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| RESEARCH ARTICLE

Simulation-Based Study on Extreme Ranked Set Sampling from Rician Distribution

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| ABSTRACT
The RSS approach is a useful method of sampling that reduces the cost and improves the representativeness of the population. It provides more efficient estimators than the competitors based on SRS. However, using RSS could be a difficult task to observe all the ranks. Thus, using only the extreme ranks eases the task and reduces the error in ranking. Samawi et al. (1996) proposed the method of Extreme Ranked Set Sampling (ERSS) to reduce errors in ranking and showed that the method gives an unbiased estimate of the population mean in the case of symmetric populations and it provides a more efficient estimator than SRS. However, the estimator of this method is biased when the distribution is skewed. Many researchers have considered ERSS, investigated several estimators, and studied their properties. In this paper, we adopt the ERSS technique when the samples are drawn from the Rician distribution. Several estimators have been studied, including arithmetic mean, geometric mean, harmonic mean, quadratic mean, median, variance, mean deviation, skewness, and kurtosis. Computer simulations were used to check the properties of these estimators and compared with the corresponding estimators using SRS. Some estimators based on ERSS are more efficient than the corresponding estimators from SRS, but some others are not.

| KEYWORDS

Extreme Ranked Set Sampling, relative efficiency, geometric mean, harmonic mean, skewness and kurtosis.

| ARTICLE INFORMATION

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1. Introduction

Ranked set sampling (RSS) is a cost-effective sampling method that provides estimators with more efficiency than simple random sampling (SRS). It is useful when the measurements of the units are difficult, but they can be easily ranked by judgment before measurement. RSS has been considered by many researchers, and the descriptive and inferential aspects were investigated. In addition, several modifications were proposed to increase the efficiency and reduce ranking errors. Extreme ranked set sampling (ERSS) is one of these modifications suggested by Samawi et al. (1996), and it used only the largest and the smallest ranked. We mention here a simple procedure of ERSS which will be used in this paper:

1. Select randomly m sets each of m units.
2. Identify from each set the unit with the minimum rank for actual quantification.
3. Repeat the above steps but for the maximum ranks.
4. Repeat the above three steps r times until the desired sample size $n = 2rm$ is obtained.

ERSS is a useful modification of RSS proposed by Samawi et al. (1996). It requires identifying only the maximum and minimum.
Case 1: If the set size, m, is even number, select from the first $m/2$ samples the smallest unit, and from the last $m/2$ samples, select the largest unit.

Case 2: If the sample size is an odd number, select from the first $(m-1)/2$ samples the smallest unit, and from the next $(m-1)/2$ samples, select the largest units. From the last sample, take the median rank.

The arithmetic means of the sample based on ERSS in the case when m is even number is

$$\hat{Y}_1 = \frac{1}{rm}\sum_{k=1}^{r}\left[\sum_{i=1}^{L} Y_{ik(m)} + \sum_{i=L+1}^{m} Y_{ik(1)}\right]$$

where $L = m/2$. The estimator is unbiased under symmetric distribution, and the variance of this estimator is

$$\begin{aligned}\operatorname{Var}(\hat{Y}_1) &= \frac{1}{rm^2}\left[\sum_{i=1}^{L}\operatorname{Var}(Y_{i(1)}) + \sum_{i=L+1}^{m}\operatorname{Var}(Y_{i(m)})\right] \\ &= \frac{1}{2rm}\left[\operatorname{Var}(Y_{(1)}) + \operatorname{Var}(Y_{(m)})\right]\end{aligned}$$

For symmetric distribution, we have $\operatorname{Var}(Y_{(1)}) = \operatorname{Var}(Y_{(m)})$. Thus, the variance is reduced to

$$\operatorname{Var}(\hat{Y}_1) = \frac{\operatorname{Var}(Y_{(1)})}{rm}$$

When m is an odd number, then the arithmetic mean is

$$\hat{Y}_2 = \frac{1}{rm}\sum_{k=1}^{r}\left[\sum_{i=1}^{t} Y_{i(1)} + \sum_{i=t+1}^{m-1} Y_{i(m)} + Y_{m(s)}\right]$$

where $t = (m-1)/2$ and $s = (m+1)/2$. The estimator is unbiased under symmetric distribution. The variance of $\hat{Y}$ is

$$\operatorname{Var}(\hat{Y}_2) = \frac{1}{rm^2}\left\{\frac{m-1}{2}\left[\operatorname{Var}(Y_{(1)}) + \operatorname{Var}(Y_{(m)})\right] + \operatorname{Var}(Y_{m(s)})\right\}$$

When the distribution of Y is symmetric, then this variance is reduced to

$$\operatorname{Var}(\hat{Y}_2) = \frac{1}{rm^2}\left\{(m-1)\operatorname{Var}(Y_{(1)}) + \operatorname{Var}(Y_{m(s)})\right\}$$

Samawi et al. (1996) showed that when the distribution of Y is symmetric, then the variance is smaller than the variance of $\bar{Y}$.

In this paper, we consider the Rice distribution or Rician distribution. Its pdf is given by

$$f(x|v,\sigma) = \frac{x}{\sigma^2} exp\left(\frac{-(x^2+v^2)}{2\sigma^2}\right) I_0\left(\frac{xv}{\sigma^2}\right), \qquad 0 \le x < \infty$$

v *and* σ^2 are two parameters and $I_0\left(\frac{xv}{\sigma^2}\right)$ is the modified Bessel function of the first kind with order zero. Figure 1.1 shows the graph of the distribution for different values of its parameter v and with $\sigma = 1$.

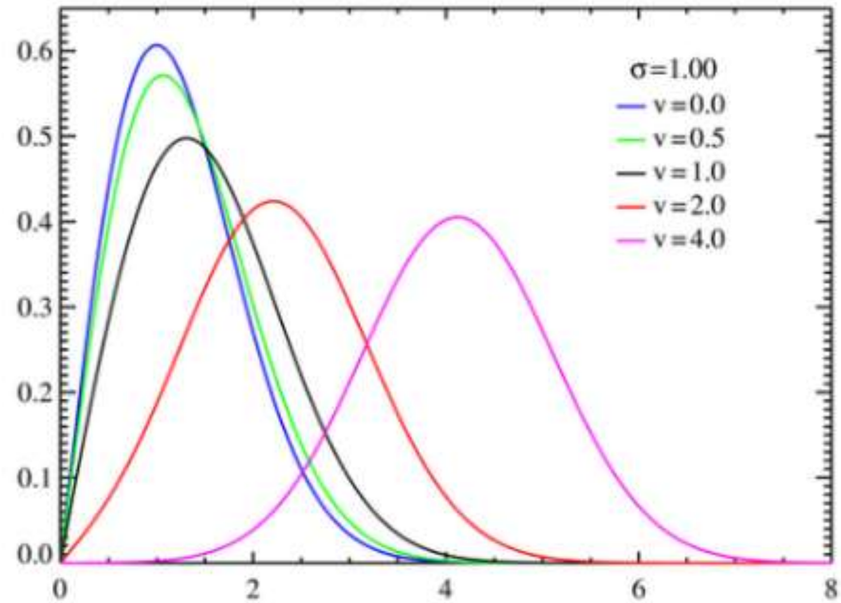

Figure 1: probability density function for Rician distribution with different parameters. Source: https://en.wikipedia.org/wiki/Rice_distribution

2. Literature Review

In some cases, measurement of the units is too expensive, but it is easier to rank the observations before measurement. The ranked set sampling (RSS) technique is the appropriate approach to consider this situation and becomes a very efficient, inexpensive technique for obtaining a more representative sample. The method was proposed by McIntyre (1952) to improve the method of estimation of a population mean. The method has been applied in many fields and has attracted many researchers.

