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| RESEARCH ARTICLE

Some Inequalities for Differentiable Functions on a Generalization of Hermite-Hadamard Inequality with Applications

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| ABSTRACT
The main identity of midpoint type inequalities is generalized and using this identity some composite midpoint type inequalities are estimated. Moreover, applications to some special means and some error estimates for the composite midpoint formula are discussed.

| KEYWORDS

Trapezoid inequality, Convex function, Hermite-Hadamard inequality, Midpoint inequalities.

| ARTICLE INFORMATION

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1. Introduction

A function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $[a,b]$, if it satisfy the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \qquad (1.1)$$

for every $x, y \in [a,b]$ and $t \in [0,1]$. Also if $f : [a,b] \subseteq R \to R$ is a convex function, then the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}, \qquad (1.2)$$

is true [Josip, 1992] and (1.2) is called Hermite-Hadamard inequality in the literature.

Some trapezoidal and Midpoint inequalities with its applications have been obtained based on the inequality (1.2) in [Dragomir, 1998, Kirmaci, 2004] The results in [Dragomir, 1998, Kirmaci, 2004] [Chu, 2016, Kirmaci, 2007] attracted the attention of many researchers in this area and it has been generalized in different ways. For more information in this regard, the reader can see [Alomari, 2016, Dragomir, 1998, Chu, 2017, Chu, 2016, Du, 2016, Du, 2017, İŞCAN, 2021, Kirmaci, 2007, Noor, 2014, Pearce, 2000] and references there in.

A generalization of (1.2) is introduced in [Alomari, 2016], as follows:

$$h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) \leq \int_a^b f(t)dt \leq \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right], \qquad (1.3)$$

where $x_i = a + ih, i =, 1, 2, 3, \cdots, n$, with $h = \frac{b-a}{n}$ and $n \in \mathbb{N}$. Also, if a function $f : [a,b] \subseteq R \to R$ is twice differentiable on (a,b) and its second derivative is bounded on(a,b), then an estimation of the composite midpoint formula is given like this

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \sum_{i=0}^{n-1}\frac{1}{n} f\left(\frac{x_{i+1}+x_i}{2}\right)\right| \leq \frac{\|f''\|_\infty}{24n^2}(b-a)^2, \qquad (1.4)$$

More details on the composite midpoint rule can be found in [Davis, 2007].

In 2021 ˙ I, scan and et al., (2021) presented the generalization of the main identity in [Kirmaci, 2004] in this wise:

Lemma 1.1. (İŞCAN , 2021) *Let $f : J^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on J° and $a, b \in J^{\circ}$, with $a < b$. If $f' \in L[a,b]$, then the following equality*

$$I_n(f,a,b)$$
$$= \sum_{i=0}^{n-1} \frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right) + f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] - \frac{1}{b-a}\int_a^b f(u)du$$
$$= \sum_{i=0}^{n-1} \frac{b-a}{2n^2}\left[\int_0^1 (1-2t) f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right], (1.5)$$

is fulfilled.

In addition, the authors in (İŞCAN, 2021) have obtained some inequalities based on identity (1.5).

The aim of this work is to find out the generalization of the main identity in [Kirmaci, 2004] by considering the difference between the mid and left-hand side terms of the inequality 1.3, and estimation of some inequalities. Also, using these results we observe some sensible applications to some special means and error term in the composite midpoint formula. The results of the midpoint type inequalities in [Kirmaci, 2004] are obtained as special cases when $n = 1$.

2. Main Results

In order to prove our main results, we need to prove the following lemma

Lemma 2.1. Let $f : J^{\circ} \subset R \to R$ be a differentiable function on J° and $a, b \in J^{\circ}$, with $a < b$. If $f' \in L[a,b]$, then the following equality holds

$$J_n(f,a,b)$$
$$= \frac{1}{b-a}\int_a^b f(u)du - \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right)$$
$$= \sum_{i=0}^{n-1} \frac{b-a}{n^2}\left[\int_0^{\frac{1}{2}} t\, f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right.$$
$$\left. + \int_{\frac{1}{2}}^1 (t-1) f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right], \quad (2.1)$$

