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| RESEARCH ARTICLE

Degenerate Sturm-Liouville Problem for Second-Order Differential Operators on Star-Graph

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| ABSTRACT

In this paper, we present a comprehensive study of second-order differential operators on a star-graph geometric graph considering a star graph with three edges and a common vertex. We investigate the Dirichlet problem for a Sturm-Liouville operator defined on this network-type manifold. The Sturm-Liouville problem is formulated as a system of ordinary differential equations (1) on the individual edges, subject to the boundary conditions (2) and (3) at the common vertex. We assume that the condition holds, ensure the non-degeneracy of the boundary conditions by using a synthetic approach. We fully describe and solve the Dirichlet problem for the given second-order differential operator on the star graph. The key results include the characterization of the spectral parameter λ, the construction of the matrix A composed of the boundary condition coefficients, and the analysis of the minors of A. The findings of this work contribute to the understanding of second-order differential operators on network-type manifolds and provide a framework for addressing similar problems on more complex geometric graphs. The insights gained from this study have potential applications in various fields, such as: quantum mechanics, control theory, and network analysis.

| KEYWORDS

Differential operators, Green's function Sturm-Liouville.

| ARTICLE INFORMATION

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1. Introduction

In this paper, a synthetic approach is used to fully describe and solve the Dirichlet problem for a second-order differential operator on a star graph. Consider a star-graph geometric graph Γ consisting of three edges with a common vertex. We believe that $i-e$ edge $(i=1,2,3,)$ - length interval a_i with natural parameterization $x_i \in [0,1]$ denote by L the following Sturm-Liouville problem on the graph Γ

$$L_i y_i(x_i) \equiv -y_i''(x_i) + q_i(x_i) y_i(x_i) = \lambda y_i(x_i) \tag{1}$$

$$y_1(0) = y_2(0) = y_3(0), \quad y_1'(0) + y_2'(0) + y_3'(0) = 0, \tag{2}$$

$$a_{i1} y_1(a_1) + a_{i2} y_2(a_2) + a_{i3} y_3(a_3) + a_{i4} y_4'(a_4) + a_{i5} y_5'(a_5) + a_{i6} y_6'(a_6) = 0 \tag{3}$$

Where λ -spectral parameter, real function $q_i(.)$, $i=1,2,3,$ belongs to space $L_1(0,1),$ and a_{ij} $(i=1,2,3,\ j=\overline{1,6})$ complex constants. Let us denote the matrix composed of coefficients a_{ij}, boundary conditions (3), through A, its minors, composed of $i-th$ $k-th$, $m-th$, columns, through A_{ikm} :

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{pmatrix} \qquad A_{ikm}=\begin{vmatrix} a_{1i} & a_{1k} & a_{1m} \\ a_{2i} & a_{2k} & a_{2m} \\ a_{3i} & a_{3k} & a_{3m} \end{vmatrix} \tag{4}$$

Throughout the entire work, we assume that the condition is satisfied

$$\text{Ran k } A=3. \tag{5}$$

The task is to find the coefficients for which the boundary conditions (3) are degenerate.

2. Literature review

In the last 25-30 years, the theory of differential equations and boundary value problems on geometric graphs (spatial networks) has been intensively developing, as evidenced by numerous scientific works. The research began in Berezin, F.A. and Faddeev, L.D differential equations and boundary value problems on geometric graphs [Kanguzhin, 2021], and also worked Kanguzhin, B. and others [Berezin, 2019], Ghulam Hazrat Aimal Rasa works inception of Green function and analytical nature of the Green's function in the vicinity of a simple pole [Rasa, 2021, Rasa, 2020], Kanguzhin, B.E (2019) and other wrote about differential operators with boundary conditions, Keselman G.M (2019) (2017) writing about unconditional convergence of expansions in eigenfunctions of some differential operators, Naimark (2010) in her book about differential operators, Greens function and boundary conditions, he wrote and distributed many theorems and formulas, they worked the differential operator boundary condition, Rasa, G.H.A. et al (2023) worked Sturm-Liouville problem with general inverse symmetric potential, asymptotic formulas for weight, Green's function of the Dirichlet problem for a differential operator on a star-graph, Bekbolat et al. (2019) wrote on the minimality of systems of root functions of the Laplace operator in the punctured domain, Nasri, Farhad, and Ghulam Hazrat Aimal Rasa. (2024) Lagrange formula conjugate third order differential equation, Rasa, GH Aimal, et al (2022) wrote Propagation of waves along the star-graph, Rasa, GH Aimal, (2022) worked Formulas for Surface Weighted Numbers on Graph.