In the Ranked set sampling method, it may be difficult to observe all the ranks, but identifying only the maximum and minimum is an easy task that reduces the error in ranking. Samawi et al. (1996) proposed the method of extreme rank set sampling (ERSS) to reduce errors in ranking and showed that the method gives an unbiased estimate of the population mean in the case of symmetric populations, and it is more efficient than SRS. However, the estimator of this method is not unbiased when the distribution is skewed. The method has become popular and considered by many researchers. The following is important research on ERSS:

Ehsan Zamanzade and M Mahdizadeh (2019) studied ERSS to estimate the population proportion. Shakeel Ahmed Javid Shabbir (2019) proposed a mixture of Extreme Ranked Set Sampling (ERSS) and Median Ranked Set Sampling (MRSS) to obtain a more representative sample and provide an unbiased estimator of the mean for symmetric distribution and gives moderate efficiency for both symmetric and asymmetric populations via simulation study. Shaibu and Muttlak (2004) proposed maximum likelihood estimators of the parameters of the normal, exponential, and gamma using median ranked set sampling (MRSS) and extreme ranked set sampling (ERSS). They also proposed some linear unbiased estimators of the same parameters. The MLE's of the normal mean and the scale parameters of the exponential and gamma distributions under MRSS are found to dominate all other estimators. The MLE's of the normal standard deviation under ERSS are the most efficient among the MLE's. Samawai & Laith J. Saeid (2004) proposed a Stratified extreme ranked set sample (SERSS). They showed that SERSS for estimating the ratios is more efficient than using stratified simple random sample (SSRS) and simple random sample (SRS). In some cases, it is more efficient than RSS and stratified ranked set sample (SRSS) when the underlying distribution is symmetric. Amer Al-Omari & Said Al-Hadhrami (2011) estimated the three parameters as well as the population mean of the modified Weibull distribution using ERSS. The MLE was investigated and compared to the corresponding one based on SRS. It was found that the MLE based on ERSS is more efficient than the MLE using SRS to estimate the three parameters of the MWD. The ERSS estimator of the population mean of the MWD is also more efficient than the SRS. Hashemi Majd & Saba (2017) proposed robust extreme double ranked set sampling (REDRSS). The method was investigated using simulation and showed that using REDRSS gives more efficient estimates of the population mean and smaller variance than the estimate by SRS. Biradar & Santosha (2015) emphasize that the estimation of the population mean based on ERSS has smaller variances than the estimator using SRS and provides more efficient estimators. Samawi et al. (2010) examined several RSS designs for the run test of symmetry. They showed that an optimal RSS design for runs test of symmetry is the ERSS. ERSS increases the power and improves the performance of the symmetry run test, hence reducing the cost and the sample size needed in the study. Hassen A. Muttlak (2010) proposed regression-type estimators based on ERSS for estimating the population. He found that the regression-type estimator of the population mean based on ERSS dominates all other estimators considered. Amjad D. Al-Nasser & Ahmed Bani Mustafa (2009) introduced a robust ERSS (RERSS) procedure for estimating the population mean. The procedure gives an unbiased estimator with a smaller variance when the distribution is symmetric. Simulation was used to investigate the performance of this procedure, and the results showed that the estimator is more efficient than the estimators based on the traditional sampling methods. Hani Samawi (2002) introduced a double extreme ranked set sampling scheme (DERSS). He showed that DERSS gives an unbiased estimator for the population mean when the underlying distribution is symmetric. DERSS is also more efficient than SRS, ERSS, and RSS. DERSS was used to estimate the population mean using the regression method of estimation, and it was shown by simulation that it is more efficient than SRS, ERSS, and RSS. Manoj Chacko (2016) proposed a modification of RSS called ordered extreme ranked set sampling (OERSS) in which an auxiliary variable X correlated with the study variate Y is used to rank the sample units. This modification of RSS is applied to obtain an estimator of the parameter associated with the study variate Y when (X, Y) follows a Morgenstern type bivariate exponential distribution. Fatma Gul Akgul (2017) obtained the maximum likelihood (ML) and the modified maximum likelihood (MML) on ERSS, MRSS, and PRSS when the distribution is Weibull distribution. She showed using simulation that the use of RSS and its modifications are highly efficient when compared to SRS for estimating the system reliability R using the ML and the MML estimators. To reduce the ranking error, she recommended using the estimators based on ERSS, MRSS, and PRSS. Monjed et al. (2020) studied the MLE of the parameters of the new Weibull-Pareto distribution under SRS, RSS, MRSS, and ERSS. All the estimators using RSS, ERSS, and MRSS have smaller biases and MSEs than the corresponding estimators based on SRS. Biases and MSEs in RSS, ERSS, and MRSS decrease as set sizes increase. The MLEs derived by RSS, ERSS, and MRSS are more efficient than SRS estimators. In addition, the widths of the confidence intervals of the parameters based on RSS, ERSS, and MRSS are narrower than the ones constructed by the SRS confidence interval. Hasanalipour, P., and Razmkhah, M. (2021) proposed using ERSS to test the skew-symmetry of data. Some special cases of skew distributions, including skew-normal, skew-Laplace, and skew-logistic distributions, were studied, and the power function of the proposed test was used. They suggested using the ERSS when the data seems to be skewed.

3. Methodology

Rician distribution does not appear in the literature review under the Extreme ranked set sampling (ERSS) scheme. Thus, ERSS is adopted in this paper and compared with Simple Random Sampling (SRS) from Rician distribution. Several estimators are considered: arithmetic, geometric, harmonic, and quadratic mean, median, variance, coefficient of skewness and kurtosis, coefficient of variation, and mean deviation. The bias, variance, and MSE of these estimators are investigated using Monte Carlo simulation. The relative efficiency of ERSS estimators with respect to SRS estimators is obtained.

The following algorithm is used to generate a random sample from Rician distribution Rice(1945).

