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| RESEARCH ARTICLE

An Application of Einsteinian-Phythagorean Theorem in Einstein Gyrovector Spaces

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| ABSTRACT
In [Ungar 2008; Ungar 2015] A.A. Ungar, employs the Einstein gyrovector spaces for the introduction of the gyrotrigonometry, Ungar's and other researcher's works play a major role in translating some theorems from Euclidean geometry to corresponding theorems in Einstein gyrovector spaces. In Euclidean geometry, the sum of the squares of the lengths of opposite sides of convex or concave quadrilaterals whose diagonals intersect perpendicularly is equal to each other. In this paper, we present this theorem in Einstein gyrovector spaces in terms of gamma factors.

| KEYWORDS

Gyrotriange, Einsteinian-Phythagorean Identities, Gamma factor.

| ARTICLE INFORMATION

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1. Introduction

Hyperbolic geometry appeared in the first half of the 19th century. It is also known as a type of non-Euclidean geometry. Although Euclidean Geometry and Hyperbolic Geometry have common concepts as distance, angle, both these geometries have many different. Hyperbolic Geometry has many models such as: Poincare' disc model, Einstein relativistic velocity model, etc.

Einstein gyrovector spaces form the algebraic setting for the Beltrami-Klein ball model of Hyperbolic Geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean Geometry.

Let c be the vacuum speed of light, and let

$$\mathbb{R}_c^3 = \{\boldsymbol{v} \in \mathbb{R}^3 : \|\boldsymbol{v}\| < c\} \tag{1.1}$$

be the c ball of all relativistically admissible velocities of material particles. Einstein addition in c-ball is given by the equation

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\left\{\boldsymbol{u} + \boldsymbol{v} + \frac{1}{c^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}\left(\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{v})\right)\right\} \tag{1.2}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_c^3$, where $\boldsymbol{u} \cdot \boldsymbol{v}$is the inner product that the ball $\mathbb{R}_c^3$ inherits from its space $\mathbb{R}^3, \boldsymbol{u} \times \boldsymbol{v}$is the vector product in $\mathbb{R}_c^3 \subset \mathbb{R}^3$ and where $\gamma_{\boldsymbol{u}}$ is the gamma factor

$$\gamma_{\boldsymbol{u}} = \frac{1}{\sqrt{1-\frac{\|\boldsymbol{u}\|^2}{c^2}}} \geq 1 \tag{1.3}$$

in the c-ball.

Owing to the vector identity,

$$(\boldsymbol{x} \times \boldsymbol{y}) \times \boldsymbol{z} = -(\boldsymbol{y} \cdot \boldsymbol{z})\boldsymbol{x} + (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} \tag{1.4}$$

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$, Einstein addition (1.2) can also be written in the form

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\{\boldsymbol{u} + \frac{1}{\gamma_{\boldsymbol{u}}}\boldsymbol{v} + \frac{1}{s^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}\} \tag{1.5}$$

which remains valid in higher dimensions. Einstein addition (1.5) of relativistically admissible velocities was introduced Einstein in 1905.

In this paper, we study in an Einstein gyrovector space who was introduced by A. A. Ungar [2008, 2009, 2015].

2. Preliminaries

Definition 2.1. A groupoid($\mathbb{G}$,$\oplus$) is a gyrogroup if its binary operation satisfies the following axioms. In $\mathbb{G}$ there is at least one element, **0,** calledleftidentity, satisfying

$$\boldsymbol{0} \oplus \boldsymbol{a} = \boldsymbol{a}$$

for all $\boldsymbol{a} \in \mathbb{G}$. There is an element $\boldsymbol{0} \in \mathbb{G}$ for each $\boldsymbol{a} \in \mathbb{G}$ there is an element $\ominus\, \boldsymbol{a} \in \mathbb{G}$, called a left inverse of $\boldsymbol{a}$**,** satisfying

$$\ominus\, \boldsymbol{a} \oplus \boldsymbol{a} = \boldsymbol{0}.$$

Moreover, for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{G}$ there exit a unique element $gyr[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in \mathbb{G}$ such that binary operation obeys the left gyroassociative law