Proof. Let $n \in \mathbb{N}$ be arbitrary, then for $i \in \{0,1,2,\ldots,n-1\}$ and integration by parts, we have

$$J_i = \int_0^{\frac{1}{2}} t\, f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt$$
$$+ \int_{\frac{1}{2}}^1 (t-1) f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt$$
$$= \frac{n}{a-b}\left[tf\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Big|_0^{\frac{1}{2}}\right]$$
$$- \frac{n}{a-b}\int_0^{\frac{1}{2}} f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt$$
$$+ \frac{n}{a-b}\left[(t-1)f\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Big|_{\frac{1}{2}}^1\right]$$
$$= \frac{n}{2(a-b)} f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right) - \frac{n^2}{(a-b)^2}\int_{\frac{(n-i-1)a+(i+1)b}{n}}^{\frac{(2n-2i-1)a+(2i+1)b}{2n}} f(u)du$$
$$+ \frac{n}{2(a-b)} f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right) - \frac{n^2}{(a-b)^2}\int_{\frac{(2n-2i-1)a+(2i+1)b}{2n}}^{\frac{(n-i)a+ib}{n}} f(u)du$$
$$= -\frac{n}{a-b} f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right) - \frac{n^2}{(a-b)^2}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(u)du, \quad (2.2)$$

Multiplying both sides of (2.2) by $\frac{b-a}{n^2}$, we obtain

$$\frac{b-a}{n^2}J_i=\frac{1}{a-b}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}}f(u)du-\frac{1}{n}f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right).$$

Therefore, we get

$$\sum_{i=0}^{n-1}\frac{b-a}{2n^2}J_i=\frac{1}{b-a}\sum_{i=0}^{n-1}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}}f(u)du-\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right).$$

$$=\frac{1}{b-a}\int_a^b f(u)du\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)a+(2i+1)b}{2n}\right).$$

Remark 2.1. *If we choose* $n=1$*, then (2.1) reduces to* [3, Lemma 2.1].

Theorem 2.1. *Let* $f: J^\circ\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* J° *and* $a,b\in J^\circ$*,with* $a<b$*. If* $|f'|^q$ *is convex for some fixed* $q\geq 1$*, then the following inequality holds*

$$|J_n(f,a,b)|$$

$$\leq\sum_{i=0}^{n-1}\frac{b-a}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+2\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(2\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right]. \qquad (2.3)$$

Proof. From Lemma 2.1 and using the well known Power-mean inequality, we have

$$|J_n(f,a,b)|$$

$$\leq\sum_{i=0}^{n-1}\frac{b-a}{n}\left[\left(\int_0^{\frac{1}{2}}t\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t\left|f'\left(t\frac{(n-i)a+ib}{n}\right)+(1-t)\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t-1|\left|f'\left(t\frac{(n-i)a+ib}{n}\right)+(1-t)\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By using the convexity of $|f'|^q$, we get

$$|J_n(f,a,b)|$$

$$\leq\sum_{i=0}^{n-1}\frac{b-a}{n}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(t\frac{(n-i)a+ib}{n}\right)^q\right|\int_0^{\frac{1}{2}}t\,dt+\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\int_0^{\frac{1}{2}}t(1-t)\,dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left|f'\left(t\frac{(n-i)a+ib}{n}\right)^q\right|\int_{\frac{1}{2}}^1|t-1|t\,dt+\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\int_{\frac{1}{2}}^1|t-1|(1-t)\,dt\right)^{\frac{1}{q}}\right]$$

$$=\sum_{i=0}^{n-1}\frac{b-a}{n}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|f'\left(\frac{(n-i)a+ib}{n}\right)^q\right|+2\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q}{24}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{2\left|f'\left(\frac{(n-i)a+ib}{n}\right)^q\right|+\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q}{24}\right)^{\frac{1}{q}}\right]$$

$$=\sum_{i=0}^{n-1}\frac{b-a}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+2\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(2\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

This completes the proof.

Remark 2.2. *If* $n=1$ *and* $q=1$ *in Theorem 2.1, then (2.3) is reduced to* [3, Theorem 2.2].