1- Generate $X \sim N(\upsilon.cos\theta, \sigma^2)$.
2- Generate $Y \sim N(\upsilon.sin\theta, \sigma^2)$. X and Y are independent.
3- $R = \sqrt{X^2 + Y^2}$. $R \sim Rice(|\upsilon|, \sigma)$

Another possible algorithm is as follows:

1- Generate $P \sim Poisson\left(\frac{\upsilon^2}{2\sigma^2}\right)$.
2- Generate $X \sim \chi^2_{(2P+2)}$.
3- $R = \sigma\sqrt{X}$.

The algorithm for the whole program for all estimators:

1. Choose some values for the set size, number of cycles, and number of iterations, as well as values for the Rician distribution parameters. Thus, the sample size is $n = mr$ where m is the set size and r is number of cycles.
2. Generate a random sample from Rician distribution using both SRS and ERSS.
3. Calculate the estimators.
4. Repeat steps (1), (2), and (3) r times.
5. Compute the average, the variance, and the mean square error of the estimators.
6. Calculate the relative efficiency of ERSS with respect to the corresponding estimators from SRS.

4. Results and Discussion

The outcomes of the simulation for each type of estimator will be illustrated in the following subsection.

4.1 Arithmetic mean

The arithmetic mean of Rician distribution is given by

$$\mu = \sigma\sqrt{\frac{\pi}{2}}L_{1/2}\left(\frac{-v^2}{2\sigma^2}\right)$$

where $L_{1/2}\left(\frac{-v^2}{2\sigma^2}\right)$ is the square of Laguerre polynomial. The general formula of this polynomial is given by

$$L_n(x) = \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!\,(r!)^2}x^r.$$

Computer simulation was used to compare the biasness, variance, and MSE of the arithmetic mean from both ERSS and SRS. The relative efficiency is calculated by the ratio MSE(SRS)/MSE(ERSS) where MSE(SRS) is the mean square error of the arithmetic mean from SRS and MSE(ERSS) is the mean square error of the arithmetic mean from ERSS. Table 1 shows the outcomes when the distribution is asymmetric, while Table 2 shows the outcomes when the distribution is symmetric.

When the distribution is asymmetric, as in Table 1, the estimators are biased, and it is less for the SRS estimator. However, when the distribution is symmetric, both estimators are unbiased. In addition, the variances and MSE of both estimators decrease as the sample sizes increase. ERSS estimator has a smaller MSE, and therefore, it is more efficient than the SRS estimator. Moreover, the relative efficiency of ERSS with respect to the SRS estimator increases as the sample size increases.

Table 1 Computer simulation outcome to estimate arithmetic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$.

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0168	0.0057	1.3158	2.3761	1.3161	2.3762	1.81
4	2	0.0408	0.0013	0.8658	1.8023	0.8675	1.8023	2.08
5	2	0.0526	0.0011	0.6192	1.4321	0.6220	1.4321	2.30
6	2	0.0673	0.0002	0.4758	1.2014	0.4803	1.2014	2.50
3	4	0.0157	0.0022	0.6654	1.2001	0.6657	1.2001	1.80
4	4	0.0366	0.0039	0.4325	0.8948	0.4338	0.8948	2.06
5	4	0.0488	0.0001	0.3100	0.7269	0.3124	0.7269	2.33
6	4	0.0649	0.0006	0.2363	0.6028	0.2405	0.6028	2.51
3	6	0.0198	0.0032	0.4419	0.8036	0.4423	0.8036	1.82
4	6	0.0354	0.0020	0.2874	0.5994	0.2886	0.5994	2.08
5	6	0.0486	0.0021	0.2052	0.4822	0.2075	0.4822	2.32
6	6	0.0657	0.0003	0.1582	0.4005	0.1625	0.4005	2.46

Table 2 Computer simulation outcome to estimate arithmetic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0000	0.0002	0.0932	0.1664	0.0932	0.1664	1.78
4	2	0.0010	0.0003	0.0615	0.1247	0.0615	0.1247	2.03
5	2	0.0017	0.0008	0.0447	0.1000	0.0447	0.1000	2.24
6	2	0.0000	0.0005	0.0345	0.0834	0.0345	0.0834	2.42
3	4	0.0008	0.0012	0.0462	0.0829	0.0462	0.0829	1.80
4	4	0.0003	0.0011	0.0307	0.0624	0.0307	0.0624	2.04
5	4	0.0005	0.0003	0.0223	0.0501	0.0223	0.0501	2.25
6	4	0.0000	0.0006	0.0174	0.0413	0.0174	0.0413	2.38
3	6	0.0008	0.0002	0.0311	0.0557	0.0311	0.0557	1.79
4	6	0.0008	0.0006	0.0205	0.0418	0.0205	0.0418	2.04
5	6	0.0004	0.0002	0.0149	0.0332	0.0149	0.0332	2.23
6	6	0.0001	0.0010	0.0116	0.0279	0.0116	0.0279	2.40

4.2 Geometric mean

The geometric mean can be used to obtain the mean of ratios and percentages, such as the growth of a population, financial indices, and interest rates. The geometric mean is defined by

$$G = \sqrt[n]{x_1 . x_2 \dots x_n}$$

It is only appropriate for positive observations. It has a smaller value than the arithmetic mean and attains equality when all the observations are equal. Tables 3 and 4 demonstrate the simulation results to investigate the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 3 and symmetric in Table 4.

Table 3 Computer simulation outcome to estimate geometric mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$ (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.1097	0.1409	1.8574	2.8366	1.8695	2.8565	1.53
4	2	0.3286	0.1092	1.3503	2.1841	1.4583	2.1960	1.51
5	2	0.5294	0.0889	1.0463	1.7450	1.3266	1.7529	1.32
6	2	0.7025	0.0757	0.8642	1.4739	1.3577	1.4796	1.09
3	4	0.1606	0.0775	0.9706	1.4694	0.9963	1.4754	1.48
4	4	0.3715	0.0515	0.6913	1.1095	0.8292	1.1122	1.34
5	4	0.5630	0.0454	0.5359	0.9017	0.8528	0.9038	1.06
6	4	0.7267	0.0380	0.4378	0.7533	0.9659	0.7547	0.78
3	6	0.1699	0.0543	0.6480	0.9912	0.6769	0.9942	1.47
4	6	0.3832	0.0358	0.4641	0.7487	0.6109	0.7500	1.23
5	6	0.5750	0.0329	0.3595	0.6003	0.6900	0.6014	0.87
6	6	0.7347	0.0246	0.2945	0.5003	0.8342	0.5009	0.60