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus gyr[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}.$$

The map $gyr: \mathbb{G} \to \mathbb{G}$ is given by $\boldsymbol{c} \mapsto gyr[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}$is an automorphism of the groupoid ($\mathbb{G}$,$\oplus$), that is,

$$gyr[\boldsymbol{a}, \boldsymbol{b}] \in \boldsymbol{Aut}(\mathbb{G}, \oplus)$$

and the automorphism $gyr[\boldsymbol{a}, \boldsymbol{b}]$ of automorphism of $\mathbb{G}$ is called the gyroautomorphism of$\mathbb{G}$ generated by $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{G}$. Finally, the gyroautomorphism of$\mathbb{G}$ generated by $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{G}$ possesses the left loop property

$$gyr[\boldsymbol{a}, \boldsymbol{b}] = gyr[\boldsymbol{a} \oplus b, \boldsymbol{b}].$$

Additionally, if the binary operation "$\oplus$ " obeys the gyrocommutative law

$$\boldsymbol{a} \oplus \boldsymbol{b} = gyr[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a})$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{G}$, then ($\mathbb{G}$,$\oplus$) is called a gyrocommutative gyrogroup.

Definition 2.2. Let $\mathbb{V}$ be a real inner product space and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\boldsymbol{v} \in \mathbb{V} : \|\boldsymbol{v}\| < s\},$$

where $s > 0$ is an arbitrary fixed constant. Einstein addition $\oplus$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\{\boldsymbol{u} + \frac{1}{\gamma_{\boldsymbol{u}}}\boldsymbol{v} + \frac{1}{s^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}\}$$

where $\gamma_{\boldsymbol{u}}$ is the gamma factor

$$\gamma_{\boldsymbol{u}} = \frac{1}{\sqrt{1-\frac{\|\boldsymbol{u}\|^2}{s^2}}} \geq 1$$

in the s-ball $\mathbb{V}_s$, and where $\cdot$ and $\|.\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

Einstein addition satisfies the mutually equivalent gamma identities

$$\gamma_{\boldsymbol{u}\oplus\boldsymbol{v}} = \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}\left(1 + \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}\right)$$

and

$$\gamma_{\ominus\boldsymbol{u}\oplus\boldsymbol{v}} = \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}\left(1 - \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}\right)$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_s^n$.

When the nonzero vectors $\boldsymbol{u}$and $\boldsymbol{v}$in the ball$\mathbb{R}_s^n$of $\mathbb{R}^n$are parallel in $\mathbb{R}^n$, $\boldsymbol{u} \parallel \boldsymbol{v}$, that is, $\boldsymbol{u} = \lambda\boldsymbol{v}$for some $0 \neq \lambda \in \mathbb{R}$, Einstein addition reduces to the Einstein addition of parallel velocities

$$\boldsymbol{u}\oplus\boldsymbol{v} = \frac{\boldsymbol{u}+\boldsymbol{v}}{1+\frac{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}{s^2}}$$

Hence,

$$\|\boldsymbol{u}\|\oplus\|\boldsymbol{v}\| = \frac{\|\boldsymbol{u}\|+\|\boldsymbol{v}\|}{1+\frac{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}{s^2}}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_s^n$. In this case, Einstein addition is both commutative and associative.

In the Newtonian limit, $s \to \infty$, s-ball $\mathbb{R}_s^n$ expands to the whole of its space $\mathbb{R}^n$, and Einstein addition $\oplus$ in $\mathbb{R}_s^n$ reduces to vector addition + in$\mathbb{R}^n$.

Theorem2.3.($\mathbb{R}_s^n$,$\oplus$) Einstein groupoid is a gyrocommutative gyrogroup.

Some gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces.