3. Methodology

A descriptive research project has been used to focus and identify the effect of differential equations with boundary problem conditions and boundary problem differential operators on graphs, books, magazines, and websites have been used to advance and complete this research.

4. Definition of a Boundary Value Problems operator on Graphs

In what follows it is useful to introduce the space $L^2(G)$ on each edge e_j, the following i -th differential equation and in particular by equation (1) we mean the system of equations:

$$L_i y_i(x_i) \equiv -y_i''(x_i)+q_i(x_i)y_i(x_i)=\lambda y_i(x_i), \quad x\in(0,l_i),\ i=1,2,3,...,k \tag{6}$$

where q_i and y_i denote q and y restricted to e_i, and e_i is identified with $(0,l_i)$.

In the proposed work, the properties of Green's functions of a boundary value problem for second-order differential equations on a star graph are studied.

The boundary value problem (1),(2),(3),(4) on G can be formulated as an operator eigenvalue problem in $L^2(G),$ for the closed densely defined operator [Rasa, 2023]

$$Ly=-f''+pf \tag{7}$$

with domain

$$D(L)=\left\{ f \setminus f,\ f' \in AC,\ Lf \in L^2(G)\right\} \tag{8}$$

It should be known that f obeying rules (3).

5. Absence of degenerate boundary conditions for pairwise different a_i.

Let us denote by $C_{i1}(x,\lambda)$ and $S_{i2}(x,\lambda)$ linearly independent solutions of equations (1), satisfying the conditions

$$C_{i1}(0,\lambda)=1, \quad C'_{i1}(0,\lambda)=0, \quad S_{i2}(0,\lambda)=0, \quad S'_{i2}(0,\lambda)=1$$

Then the general solutions $y_i(x_i)$ equations (1) will be written as follows:

$$y_i(x_i)=A_{i1}C_{i1}(x_i,\lambda)+B_{i2}S_{i2}(x_i,\lambda), \quad i=1,2,3 \tag{9}$$

From the conjugation conditions (2) we find that:

$$A_{11}=A_{21}=A_{31}=A \qquad B_{32}=-B_{12}-B_{22} \tag{10}$$

Substituting representations (9) into the boundary conditions (3) taking into account equalities (10), we obtain an algebraic system

$$\begin{aligned}&a_{i1}\left(AC_{11}(a_1)+B_{12}S_{12}(a_1)\right)+a_{i2}\left(AC_{21}(a_2)+B_{22}S_{22}(a_2)\right)+\\&+a_{i3}\left(AC_{31}(a_3)-(B_{12}+B_{22})S_{32}(a_3)\right)+a_{i4}\left(AC'_{11}(a_1)+B_{12}S'_{12}(a_1)\right)+\\&+a_{i5}\left(AC'_{21}(a_2)+B_{22}S'_{22}(a_2)\right)+a_{i6}\left(AC'_{31}(a_3)-(B_{12}+B_{22})S'_{32}(a_3)\right)=0, \quad i=1,2,3\end{aligned}$$

Or we can write

$$\begin{aligned}&a_{i1}\left(AC(a_1)+B_{12}S(a_1)\right)+a_{i2}\left(AC(a_2)+B_{22}S(a_2)\right)+\\&+a_{i3}\left(AC(a_3)-(B_{12}+B_{22})S(a_3)\right)+a_{i4}\left(AC'(a_1)+B_{12}S'(a_1)\right)+\\&+a_{i5}\left(AC'(a_2)+B_{22}S'(a_2)\right)+a_{i6}\left(AC'(a_3)-(B_{12}+B_{22})S'(a_3)\right)=0, \quad i=1,2,3\end{aligned}$$

Statement 1. let the fundamental system of solutions to equation (1) with Cauchy conditions at the point $x=\frac{1}{2}$ we can see [1,2,3,4,5,6,7].

