 z_2{}^2 + \frac{24\pi\sqrt{2} z_1 z_2{}^2}{5}$$
$$+\frac{3}{4} z_2^4 + \left(-2\beta\pi^2 + \frac{\gamma}{2}\right) z_2^2.$$

For the function $\widetilde{W}$, its principal part W_2 fully determines the geometrical shape of critical points' bifurcations and the branches' initial asymptotic of bifurcating. W_1 and $W_2$are contact equivalences if we replace z_1 by $\frac{z_1}{\sqrt[4]{3}}$ and z_2 by $\frac{z_2}{\sqrt[4]{3}}$ in the function W_2. This is because they share the same germ (the same principal part), $W_0 = \frac{z_1^4}{4} + \frac{z_2^4}{4}$, and the deformation in this instance. Consequently, the function W_2's and the function W_1's caustics correspond [Marsden, 1983].

Thereby, the function W_1 possesses all of the functional (4)'s topological and analytical properties, meaning that studying the bifurcation of equation (18) is the same as studying the bifurcation of the function W_1. This demonstrates that it is equal to investigating the bifurcating solutions of equation (18) to investigate the bifurcating solutions of function (3).

6. Conclusion

In this paper, we found the functional (4) that satisfies the variational property of (18) (as an operator). We found the key function corresponding to functional (4) in Theorem 3. We proved that function (3) of the fourth degree is equivalent to the key function. We found the bifurcation solution regions of (18) (as an equation), which are the critical points of function (3) spread in the branching diagram (caustic). Also, the parametric equation was found. The critical points were classified and their regions of existence in the diagram were found. Finally, an application of this work was given.

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