Definition 2.4. A ($\mathbb{G}$,$\oplus$,$\otimes$) gyrovector space is a gyrocommutative gyrogroup ($\mathbb{G}$,$\oplus$)that obeys the following axioms:

1. $gyr[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a} \cdot gyr[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{b}$for all points $\boldsymbol{a},\boldsymbol{b},\boldsymbol{u},\boldsymbol{v} \in \mathbb{G}$.
2. $\mathbb{G}$ admits a scalar multiplication, $\otimes$, possessing following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points and$\boldsymbol{a} \in \mathbb{G}$:
   - $1\otimes\boldsymbol{a} = \boldsymbol{a}$
   - $(r_1+r_2)\otimes\boldsymbol{a} = (r_1\otimes\boldsymbol{a})\oplus(r_2\otimes\boldsymbol{a})$
   - $(r_1r_2)\otimes\boldsymbol{a} = r_1\otimes(r_2\otimes\boldsymbol{a})$
   - $\frac{|r|\otimes\boldsymbol{a}}{\|r\otimes\boldsymbol{a}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\boldsymbol{a} \neq \boldsymbol{0}$, $r \neq 0$
   - $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](r\otimes\boldsymbol{a}) = r\otimes\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{a})$
   - $\mathrm{gyr}[r_1\otimes\boldsymbol{v}, r_2\otimes\boldsymbol{v}] = I$
3. Real vector space structure ($\|\mathbb{G}\|$,$\oplus$,$\otimes$) for the set $\|\mathbb{G}\|$ of one-dimensional "vectors"
   $$\|\mathbb{G}\| := \{\mp\|\boldsymbol{a}\|: \boldsymbol{a} \in \mathbb{G}\} \subset \mathbb{R}$$
   with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\boldsymbol{a},\boldsymbol{b} \in \mathbb{G}$,
   - $\|r\otimes\boldsymbol{a}\| = |r|\otimes\|\boldsymbol{a}\|$
   - $\|\boldsymbol{a}\oplus\boldsymbol{b}\| \leq \|\boldsymbol{a}\|\oplus\|\boldsymbol{b}\|$

Theorem2.5. An Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is an Einstein gyrocommutative gyrogroup $(\mathbb{R}_s^n, \oplus)$ with scalar multiplication $\otimes$ given by

$$r\otimes\boldsymbol{v} = s\frac{\left(1+\frac{\|\boldsymbol{v}\|}{s}\right)^r - \left(1-\frac{\|\boldsymbol{v}\|}{s}\right)^r}{\left(1+\frac{\|\boldsymbol{v}\|}{s}\right)^r + \left(1-\frac{\|\boldsymbol{v}\|}{s}\right)^r}\frac{v}{\|\boldsymbol{v}\|} = stanh(rtanh^{-1}\frac{\|\boldsymbol{v}\|}{s})\frac{v}{\|\boldsymbol{v}\|}$$

Definition 2.6. Let $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ be an Einstein gyrovector space. Its gyrometric is given by the gyrodistance function $d_\oplus: \mathbb{R}_s^n \times \mathbb{R}_s^n \longrightarrow \mathbb{R}^{\geq 0} := \{r \in \mathbb{R}: r \geq 0\}$,

$$d_\oplus(\boldsymbol{a},\boldsymbol{b}) = \|\ominus\,\boldsymbol{a}\oplus\boldsymbol{b}\| = \|\boldsymbol{b}\ominus\boldsymbol{a}\|$$

where $d_\oplus(\boldsymbol{a},\boldsymbol{b})$ is the gyrodistance of $\boldsymbol{a}$and $\boldsymbol{b}$**.**

The unique Einstein gyrolineL_{AB} that passes two given points A and B in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is represented by the equation

$$L_{AB} = A \oplus (\ominus A \oplus B)\otimes t$$

$t \in \mathbb{R}$. Gyrolines in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ turn out to be well-known geodesic of the Beltrami Klein ball model of hyperbolic geometry.