Corollary 2.1. *If $n = 2$ in Theorem 2.1, then the following inequality holds*

$$\left|\int_a^b f(u)du - \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\right|$$

$$\leq \frac{b-a}{16}\left(\frac{2+2^{\frac{1-q}{q}}+2^{\frac{1-q}{q}}}{3}\right)^q [|f'(a)|+|f'(b)|] \qquad (2.4)$$

Theorem 2.2. *Let $f : J^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on J° and $a, b \in J^\circ$, with $a < b$. If $|f'|^q$ is convex for some fixed $q > 1$, then the following inequality*

$|J_n(f,a,b)|$

$$\leq \sum_{i=0}^{n-1} \frac{b-a}{16n}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + 3\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(3\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right]. \qquad (2.5)$$

where $\frac{1}{p} + \frac{1}{q} = 1$, is true.

Proof. From Lemma 2.1 and using Holder's integral inequality, we have

$|J_n(f,a,b)|$

$$\leq \sum_{i=0}^{n-1} \frac{b-a}{n}\left(\int_0^{\frac{1}{2}} t^p\, dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.\left(\int_{\frac{1}{2}}^1 |t-1|^p\, dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By using the convexity of $|f'|^q$, we obtain

$|J_n(f,a,b)|$

$$\leq \sum_{i=0}^{n-1} \frac{b-a}{n}\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + 3\left|f'\frac{(n-i-1)a+(i+1)b}{n}\right|^q}{8}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left|f'\frac{(n-i-1)a+(i+1)b}{n}\right|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$= \sum_{i=0}^{n-1} \frac{b-a}{16n}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + 3\left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(3\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q + \left|f'\left(\frac{(n-1-i)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

This completes the proof.

Remark 2.3. *If we take n = 1 in Theorem 2.1, then (2.3) is reduced to* [3, Theorem 2.3].

3. Application to Special Mains

As in [Pearce, 2000], we consider extension of arithmetic, Logarithmic, hermonic and generalized logarithmic means for arbitrary real numbers.

Let $a, b \in \mathbb{R}$.

$$A = A(a,b) = \frac{a+b}{2}, \quad a,b \geq 0,$$

$$H = H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

$$L = L(a,b) = \begin{cases} a & , \text{ if } a = b \\ \frac{b-a}{\ln b - \ln a} & , \text{ if } a \neq b \end{cases} \quad , \quad a,b > 0,$$

$$L_t = L_t(a,b) = \begin{cases} a & , \text{ if } a = b \\ \left(\frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}\right)^{\frac{1}{t}} & , \text{ if } a \neq b \end{cases} \quad , \quad t \in \mathbb{R} \setminus \{0.-1\}, \ a,b > 0.$$

Now, we shall use the results of section 2 to derive the following inequalities for arbitrary real numbers.

Proposition 3.1. *Let $a,b \in \mathbb{R}$ such that $a < b$ and $m \in \mathbb{N}, m \geq 2$. Then the following inequality is true for all $q \geq 1$.*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A^m\left(\frac{(n-i)a+ib}{n}, \frac{(n-1-i)a+(i+1)b}{n}\right) - L_m^m(a,b)\right|$$

$$\leq \sum_{i=0}^{n-1} \frac{m(b-a)}{8n}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\left|\frac{(n-i)a+ib}{n}\right|^{(m-1)q} + 2\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{(m-1)q}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(2\left|\frac{(n-i)a+ib}{n}\right|^{(m-1)q} + 2\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{(m-1)q}\right)^{\frac{1}{q}}\right]. \tag{3.1}$$

Proof. The proof is clear from Theorem 2.1 with $f(x) = x^m, x \in [a,b], m \in \mathbb{N}$ and $m \geq 2$.

Proposition 3.2. *Let $a,b \in \mathbb{R}$ such that $a < b$ and $m \in \mathbb{N}, m \geq 2$. Then for all $p > 1$, we have*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A^m\left(\frac{(n-i)a+ib}{n}, \frac{(n-1-i)a+(i+1)b}{n}\right) - L_m^m(a,b)\right|$$

$$\leq \sum_{i=0}^{n-1} \frac{m(b-a)}{16n}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\left|\frac{(n-i)a+ib}{n}\right|^{(m-1)q} + 3\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{(m-1)q}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(3\left|\frac{(n-i)a+ib}{n}\right|^{(m-1)q} + \left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{(m-1)q}\right)^{\frac{1}{q}}\right]. \tag{3.2}$$

Proof. The proof is clear from Theorem 2.2 with $f(x) = x^m, x \in [a,b], m \in \mathbb{N}$ and $m \geq 2$.