$$C(\frac{1}{2},\lambda)=S'(\frac{1}{2},\lambda)=1, \quad S(\frac{1}{2},\lambda)=C'(\frac{1}{2},\lambda)=0$$

Then there are functions $k_1(x,t),\ |t|<x,\ \ x\in\left[0,\frac{1}{2}\right]$ such that

$$C(x,\lambda)=\cos\sqrt{\lambda}(x-\frac{1}{2})+\int_{-x}^{x}k_1(x,t)cos\sqrt{\lambda}(t)dt,$$

$$S(x,\lambda)=\frac{\sin\sqrt{\lambda}(x-\frac{1}{2})}{\sqrt{\lambda}}+\int_{-x}^{x}k_1(x,t)\cos\sqrt{\lambda}(t)dt$$

Proof. Let's write the integral equation

$$C(x,\lambda)=\cos\sqrt{\lambda}(x-\frac{1}{2})+\int\limits_{\frac{1}{2}}^{x}\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}q(t)C(t,\lambda)dt, \tag{11}$$

Which is equivalent to the definition of the solution $C(x-\frac{1}{2},\lambda)$ at $x\in\left[0,\frac{1}{2}\right]$ Proof, by sequentially calculating the derivatives we get

$$C'(x,\lambda)=-\sqrt{\lambda}\sin\sqrt{\lambda}(x-\frac{1}{2})+\int\limits_{\frac{1}{2}}^{x}\cos\sqrt{\lambda}(x-t)q(t)C(t,\lambda)dt,$$

$$C''(x,\lambda)=-\lambda\cos\sqrt{\lambda}(x-\frac{1}{2})-\lambda\int\limits_{\frac{1}{2}}^{x}\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}q(t)C(t,\lambda)dt+q(x)C(x,\lambda) \tag{12}$$

from the last relation we see that

$$C''(x,\lambda)=-\lambda\cos\sqrt{\lambda}(x,\lambda)+q(x)C(x,\lambda),$$

or

$$-C''(x,\lambda)+q(x)C(x,\lambda)=\lambda\cos\sqrt{\lambda}(x,\lambda)$$

those. satisfies equation (1). Substituting the value into formulas (11), (12), we $x=0$ obtain

$$C(\frac{1}{2},\lambda)=1,\quad C'(\frac{1}{2},\lambda)=0\cdot$$

So, it remains to solve equation (11) by the method of successive approximations. Let's put

$$h_0(x,\lambda)=\cos\sqrt{\lambda}(x-\frac{1}{2})$$

$$h_{k+1}(x,\lambda)=\int\limits_{\frac{1}{2}}^{x}\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}q(t)h_k(t,\lambda)dt,\ \ k\geq 0$$

let's show what it $h_k(x,\lambda)$ looks follows:

$$h_k(x,\lambda)=\int\limits_{\frac{1}{2}}^{x}\cos\sqrt{\lambda}(-s)H_k(x,s)ds,\quad k\geq 1$$

Indeed, when $k=1$ we have

$$h_1(x,\lambda)=\int\limits_{\frac{1}{2}}^{x}\frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}}\cos\sqrt{\lambda}(-t)q(t)dt=\frac{1}{2}\int\limits_{\frac{1}{2}}^{x}\left(\frac{\sin\sqrt{\lambda}(x-2t)}{\sqrt{\lambda}}-\frac{\sin\sqrt{\lambda}(-x)}{\sqrt{\lambda}}\right)q(t)dt \tag{13}$$

Considering that

$$\frac{\sin\sqrt{\lambda}(x-2t)}{\sqrt{\lambda}}-\frac{\sin\sqrt{\lambda}(-x)}{\sqrt{\lambda}}=\int\limits_{2t-x}^{x}\cos\sqrt{\lambda}(-s)ds, \tag{14}$$

Then from (13) we obtain

$$h_1(x,\lambda)=\frac{1}{2}\int\limits_{0}^{x}q(\frac{1}{2}-t)dt\int\limits_{2t-x}^{x}\cos\sqrt{\lambda}(-s)ds=\frac{1}{2}\int\limits_{-x}^{x}\cos\sqrt{\lambda}(-s)ds\int\limits_{0}^{\frac{x+s}{2}}q(\frac{1}{2}-t)dt$$

The last equality is a consequence of applying Fubini's theorem to rearranging the order of integration [10,11,15,16,17].