Table 4 Computer simulation outcome to estimate geometric mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0064	0.0060	0.0944	0.1674	0.0944	0.1674	1.77
4	2	0.0178	0.0041	0.0624	0.1255	0.0628	0.1255	2.00
5	2	0.0244	0.0027	0.0456	0.1006	0.0462	0.1006	2.18
6	2	0.0341	0.0024	0.0353	0.0839	0.0364	0.0839	2.30
3	4	0.0087	0.0040	0.0468	0.0835	0.0469	0.0835	1.78
4	4	0.0177	0.0034	0.0312	0.0629	0.0315	0.0629	2.00
5	4	0.0265	0.0021	0.0227	0.0504	0.0234	0.0504	2.15
6	4	0.0346	0.0009	0.0178	0.0416	0.0190	0.0416	2.19
3	6	0.0093	0.0021	0.0315	0.0561	0.0316	0.0561	1.78
4	6	0.0191	0.0020	0.0209	0.0421	0.0212	0.0421	1.98
5	6	0.0267	0.0010	0.0152	0.0334	0.0159	0.0334	2.10
6	6	0.0350	0.0000	0.0119	0.0281	0.0131	0.0281	2.13

The simulation outcomes in Tables 3 and 4 demonstrate the following:

- Both estimators are biased, and the ERSS estimator has a bigger bias for a set size bigger than 3. The bias of SRS estimator decreases as the sample size increases. However, ERSS estimator inflates with sample size.
- The amount of bias when the distribution is symmetric is smaller than for asymmetric distribution.
- The variances and MSEs of both estimators get smaller when the sample size gets bigger.
- The MSE of ERSS estimator is less than the counterpart estimator from SRS in most of the cases.
- When the distribution is asymmetric, as in Table 3, the ERSS estimator is more efficient than the SRS estimator when the set size is less than 6. The relative efficiency decreases when either the set size or the number of cycles increases.
- When the distribution is symmetric, the relative efficiency is always greater than 1. It also increases as the set size increases.
- The relative efficiency is higher when the distribution is symmetric than when the distribution is asymmetrical.

4.3 Harmonic mean

The harmonic mean is defined by

$$H = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$$

The harmonic mean has a smaller value than other means and has the same value if all observations are of the same value. Tables 5 and 6 display the simulation results to investigate the estimators from ERSS and SRS. The samples are selected from the Rician distribution when the parameters make the distribution asymmetric in Table 7 and symmetric in Table 8.

Table 5 Computer simulation outcome to estimate harmonic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$(asymmetric case).

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0737	0.5959	3.3180	4.0019	3.3234	4.3570	1.31
4	2	0.4202	0.4912	2.8418	3.3345	3.0183	3.5759	1.18
5	2	0.8481	0.4237	2.4550	2.8095	3.1743	2.9890	0.94
6	2	1.2150	0.3755	2.2221	2.5072	3.6982	2.6482	0.72
3	4	0.1379	0.3762	2.2213	2.4906	2.2404	2.6321	1.17
4	4	0.6182	0.2980	1.8694	2.0487	2.2516	2.1375	0.95
5	4	1.0286	0.2581	1.6171	1.7655	2.6752	1.8322	0.68
6	4	1.3740	0.2248	1.4447	1.5601	3.3325	1.6106	0.48
3	6	0.2149	0.2865	1.7045	1.8576	1.7507	1.9397	1.11
4	6	0.6955	0.2217	1.4519	1.5567	1.9357	1.6058	0.83
5	6	1.1093	0.1954	1.2698	1.3065	2.5004	1.3447	0.54
6	6	1.4441	0.1681	1.1153	1.1400	3.2006	1.1682	0.37

Table 6 Computer simulation outcome to estimate harmonic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0126	0.0119	0.0966	0.1691	0.0968	0.1693	1.75
4	2	0.0345	0.0086	0.0644	0.1269	0.0655	0.1269	1.94
5	2	0.0504	0.0063	0.0473	0.1017	0.0499	0.1018	2.04
6	2	0.0681	0.0054	0.0369	0.0848	0.0415	0.0849	2.05
3	4	0.0166	0.0070	0.0480	0.0844	0.0482	0.0844	1.75
4	4	0.0356	0.0057	0.0322	0.0636	0.0335	0.0636	1.90
5	4	0.0533	0.0039	0.0236	0.0510	0.0264	0.0510	1.93
6	4	0.0692	0.0024	0.0185	0.0421	0.0233	0.0421	1.80
3	6	0.0177	0.0041	0.0323	0.0567	0.0326	0.0567	1.74
4	6	0.0375	0.0034	0.0215	0.0426	0.0229	0.0426	1.86
5	6	0.0538	0.0022	0.0158	0.0338	0.0187	0.0338	1.81
6	6	0.0699	0.0011	0.0125	0.0284	0.0174	0.0284	1.63

Tables 5 and 6 show the following:

- The estimators are biased, and the amount of bias for the ERSS estimator is bigger than the SRS estimator in most cases except when the set size is 3. In addition, the bias of ERSS becomes bigger as the set size becomes bigger.
- When the distribution is symmetric, as in Table 6, the amount of bias is smaller than that for asymmetric distribution, as shown in Table 5.
- In all cases, the variance of the ERSS estimator is less than the SRS estimator. The variance of both estimators gets smaller when the sample size gets bigger.
- When the distribution is asymmetric, the MSE of the SRS estimator is less than the ERSS estimator except in the case of a set size of 3. Thus, the relative efficiency of the ERSS estimator is greater than 1 only with a set size of 3.
- When the distribution is symmetric, the ERSS estimator has higher relative efficiency than in the case of asymmetric distribution. The relative efficiency also increases as the set size increases.

4.4 Quadratic mean

The quadratic mean is defined by

$$Q = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

It gives greater weight to larger items in the data, and it is greater than arithmetic, geometric, and harmonic means. Tables 7 and 8 demonstrate the simulation results to investigate the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 7 and symmetric in Table 8.

Table 7 Computer simulation outcome to estimate quadratic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.1256	0.1081	1.3633	2.3332	1.3791	2.3449	1.70
4	2	0.3324	0.0788	0.9222	1.7685	1.0327	1.7747	1.72
5	2	0.5017	0.0636	0.6742	1.4024	0.9259	1.4065	1.52
6	2	0.6511	0.0526	0.5260	1.1782	0.9499	1.1809	1.24
3	4	0.1546	0.0498	0.6925	1.1775	0.7164	1.1800	1.65
4	4	0.3496	0.0414	0.4624	0.8740	0.5846	0.8757	1.50
5	4	0.5121	0.0311	0.3379	0.7106	0.6002	0.7116	1.19
6	4	0.6587	0.0253	0.2621	0.5881	0.6960	0.5887	0.85
3	6	0.1662	0.0312	0.4618	0.7873	0.4895	0.7882	1.61
4	6	0.3539	0.0285	0.3074	0.5863	0.4326	0.5871	1.36
5	6	0.5174	0.0187	0.2245	0.4722	0.4922	0.4726	0.96
6	6	0.6631	0.0169	0.1759	0.3919	0.6157	0.3922	0.64