Remark 3.1. *If we take $n = 1$ in Proposition 3.2, then we have* [3, Proposition 3.2].

Proposition 3.3. *Suppose $a,b \in \mathbb{R}$ such that $a < b$ and $0 \notin [a,b]$. Then the following inequality*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A^{-1}\left(\frac{(n-i)a+ib}{n}, \frac{(n-1-i)a+(i+1)b}{n}\right) - L^{-1}(a,b)\right|$$

$$\leq \sum_{i=0}^{n-1} \frac{b-a}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|\frac{(n-i)a+ib}{n}\right|^{-2q} + 2\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{-2q}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(2\left|\frac{(n-i)a+ib}{n}\right|^{-2q} + \left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{-2q}\right)^{\frac{1}{q}}\right]. \tag{3.3}$$

is fulfilled for all $r \geq 1$.

Proof. The proof is immediate from Theorem 2.1 with $f(x) = x^{-1}, x \in [a,b]$.

Proposition 3.4. *Suppose $a,b \in \mathbb{R}$ such that $a < b$ and $0 \notin [a,b]$. Then for all $q > 1$ the following inequality holds*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} A^{-1}\left(\frac{(n-i)a+ib}{n}, \frac{(n-1-i)a+(i+1)b}{n}\right) - L^{-1}(a,b)\right|$$

$$\leq \sum_{i=0}^{n-1} \frac{b-a}{16n}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\left|\frac{(n-i)a+ib}{n}\right|^{-2q} + 3\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{-2q}\right)^{\frac{1}{q}}\right.$$

$$+\left(3\left|\frac{(n-i)a+ib}{n}\right|^{-2q}+\left|\frac{(n-i-1)a+(i+1)b}{n}\right|^{-2q}\right)^{\frac{1}{q}}\Bigg]. \qquad (3.3)$$

Proof. The proof is immediate from Theorem 2.2 with $f\,(x)\; = x^{-1}, x\; \in\; [a,b]$.

Remark 3.2. *If we take $n = 1$ in Proposition 3.4, then we have* [3, Proposition 3.5].

4. The Composition Midpiont Formula

Let $f\,:\,[a,b]\;\to \mathbb{R}$ be integral able function, $p_m\,:\,a\,=x_0\,<\,x_1\,<,\ldots,<\,x_m\,=\,b$ be a partition of $[a,b]$ and $\Delta_j\;=\;x_{j+1}\;-\;x_j\;, j=0,1,2,\ldots,m\;-\;1$. As in [ ], the authors described the following notations

$$M(f,P_m)=\sum_{j=0}^{m-1}f\left(\frac{x_j+x_{j+1}}{2}\right)\Delta_j,\;\;(the\; misppint\; formula),$$

and the approximation error of $\int_a^b f(u)du$ by $M(f,P_m)$ in this wise

$$M(f,P_m)=\int_a^b f(u)du-M(f,P_m).$$

Moreover, Pearce and Pecari (2000) obtained the following result which is approximation errors for the trapezoidal and midpoint formulas.

Proposition 4.1. *Let $f\,:J^{\circ}\;\subset\;\mathbb{R}\to\mathbb{R}$ be a differential function on $J^{\circ}, a,b\;\in\;J$, with $a\;<\;b$ and $q\;\geq\;1$. If $|f|^q$ is convex on $[a,b]$, then the following inequalities are*

$$|F(f,P_m)|\leq\frac{1}{4}\sum_{j=0}^{m-1}\left(\frac{|f'(x_j)|^q+|f'(x_{j+1})|^q}{2}\right)\Delta_j$$

$$\leq\frac{\max\{|f'(a)|,|f'(b)|\}}{4}\sum_{j=0}^{m-1}\Delta_j^2. \qquad (4.1)$$

We have the following propositions

Proposition 4.2. *Suppose the conditions of Theorem 2.1 holds, then the following inequality*

$$\left|\int_a^b f(u)du-\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)x_j+(2i+1)x_{j+1}}{2n}(x_{j+1}-x_j)\right)\right|$$

$$\leq\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{(x_{j+1}-x_j)^2}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\Bigg[\left(\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+2\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(2\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

$$\leq\frac{\max\{|f'(a)|,|f'(b)|\}}{4}\sum_{j=0}^{m-1}(x_{j+1}-x_j)^2. \qquad (4.2)$$

is fulfilled.