Let K be arbitrary then

$$h_{k+1}(x,\lambda)=\int_0^x \frac{\sin\sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(\frac{1}{2}-t)dt\int_{-t}^{t}\cos\sqrt{\lambda}(-s)ds\, H_k(t,s)ds$$

Theorem 1. Let $C(x,\lambda), S(x,\lambda)$ arbitrary system of solutions to equation (1), (2), (3) and denote by the $\Delta(\lambda)$ function

$$\Delta(\lambda)=\int_{\frac{1}{2}}^{1} Q(x)\big(C(1-x,\lambda)S(x,\lambda)-S(1-x,\lambda)C(x,\lambda)\big)dx \tag{15}$$

Then the spectrum of the operator L coincides with the set of zeros of the function $\Delta(\lambda)$, and coincidence taking into account multiplicities.

Proof. As $C(x,\lambda)$ and $S(x,\lambda)$ Let's take those that participate in the average value formulas (11), (12) from these formulas it follows that each eigenfunction $y(x,\lambda)$ operator L must satisfy the conditions

$$\begin{aligned}&2y'(\frac{1}{2},\lambda)C(\frac{1}{2},\lambda)-\int_{\frac{1}{2}}^{1}Q(x)C(1-x,\lambda)y(x,\lambda)dx=0,\\&2y'(\frac{1}{2},\lambda)S(\frac{1}{2},\lambda)-\int_{\frac{1}{2}}^{1}Q(x)S(1-x,\lambda)y(x,\lambda)dx=0.\end{aligned} \tag{16}$$

Because the

$$y(x,\lambda)=y(\frac{1}{2},\lambda)V_1(x)+y'(\frac{1}{2},\lambda)V_2(x),$$

where $V_k(x)$ solution of equation (12) with the conditions

$$V_1(\frac{1}{2})=V_2'(\frac{1}{2})=1,\qquad V_1'(\frac{1}{2})=V_2(\frac{1}{2})=0.$$

Then the matrix-vector recording of system (15) will take the form

$$\begin{bmatrix}2C(\frac{1}{2})-\int_{\frac{1}{2}}^{1}Q(x)C(1-x)V_2(x)dx & \int_{\frac{1}{2}}^{1}Q(x)C(1-x)V_1(x)dx\\ 2C(\frac{1}{2})-\int_{\frac{1}{2}}^{1}Q(x)S(1-x)V_2(x)dx & \int_{\frac{1}{2}}^{1}Q(x)S(1-x)V_1(x)dx\end{bmatrix}\begin{bmatrix}y'(\frac{1}{2})\\ -y(\frac{1}{2})\end{bmatrix}=0 \tag{17}$$

A homogeneous system of linear algebraic equations has nontrivial solutions if and only $\Delta(\lambda)=0,$ where $2\Delta(\lambda)$ determinant of system (17) we find the integral representation of the characteristic determinant $\Delta(\lambda)$ system related (17)

$$\Delta(\lambda)=\int_{\frac{1}{2}}^{1}Q(x)V_1(x)\begin{vmatrix}C(\frac{1}{2}) & C(1-x)\\ S(\frac{1}{2}) & S(1-x)\end{vmatrix}dx-\frac{1}{2}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}V_2(x_1)V_1(x_2)Q(x_1)Q(x_2)\begin{vmatrix}C(1-x_1) & C(1-x_2)\\ S(1-x_1) & S(1-x_2)\end{vmatrix}dx_1dx_2 \tag{18}$$

It's not hard to understand that

$$V_2(x) = -\begin{vmatrix} C(\frac{1}{2}) & C(1-x) \\ S(\frac{1}{2}) & S(1-x) \end{vmatrix} - \int_{\frac{1}{2}}^{x} Q(\tau)V_2(\tau)\begin{vmatrix} C(1-\tau) & C(1-x) \\ S(1-\tau) & S(1-x) \end{vmatrix} d\tau$$

$$V_1(x) = -\begin{vmatrix} C'(\frac{1}{2}) & C(1-x) \\ S'(\frac{1}{2}) & S(1-x) \end{vmatrix} - \int_{\frac{1}{2}}^{x} Q(\tau)V_1(\tau)\begin{vmatrix} C(1-\tau) & C(1-x) \\ S(1-\tau) & S(1-x) \end{vmatrix} d\tau \qquad (19)$$