Table 8 Computer simulation outcome to estimate quadratic mean from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0063	0.0054	0.0931	0.1661	0.0932	0.1661	1.78
4	2	0.0156	0.0046	0.0615	0.1244	0.0618	0.1244	2.01
5	2	0.0275	0.0042	0.0447	0.0997	0.0454	0.0997	2.20
6	2	0.0338	0.0035	0.0345	0.0832	0.0356	0.0832	2.34
3	4	0.0071	0.0016	0.0461	0.0827	0.0462	0.0827	1.79
4	4	0.0181	0.0011	0.0307	0.0623	0.0310	0.0623	2.01
5	4	0.0271	0.0014	0.0223	0.0500	0.0230	0.0500	2.17
6	4	0.0344	0.0020	0.0174	0.0413	0.0186	0.0413	2.22
3	6	0.0076	0.0016	0.0311	0.0555	0.0311	0.0555	1.78
4	6	0.0174	0.0008	0.0206	0.0417	0.0209	0.0417	2.00
5	6	0.0272	0.0013	0.0148	0.0332	0.0156	0.0332	2.13
6	6	0.0345	0.0020	0.0116	0.0278	0.0128	0.0278	2.17

Remarks from Tables 7 and 8 are given below:

- The estimators are biased, and the SRS estimator has a smaller bias than the ERSS estimator. While the bias of the SRS estimator gets smaller with the increase in sample size, the bias of the ERSS estimator gets bigger.
- The bias of ERSS estimator when the distribution is symmetric is less than when the distribution is asymmetric.
- The variance of ERSS estimator is less than SRS estimator. Both variances decrease when the sample size increases.
- When the distribution is asymmetric, the MSE of the ERSS estimator is less than the SRS estimator in most of the cases, and hence, the relative efficiency is greater than 1 for most of the cases. The relative efficiency gets smaller as either the set size or the cycles get bigger.
- The relative efficiency of ERSS for symmetric distribution is bigger than that for asymmetric distribution. The relative efficiency increases as the set size increases.

4.5 Median

Unlike the previous types of mean, the data is arranged in ascending or descending order to identify the median. It is defined as the mid value in a set of observations so that 50% of the observations are below the median. It is also known as the 50th percentile. It is a measure of central tendency, although it utilizes only the values in the middle and thus loses much available information in the sample. Tables 9 and 10 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 9 and symmetric in Table 10.

Table 9 Computer simulation outcome to estimate median from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0166	0.0052	1.9408	3.1541	1.9410	3.1541	1.62
4	2	0.0162	0.0112	1.4358	2.4807	1.4361	2.4809	1.73
5	2	0.0133	0.0083	1.1425	2.0425	1.1427	2.0426	1.79
6	2	0.0180	0.0030	0.9496	1.7276	0.9499	1.7276	1.82
3	4	0.0028	0.0094	1.2007	1.7310	1.2007	1.7311	1.44
4	4	0.0046	0.0027	0.9539	1.3266	0.9539	1.3266	1.39
5	4	0.0049	0.0037	0.7976	1.0895	0.7977	1.0895	1.37
6	4	0.0033	0.0054	0.6870	0.9150	0.6870	0.9150	1.33
3	6	0.0067	0.0048	0.8671	1.2022	0.8671	1.2022	1.39
4	6	0.0022	0.0014	0.7185	0.9041	0.7185	0.9041	1.26
5	6	0.0018	0.0041	0.6348	0.7438	0.6348	0.7438	1.17
6	6	0.0036	0.0013	0.5655	0.6212	0.5655	0.6212	1.10

Table 10 Computer simulation outcome to estimate median from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0013	-0.0007	0.1312	0.2138	0.1312	0.2138	1.63
4	2	0.0008	0.0005	0.0969	0.1678	0.0969	0.1678	1.73
5	2	-0.0014	0.0000	0.0776	0.1378	0.0776	0.1378	1.78
6	2	0.0005	0.0006	0.0646	0.1178	0.0646	0.1178	1.82
3	4	0.0001	-0.0005	0.0811	0.1175	0.0811	0.1175	1.45
4	4	0.0010	-0.0012	0.0643	0.0907	0.0643	0.0907	1.41
5	4	-0.0010	0.0002	0.0539	0.0732	0.0539	0.0732	1.36
6	4	0.0009	0.0006	0.0461	0.0614	0.0461	0.0614	1.33
3	6	0.0017	-0.0004	0.0582	0.0816	0.0582	0.0816	1.40
4	6	0.0017	-0.0004	0.0486	0.0621	0.0486	0.0621	1.28
5	6	-0.0011	-0.0008	0.0423	0.0499	0.0423	0.0499	1.18
6	6	0.0001	0.0006	0.0383	0.0423	0.0383	0.0423	1.10

Tables 9 and 10 show the following:

- The amount of bias of ERSS is small and gets smaller as the sample size gets bigger.
- The amount of bias when the distribution is symmetric is smaller than for asymmetric distribution.
- In all cases, the ERSS estimator has a smaller variance and MSE than the SRS estimator.
- The variances of both methods decrease as the sample size increases.
- The relative efficiency is greater than 1 but gets smaller as the number of cycles gets bigger. That is ERSS estimator performs better than the SRS estimator and better for small number of cycles.
- The ERSS estimator has almost the same relative efficiency for both symmetric and asymmetric cases.

4.6 Variance

The variance of Rician distribution is

$$Var(x) = 2\sigma^2 + v^2 - \left(\frac{\pi\sigma^2}{2}\right) L_{1/2}^2\left(\frac{-v^2}{2\sigma^2}\right)$$

where $L_{1/2}^2\left(\frac{-v^2}{2\sigma^2}\right)$ is the square of Laguerre polynomial. The general formula of this polynomial is given by $L_n(x) = \sum_{r=0}^{n}(-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r$.

Tables 11 and 12 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from Rician distribution when the parameters make the distribution asymmetric in Table 11 and symmetric in Table 12.