Proof. By applying Theorem 2.1 on $[x_j\,,x_{j+1}], j\;=\;0,1,2,\ldots,m\;-\;1$, we get

$$\left|\int_{x_j}^{x_{j+1}} f(u)du-\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)x_j+(2i+1)x_{j+1}}{2n}(x_j-x_{j+1})\right)\right|$$

$$\leq\sum_{i=0}^{n-1}\frac{(x_j-x_{j+1})^2}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\Bigg[\left(\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+2\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(2\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\Bigg]$$

Using the convexity of $|f|^q$ and summing over j from 0 to $m\;-\;1$, we get

$$\left|\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} f(u)du-\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)x_j+(2i+1)x_{j+1}}{2n}(x_{j+1}-x_j)\right)\right|$$

$$\leq \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{(x_{j+1}-x_j)^2}{8n}\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+2\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(2\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

$$\leq \sum_{j=0}^{m-1}\frac{(x_j-x_{j+1})^2}{4n}\sum_{i=0}^{n-1}\left(\frac{(3n-3i-1)}{3n}\left|f'(x_j)\right|^q+\frac{(3i+1)}{3n}\left|f'(x_{j+1})\right|^q\right)^{\frac{1}{q}}$$

$$\leq \sum_{j=0}^{m-1}\frac{(x_j-x_{j+1})^2}{4n}\sum_{i=0}^{n-1}\left(\frac{(3n-3i-1)}{3n}\left(\max\left\{\left|f'(x_j)\right|^q,\left|f'(x_{j+1})\right|^q\right\}\right)\right.$$

$$\left.+\frac{(3i+1)}{3n}\left(\max\left\{\left|f'(x_j)\right|^q,\left|f'(x_{j+1})\right|^q\right\}\right)\right)^{\frac{1}{q}}$$

$$=\sum_{j=0}^{m-1}\frac{(x_j-x_{j+1})^2}{4n}\times\sum_{i=0}^{n-1}\max\left\{\left|f'(x_j)\right|^q,\left|f'(x_{j+1})\right|^q\right\}$$

$$\leq \frac{1}{4}\sum_{j=0}^{m-1}(x_j-x_{j+1})^2\max\left\{\left|f'(x_j)\right|^q,\left|f'(x_{j+1})\right|^q\right\}$$

$$\leq \frac{1}{4}\sum_{j=0}^{m-1}(x_j-x_{j+1})^2\max\{|f'(a)|^q,|f'(b)|^q\}.$$

Remark 4.1. *If we take $n=1$ and $q=1$ in Proposition 4.2, then we have* [3, Proposition 4.1].
Proposition 4.3. *Suppose the conditions of Theorem 2.2 holds, then the following inequality holds*

$$\left|\int_a^b f(u)du-\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{(2n-2i-1)x_j+(2i+1)x_{j+1}}{2n}(x_{j+1}-x_j)\right)\right|$$

$$\leq \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}\frac{b-a}{16n}\left(\frac{4}{p+1}\right)^{1-\frac{1}{q}}\left[\left(\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+3\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(3\left|f'\left(\frac{(n-i)x_j+ix_{j+1}}{n}\right)\right|^q+\left|f'\left(\frac{(n-i-1)x_j+(i+1)x_{j+1}}{n}\right)\right|^q\right)^{\frac{1}{q}}\right] \qquad (4.3)$$

Proof. The proof uses Theorem 2.2 and is similar to that of Proposition 4.2.

Remark 4.2. *If we take $n=1$ in Proposition 4.3, then we have* [3, Proposition 4.2].

Conclusion 1. In this study, we obtained the generalized form of the midpoint identity in [Kirmaci, 2004], and by using it we found some estimations of composite midpoint type inequality. The results of this work may be used to find the approximations of some composite Simpson's and Newton's type inequalities.

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