By direct verification we are convinced that $V_1(x),\ V_2(x)$ satisfy

equation (9) and the initial conditions at the point $\frac{1}{2}$. Now we transform (17), Considering (18), (19)

$$\Delta(\lambda) = \int_{\frac{1}{2}}^{1} Q(x)V_1(x)\begin{vmatrix} C(\frac{1}{2}) & C(1-x) \\ S(\frac{1}{2}) & S(1-x) \end{vmatrix} dx - \frac{1}{2}\int_{\frac{1}{2}}^{1} dx_1 \int_{\frac{1}{2}}^{1} dx_2 Q(x_1)V_1(x_1)Q(x_2)V_2(x_2)\begin{vmatrix} C(1-x_1) & C(1-x_2) \\ S(1-x_1) & S(1-x_2) \end{vmatrix} =$$

$$= \int_{\frac{1}{2}}^{1} Q(x)V_1(x)\begin{vmatrix} C(\frac{1}{2}) & C(1-x) \\ S(\frac{1}{2}) & S(1-x) \end{vmatrix} dx -$$

$$-\frac{1}{2}\int_{\frac{1}{2}}^{1} dx_1 \int_{\frac{1}{2}}^{x_1} Q(x_1)V_1(x_1)Q(x_2)V_2(x_2)\begin{vmatrix} C(1-x_1) & C(1-x_2) \\ S(1-x_1) & S(1-x_2) \end{vmatrix} dx_2 +$$

$$+\frac{1}{2}\int_{\frac{1}{2}}^{1} dx_2 \int_{\frac{1}{2}}^{x_2} Q(x_1)V_1(x_1)Q(x_2)V_2(x_2)\begin{vmatrix} C(1-x_1) & C(1-x_2) \\ S(1-x_1) & S(1-x_2) \end{vmatrix} dx_2 =$$

$$= \int_{\frac{1}{2}}^{1} Q(x)V_1(x)\begin{vmatrix} C(\frac{1}{2}) & C(1-x) \\ S(\frac{1}{2}) & S(1-x) \end{vmatrix} dx -$$

$$-\frac{1}{2}\int_{\frac{1}{2}}^{1} Q(x_1)V_1(x_1)\left(V_2(x_1) + \begin{vmatrix} C(\frac{1}{2}) & C(1-x_1) \\ S(\frac{1}{2}) & S(1-x_1) \end{vmatrix} \right) dx_1 - \int_{\frac{1}{2}}^{1} Q(x_2)V_2(x_2)\left(-\begin{vmatrix} C'(\frac{1}{2}) & C(1-x_2) \\ S'(\frac{1}{2}) & S(1-x_2) \end{vmatrix} - V_1(x_2) \right) dx_2 =$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{1} dx_1 Q(x)\begin{vmatrix} C(\frac{1}{2})V_1(x) + C'(\frac{1}{2})V_2(x) & C(1-x) \\ S(\frac{1}{2})V_1(x) + S'(\frac{1}{2})V_2(x) & S(1-x) \end{vmatrix} = \int_{\frac{1}{2}}^{1} Q(x)\begin{vmatrix} C(x) & C(1-x) \\ S(x) & S(1-x) \end{vmatrix} dx$$

Remark 1. Since the conjugate operator to the operator is defined as follows:

$$L_i^* y_i(x_i) = -y_i''(x_i) + q_i(x_i)y_i(x_i) = \lambda y_i(x_i)$$

$$C(0,\lambda) = -C(1,\lambda),\quad S(0,\lambda) = -S(1,\lambda),\quad C'(0,\lambda) = C'(1,\lambda),\quad S'(0,\lambda) = S'(1,\lambda)$$

That's the spectrum L^* also defined using the function $\Delta(\lambda)$ introduced in theorem 1. Representation (1) contains the fundamental system solution $C(x,\lambda), S(x,\lambda)$, which implicitly depend on $q(x)$ and λ. the following theorem gives an explicit dependence of the characteristic determinant $\Delta(\lambda)$ on the spectral parameter [1,2,3,16,17].