Table 11 Computer simulation outcome to estimate variance from ERSS and SRS samples selected from Rician distribution with $v = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	5.9794	0.0067	110.7177	78.4490	146.4711	78.4491	0.54
4	2	9.9616	0.0135	89.4374	56.6259	188.6717	56.6260	0.30
5	2	13.4787	0.0255	74.3782	43.1254	256.0525	43.1261	0.17
6	2	16.5492	0.0266	63.4723	35.4175	337.3487	35.4182	0.10
3	4	4.8675	0.0138	47.6484	35.2898	71.3406	35.2900	0.49
4	4	8.8428	0.0146	39.5098	26.0064	117.7042	26.0066	0.22
5	4	12.3337	0.0134	33.4697	20.2048	185.5904	20.2050	0.11
6	4	15.4268	0.0096	29.1531	16.8177	267.1385	16.8178	0.06
3	6	4.4861	0.0022	30.1952	22.9514	50.3202	22.9514	0.46
4	6	8.4609	0.0251	25.1772	16.7648	96.7633	16.7654	0.17
5	6	11.9983	0.0073	21.7821	13.4296	165.7409	13.4296	0.08
6	6	15.0842	0.0090	18.9426	11.0731	246.4767	11.0732	0.04

Table 12 Computer simulation outcome to estimate variance from ERSS and SRS samples selected from Rician distribution with $v = 14.5$ and $\sigma = 1$. (symmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.4204	0.0003	0.5794	0.3985	0.7561	0.3985	0.53
4	2	0.6969	0.0014	0.4635	0.2831	0.9491	0.2831	0.30
5	2	0.9490	0.0006	0.3911	0.2210	1.2918	0.2210	0.17
6	2	1.1658	0.0022	0.3417	0.1796	1.7008	0.1796	0.11
3	4	0.3413	0.0018	0.2466	0.1825	0.3630	0.1825	0.50
4	4	0.6197	0.0025	0.2065	0.1321	0.5906	0.1321	0.22
5	4	0.8709	0.0001	0.1797	0.1043	0.9382	0.1043	0.11
6	4	1.0887	0.0006	0.1560	0.0862	1.3412	0.0862	0.06
3	6	0.3175	0.0005	0.1552	0.1165	0.2560	0.1165	0.45
4	6	0.5962	0.0017	0.1330	0.0863	0.4885	0.0863	0.18
5	6	0.8434	0.0007	0.1149	0.0687	0.8263	0.0687	0.08
6	6	1.0653	0.0015	0.1015	0.0562	1.2365	0.0562	0.05

Tables 11 and 12 demonstrate that the ERSS estimator of variance performs worse than the variance from SRS, as the following points show:

- The bias of the ERSS estimator is high compared to that of the SRS estimator. The SRS estimator is unbiased, but the fluctuation is due to simulation error, and it gets smaller as the sample size gets bigger.

- The amount of bias when the distribution is symmetric is less than when the distribution is asymmetric.
- The variances of both estimators get smaller as the sample size becomes bigger.
- The ERSS estimator shows a larger MSE than the SRS estimator.
- For both cases, symmetric and asymmetric distribution, the relative efficiency is always smaller than 1 and gets smaller as the sample size increases. So, ERSS is not recommended to estimate the variance parameter.

4.7 Skewness

Skewness is a measure of the skewness of a distribution. If one tail of a distribution is longer than another, the distribution is skewed or asymmetric. On the other hand, if the two halves of the distribution are identical, it is symmetric. Several methods can be used to check the skewness of a distribution. Plotting the data is a way to visualize the skewness. Coefficients are also available to measure the skewness, such as using the third moment around the average divided by the cube value of the standard deviation. If the coefficient is zero or around zero, then the distribution is symmetric. Otherwise, if it is negative or positive, then the distribution is skewed to the left or skewed to the right, respectively. Tables 13 and 14 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 13 and symmetric in Table 14.

Table 13 Computer simulation outcome to estimate skewness from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0808	0.0819	0.2387	0.3735	0.2453	0.3802	1.55
4	2	0.0670	0.0746	0.1324	0.3478	0.1369	0.3534	2.58
5	2	0.0628	0.0643	0.0746	0.3185	0.0785	0.3227	4.11
6	2	0.0605	0.0539	0.0463	0.2863	0.0499	0.2892	5.79
3	4	0.0530	0.0551	0.1709	0.2886	0.1737	0.2916	1.68
4	4	0.0466	0.0425	0.0823	0.2367	0.0844	0.2385	2.83
5	4	0.0476	0.0388	0.0442	0.2000	0.0464	0.2015	4.34
6	4	0.0482	0.0329	0.0267	0.1740	0.0291	0.1751	6.02
3	6	0.0432	0.0390	0.1260	0.2160	0.1279	0.2175	1.70
4	6	0.0385	0.0339	0.0581	0.1726	0.0596	0.1738	2.92
5	6	0.0420	0.0264	0.0309	0.1421	0.0326	0.1428	4.38

Table 14 Computer simulation outcome to estimate skewness from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$ (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0028	0.0007	0.2502	0.3782	0.2503	0.3782	1.51
4	2	0.0014	0.0000	0.1433	0.3632	0.1433	0.3632	2.53
5	2	0.0006	0.0003	0.0869	0.3377	0.0869	0.3377	3.88
6	2	0.0015	0.0027	0.0553	0.3091	0.0553	0.3091	5.59
3	4	0.0011	0.0007	0.1889	0.3082	0.1889	0.3083	1.63
4	4	0.0012	0.0028	0.0961	0.2614	0.0961	0.2614	2.72
5	4	0.0001	0.0021	0.0540	0.2231	0.0540	0.2231	4.13
6	4	0.0001	0.0008	0.0335	0.1940	0.0335	0.1940	5.79
3	6	0.0022	0.0020	0.1434	0.2425	0.1434	0.2425	1.69
4	6	0.0000	0.0007	0.0694	0.1954	0.0694	0.1954	2.82
5	6	0.0004	0.0022	0.0388	0.1660	0.0388	0.1660	4.28
6	6	0.0005	0.0009	0.0237	0.1406	0.0237	0.1406	5.94

From Tables 13 and 14, we notice the following:

- The estimators are biased, but the amount of bias decreases as the set size increases. The bias when the distribution is symmetric is less than that for skewed.
- The variances of the estimators decrease as the sample size increases.
- The variance and the MSE of ERSS estimator are less than the counterpart based on SRS.
- The relative efficiency of ERSS estimator is bigger than 1 and increases with the sample size. When the distribution is skewed, the relative efficiency is slightly better than that when the distribution is symmetric. In conclusion, ERSS provides a better estimator than SRS when estimating the skewness.

4.8 Kurtosis

Kurtosis is a measure of the shape of a distribution, whether it is normal (mesokurtic), heavy-tailed (leptokurtic), or light-tailed (platykurtic). The kurtosis of the data can be checked using graphs such as histograms or by using formulas such as the ratio of the fourth moment around the average and the fourth power of the standard deviation. If the coefficient of kurtosis is 3, then it is close to the normal distribution, which is called mesokurtic. If the coefficient is lower than 3, then the distribution is platykurtic, while with a high coefficient, it is leptokurtic. Tables 15 and 16 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 15 and symmetric in Table 16.

Table 15 Computer simulation outcome to estimate kurtosis from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.9573	0.7343	0.2725	0.3519	1.1889	0.8911	0.75
4	2	1.0400	0.5578	0.2154	0.4802	1.2971	0.7913	0.61
5	2	1.1480	0.4437	0.1422	0.5364	1.4601	0.7333	0.50
6	2	1.2348	0.3721	0.0961	0.5528	1.6210	0.6913	0.43
3	4	0.6837	0.3676	0.3407	0.5618	0.8082	0.6969	0.86
4	4	0.8839	0.2738	0.1870	0.5444	0.9683	0.6194	0.64
5	4	1.0435	0.2210	0.1085	0.5021	1.1974	0.5510	0.46
6	4	1.1611	0.1791	0.0668	0.4650	1.4149	0.4971	0.35
3	6	0.5921	0.2432	0.2986	0.5213	0.6492	0.5805	0.89
4	6	0.8310	0.1786	0.1484	0.4717	0.8390	0.5036	0.60
5	6	1.0102	0.1471	0.0819	0.4121	1.1024	0.4337	0.39

Table 16 Computer simulation outcome to estimate kurtosis from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$(symmetric case).

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	1.0723	0.8592	0.2799	0.3526	1.4297	1.0908	0.76
4	2	1.1478	0.6687	0.2296	0.4978	1.5469	0.9450	0.61
5	2	1.2432	0.5434	0.1624	0.5776	1.7078	0.8728	0.51
6	2	1.3271	0.4592	0.1127	0.5991	1.8738	0.8100	0.43
3	4	0.7712	0.4605	0.3815	0.6048	0.9763	0.8169	0.84
4	4	0.9650	0.3524	0.2147	0.6107	1.1459	0.7350	0.64
5	4	1.1224	0.2861	0.1286	0.5732	1.3885	0.6551	0.47
6	4	1.2374	0.2423	0.0819	0.5404	1.6131	0.5991	0.37
3	6	0.6661	0.3154	0.3418	0.6011	0.7854	0.7006	0.89
4	6	0.9028	0.2438	0.1777	0.5365	0.9927	0.5959	0.60
5	6	1.0786	0.1887	0.1019	0.5031	1.2653	0.5387	0.43
6	6	1.2065	0.1673	0.0628	0.4373	1.5185	0.4653	0.31

The results provided in Tables 15 and 16 show that:

- The estimators are biased, and the bias for the ERSS estimator is higher than that for the SRS estimator. However, the amount of bias decreases as the number of cycles increases and increases with the increase of the set size. Also, the bias when the distribution is asymmetric is less than that for symmetric.
- The variances of the estimators decrease with the increase of the set size. However, the variance and MSE of the ERSS estimator are bigger than those based on SRS.
- The relative efficiency of ERSS estimator with respect to the SRS estimator is always less than 1 and, hence, worse performance. The efficiencies are of the same values for both symmetric and asymmetric distributions.

4.9 Coefficient of variation

Coefficient of variation is the ratio of the standard deviation to the mean. So, it is unit-free and suitable for comparing the variation of two quantitative data sets of different units.

Tables 17 and 18 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from Rician distribution when the parameters make the distribution asymmetric in Table 17 and symmetric in Table 18.

Table 17 Computer simulation outcome to estimate coefficient of variation from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$. (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0553	0.0101	0.0128	0.0133	0.0159	0.0134	0.84
4	2	0.0974	0.0075	0.0086	0.0097	0.0181	0.0097	0.54
5	2	0.1307	0.0058	0.0062	0.0075	0.0233	0.0075	0.32
6	2	0.1572	0.0048	0.0048	0.0061	0.0295	0.0061	0.21
3	4	0.0495	0.0047	0.0058	0.0061	0.0082	0.0061	0.74
4	4	0.0903	0.0029	0.0040	0.0045	0.0121	0.0045	0.37
5	4	0.1229	0.0026	0.0029	0.0035	0.0180	0.0035	0.20
6	4	0.1494	0.0022	0.0023	0.0029	0.0246	0.0029	0.12
3	6	0.0470	0.0030	0.0037	0.0040	0.0059	0.0040	0.67
4	6	0.0877	0.0023	0.0026	0.0029	0.0103	0.0029	0.29
5	6	0.1205	0.0018	0.0019	0.0023	0.0164	0.0023	0.14

Table 18 Computer simulation outcome to estimate coefficient of variation from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$(symmetric case).

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.0103	0.0033	0.0005	0.0005	0.0006	0.0005	0.79
4	2	0.0191	0.0024	0.0003	0.0003	0.0007	0.0003	0.49
5	2	0.0260	0.0019	0.0002	0.0003	0.0009	0.0003	0.28
6	2	0.0316	0.0016	0.0002	0.0002	0.0012	0.0002	0.18
3	4	0.0095	0.0015	0.0002	0.0002	0.0003	0.0002	0.70
4	4	0.0179	0.0012	0.0002	0.0002	0.0005	0.0002	0.33
5	4	0.0247	0.0009	0.0001	0.0001	0.0007	0.0001	0.17
6	4	0.0302	0.0007	0.0001	0.0001	0.0010	0.0001	0.10
3	6	0.0093	0.0010	0.0001	0.0001	0.0002	0.0001	0.61
4	6	0.0176	0.0007	0.0001	0.0001	0.0004	0.0001	0.25
5	6	0.0242	0.0006	0.0001	0.0001	0.0007	0.0001	0.12
6	6	0.0298	0.0005	0.0001	0.0001	0.0009	0.0001	0.07

Tables 17 and 18 give the following remarks:

- The estimators are biased, and the amount of bias for the ERSS estimator is higher than that for the SRS estimator. The bias of the ERSS estimator increases with the set size but decreases with the number of cycles.

- The bias when the distribution is symmetric is less than that for asymmetric.
- The variances of the estimators decrease with the increase of the set size.
- The variance and the MSE of the ERSS estimator are bigger than those based on SRS.

We conclude from the above remarks that ERSS performs worse than the corresponding estimator from SRS. The relative efficiency of the ERSS estimator with respect to the SRS estimator is always less than 1. The relative efficiency when the distribution is asymmetric is slightly higher than when it is symmetric.

4.10 Mean deviation

Mean deviation is a measure of variation and defined as the average of the absolute deviation of the data from their average. It is similar to the variance but uses the absolute value rather than the squares. Tables 19 and 20 show the simulation results of the estimators from ERSS and SRS. The samples are selected randomly from the Rician distribution when the parameters make the distribution asymmetric in Table 19 and symmetric in Table 20.