Theorem 2. For the characteristic determinant $\Delta(\lambda)$ integral is valid performance

$$\Delta(\lambda)=\int_0^1 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}D(x)dx \tag{20}$$

formulas for $D(x)$ are given below.
Proof. according to the statement

$$\begin{gathered} V_1(x-\frac{1}{2},\lambda)=\cos\sqrt{\lambda}x+\int_{-x}^{x} k_1(x,t)\cos\sqrt{\lambda}\,tdt, \\ V_2(x-\frac{1}{2},\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_{-x}^{x} k_1(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, \end{gathered} \tag{21}$$

where $V_1(t,\lambda),V_2(t,\lambda)$, were introduced in the proof of theorem 1 of this section since the determinant

$$\Psi(t,\lambda)=\begin{vmatrix} C(1-x,\lambda) & S(1-x,\lambda) \\ C(x,\lambda) & S(x,\lambda) \end{vmatrix}$$

when moving from one fundamental system of solutions to another system with the same Wronskian does not change, then instead $C(x,\lambda),S(x,\lambda)$ we can take a fundamental system of solutions $V_1(t,\lambda),V_2(t,\lambda)$

$$\Psi(x,\lambda)=\begin{vmatrix} \cos\sqrt{\lambda}(1-x)+\int_{-x}^{x} k_1(x,t)\cos\sqrt{\lambda}\,tdt & \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}+\int_{-x}^{x} k_1(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt \\ \cos\sqrt{\lambda}x+\int_{-x}^{x} k_1(x,t)\cos\sqrt{\lambda}\,tdt & \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int_{-x}^{x} k_1(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt \end{vmatrix}$$

Since the line determinant is relative to its lines, then when $x\in\left[0,\frac{1}{2}\right]$ we have.

6. Discussion

The investigation of second-order differential operators on network-type manifolds, such as the star-graph geometric graph considered in this study, is a topic of growing interest in various scientific and mathematical disciplines. The unique structure of these network-like domains presents both challenges and opportunities in the analysis of differential operators and the associated boundary value problems.

The Dirichlet problem addressed in this work is a fundamental and well-studied problem in the field of partial differential equations. However, the extension of this problem to network-type manifolds introduces additional complexities that require a more sophisticated treatment. The synthetic approach employed in this paper allows for a comprehensive and rigorous analysis of the problem, leading to a complete characterization of the spectral parameter and the boundary condition coefficients. The non-degeneracy condition (3) plays a crucial role in ensuring the well posed ness of the Dirichlet problem. By analyzing the rank of the matrix A composed of the boundary condition coefficients, we can determine the cases where the problem has a unique solution. This understanding is essential for the practical applications of the theory, as it provides a clear criterion for the admissibility of the boundary conditions. Furthermore, the insights gained from the study of the minors of the matrix A offer valuable information about the underlying structure of the boundary value problem. These minors can be interpreted as generalized Wronskians, revealing intricate relationships between the solutions of the differential equations on the individual edges of the star graph. The techniques developed in this paper can serve as a template for addressing similar problems on more complex network-type manifolds, such as graphs with a larger number of edges or more intricate topological structures. By extending the current framework, researchers can explore the behavior of second-order differential operators in increasingly sophisticated geometric settings, potentially leading to new applications and theoretical advancements. Overall, the discussion presented in this work highlights the significance of the Dirichlet problem for second-order differential operators on network-type manifolds and underscores the importance of the synthetic approach employed in this study. The findings contribute to the broader understanding of these problems and open up avenues for further exploration in the field.

7. Conclusion

In this work, we have provided a comprehensive analysis and solution to the Dirichlet problem for a second-order differential operator defined on a star-graph geometric graph. By employing a synthetic approach, we have fully characterized the spectral parameter, constructed the matrix of boundary condition coefficients, and examined its minors to ensure the non-degeneracy of the problem. The insights gained from this study contribute to the broader understanding of second-order differential operators on network-type manifolds and have potential applications in diverse fields, such as quantum mechanics, control theory, and network analysis. The techniques developed in this paper can serve as a foundation for addressing similar problems on more complex geometric graph structures, further expanding the scope of research in this area.

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