Table 19 Computer simulation outcome to estimate mean deviation from ERSS and SRS samples selected from Rician distribution with $\upsilon = 10$ and $\sigma = 4$ (asymmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.4147	0.2696	0.9981	0.8371	1.1701	0.9098	0.78
4	2	0.9602	0.2025	0.7581	0.6389	1.6802	0.6799	0.40
5	2	1.4133	0.1615	0.5825	0.5061	2.5799	0.5322	0.21
6	2	1.7837	0.1342	0.4597	0.4244	3.6412	0.4424	0.12
3	4	0.4576	0.1341	0.4927	0.4236	0.7021	0.4415	0.63
4	4	0.9807	0.0976	0.3729	0.3197	1.3347	0.3293	0.25
5	4	1.4181	0.0800	0.2887	0.2526	2.2998	0.2590	0.11
6	4	1.7840	0.0666	0.2293	0.2133	3.4120	0.2177	0.06
3	6	0.4673	0.0887	0.3272	0.2846	0.5455	0.2925	0.54
4	6	0.9834	0.0689	0.2466	0.2117	1.2137	0.2164	0.18
5	6	1.4218	0.0529	0.1920	0.1714	2.2135	0.1742	0.08

Table 20 Computer simulation outcome to estimate mean deviation from ERSS and SRS samples selected from Rician distribution with $\upsilon = 14.5$ and $\sigma = 1$(symmetric case)

Set size	cycles	$Bias$ $(ERSS)$	$Bias$ (SRS)	Var $(ERSS)$	Var (SRS)	MSE $(ERSS)$	MSE (SRS)	eff
3	2	0.1111	0.0693	0.0721	0.0594	0.0845	0.0642	0.76
4	2	0.2548	0.0519	0.0541	0.0446	0.1190	0.0473	0.40
5	2	0.3753	0.0411	0.0418	0.0360	0.1827	0.0377	0.21
6	2	0.4737	0.0349	0.0337	0.0297	0.2581	0.0309	0.12
3	4	0.1214	0.0335	0.0351	0.0301	0.0498	0.0312	0.63
4	4	0.2598	0.0262	0.0267	0.0225	0.0941	0.0232	0.25
5	4	0.3772	0.0202	0.0210	0.0179	0.1633	0.0183	0.11
6	4	0.4739	0.0169	0.0166	0.0150	0.2412	0.0153	0.06
3	6	0.1251	0.0222	0.0231	0.0199	0.0387	0.0204	0.53
4	6	0.2614	0.0159	0.0178	0.0150	0.0861	0.0152	0.18
5	6	0.3768	0.0139	0.0138	0.0120	0.1558	0.0122	0.08
6	6	0.4744	0.0114	0.0111	0.0099	0.2362	0.0101	0.04

The Tables 19 and 20 demonstrate the following:

- The estimators are biased, and the amount of bias for the ERSS estimator is higher than that for the SRS estimator in both symmetric and asymmetric cases. Also, the bias in the case of ERSS increases with the set size.
- The bias when the distribution is symmetric is less than that for asymmetric.

- The variances of the estimators decrease as the set size increases.
- The variance and the MSE of the ERSS estimator are bigger than that based on SRS.
  In conclusion, the ERSS estimator performs worse than the SRS estimator when estimating the mean deviation, as the relative efficiency is always less than 1 and gets worse with an increase in sample size.

5. Conclusion remarks

Extreme ranked set sampling (ERSS) is considered when the samples are selected from the Rician distribution. ERSS is a modification of ranked set sampling (RSS), and it requires identification of the minimum and maximum of a set of items before measurement and then measuring only the two extreme cases. We considered ten estimators: arithmetic mean, geometric mean, quadratic mean, harmonic mean, variance, median, skewness, kurtosis, coefficient of variation, and mean deviation. We investigated the properties of these estimators using computer simulation using the R-studio program and compared them with the corresponding estimators from simple random sampling (SRS). The comparison included the biasness, variance, MSE, and relative efficiency of the estimators.

Based on the simulation results, we found that arithmetic mean, geometric mean, quadratic mean, harmonic mean, median, and skewness using ERSS from Rician distribution are more efficient than the corresponding estimators by using SRS. On the other hand, estimators of the variance, kurtosis, coefficient of variation, and mean deviation using SRS are more efficient than those from ERSS.

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References

[1] AL-NASSER AD, & MUSTAFA AB. (2009.) Robust extreme ranked set sampling. *Journal of Statistical Computation and Simulation*, 79(7), 859-867.
[2] Amer A & Said A (2011). On Maximum Likelihood Estimators of the Parameters of a Modified Weibull Distribution Using Extreme Ranked Set Sampling. *Journal of Modern Applied Statistical Methods*. (10),2
[3] Biradar, B. S. & Santosha C. D.(2015). Estimation of the Population Mean Based on Extremes Ranked Set Sampling. American *Journal of Mathematics and Statistics* 5(1): 32-36.
[4] Ehsan Z and Mahdizadeh M (2019).Using ranked set sampling with extreme ranks in estimating the population proportion. *Sage journal.* 29, 1
[5] Fatma G A (2017). Estimation of using Modifications of Ranked Set Sampling for Weibull Distribution.*Pakistan Journal of statistics and operation research*. (13), 4.
[6] Hani M. Samawai L and Saeid J. (2004). Stratified Extreme Ranked Set Sample With Application To Ratio Estimators. *Journal of Modern Applied Statistical Methods* 3 | 1.
[7] Hani M. Samawi, R, Vogel L., and Christopher K.S (2010). On Distribution-Free Runs Test for Symmetry Using Extreme Ranked Set Sampling with an Application Involving Base Deficit Score. *Journal of Statistical Theory and Practice*. 4, 2. 289-301
[8] Hani, S (2002). On double extreme rank set sample with application to regression estimator. *International Journal of statistics*. 50-63
[9] Hasanalipour, P., and Razmkhah, M. (2021).Testing skew-symmetry based on extreme ranked set sampling. Stat Papers 62, 2311–2332.
[10] Hassen A. M (2010). Regression estimators in extreme and median ranked set samples. *Journal of applied statistics*. 28, 8. 1003-1017
[11] Hashemi M M. H. & Aliakbari S R. (2017). Robust extreme double ranked set sampling. *Journal of Statistical Computation and Simulation*. 88,9 1749-1758.
[12] Manoj C (2016). Ordered extreme ranked set sampling and its application in parametric estimation. *Journal of Statistical Theory and Applications.* 15, 3. 248 – 258
[13] McIntyre, G.A. (1952). A Method for Unbiased Selective Sampling Using Ranked Sets. *Australian Journal of Agricultural Research, 3*, 385-390.
[14] Monjed H. Samuh, Hebron, Amer I. Al-Omari & Nursel Koyuncu (2020). Estimation of the parameters of the new Weible estimation of the parameters of the new Weibull-Pareto distribution using RSS. STATISTICA
[15] Rice, S.O. (1945). Mathematical Analysis of Random Noise. *Bell System Technical Journal*. 24 (1). Bell Labs: 46–156.
[16] Samawi, H., Ahmed. M & Abu-Dayyeh. W (1996).Estimating the Population Mean Using Extreme Ranked Set Sampling. *Biometrical Journal*.38, 5 577-586
[17] Shaibu, A. B. and Muttlak, H. A. (2004). Estimating the parameters of the normal, exponential, and gamma distributions using median and extreme ranked set samples. Statistica, 64(1), 75-98
[18] Shakeel A and Javid S (2019). Extreme-cum-median ranked set sampling. Braz. J. Probab. Stat. 33(1): 24-38.