3. Some Gyrotrigonometric Identities

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$ be three distinct points and $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{A} \oplus \boldsymbol{C}$be two rooted gyrovectors with a common tail $\boldsymbol{A}$. They include the gyroangle$\boldsymbol{\alpha} = \angle \boldsymbol{BAC} = \angle \boldsymbol{CAB}$, the radian measure of which is given by the equation

$$\boldsymbol{cos\alpha} = \frac{\ominus \boldsymbol{A} \oplus \boldsymbol{B}}{\|\ominus \boldsymbol{A} \oplus \boldsymbol{B}\|} \cdot \frac{\ominus \boldsymbol{A} \oplus \boldsymbol{C}}{\|\ominus \boldsymbol{A} \oplus \boldsymbol{C}\|}. \quad (3.1)$$

Definition 3.1. A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a object formed by the three points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$, called the vertices of the triangle, and the gyrovectors $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{B} \oplus \boldsymbol{C}and\ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,** called the sides of the triangle.These are respectively, the sides opposite to the vertices**C,A**and **B.** The gyrotriangle sides generate the threegyrotrianglegyroangles$\boldsymbol{\alpha}, \boldsymbol{\beta}and\boldsymbol{\gamma}$ at the respective vertices **A, B** and **C**.

Gyrotrianglegyroanglesum in hyperbolicgeometryislessthanπ. The difference, δ,

$$\delta = \pi - (\alpha + \beta + \gamma) \quad (3.2)$$

being the gyrotriangulardefect.

Theorem 3.2. Let $\boldsymbol{ABC}$be a gyrotriangle in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$,and sides$\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}, \boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}and\boldsymbol{b} = \ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,**withgyroangles$\boldsymbol{\alpha}, \boldsymbol{\beta}and\boldsymbol{\gamma}$ at the vertices **A, B** and **C**.Thenwe have the law of cosines

$$\gamma_c = \gamma_a \gamma_b (1 - b_s c_s cos\gamma) \quad (3.3)$$

where$a = \|\boldsymbol{a}\|, b = \|\boldsymbol{b}\|, c = \|\boldsymbol{c}\|andb_s = b/s$, etc.

Definition 3.3. A right gyroangleγ is a gyroangle measuring $\frac{\pi}{2}$ radians.

Theorem 3.4.A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a right gyrotriangle with gyrolegs $\boldsymbol{a}, \boldsymbol{b}$and gyrohypotenuse$\boldsymbol{c}$, if and only if

$$\gamma_c = \gamma_a \gamma_b. \quad (3.4)$$

Theorem 3.5. Let $\boldsymbol{ABC}$be a right gyrotriangle in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ with the right gyroangle$\gamma = \pi/2$. Thenwe have two distinct Einsteinian-Phytagoreanidentities

$$a^2 + \left(\frac{\gamma_b}{\gamma_c}\right)^2 b^2 = c^2 \quad (3.5)$$

$$\left(\frac{\gamma_a}{\gamma_c}\right)^2 a^2 + b^2 = c^2 \quad (3.6)$$

with hypotenuse$c$ and legs a and b.

4. Application of Einsteinian-PhtagoreanIdentities

As an application of gyrotrigonometryin an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, we verify the following theorem:

Theorem 4.1. Let $\boldsymbol{ABC}$be a gyrotriangle in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$,and sides$\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}, \boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}and\boldsymbol{b} = \ominus \boldsymbol{C} \oplus \boldsymbol{A}$.We suppose that the orthogonal projections ofsides$\boldsymbol{b}$and $\boldsymbol{a}$ on side $\boldsymbol{c}$of the gyrotriangleare$\boldsymbol{c_1} = \ominus \boldsymbol{A} \oplus \boldsymbol{D}$ and $\boldsymbol{c_2} = \ominus \boldsymbol{B} \oplus \boldsymbol{D}$,respectively, and $\boldsymbol{E}$is an arbitrary point on the gyroline segment $\boldsymbol{CD}$**.**Then

$$(a^2 \ominus m^2) \otimes \gamma_a^2 = (b^2 \ominus k^2) \otimes \gamma_b^2.$$

where$a = \|\ominus \boldsymbol{B} \oplus \boldsymbol{C}\|, b = \|\ominus \boldsymbol{C} \oplus \boldsymbol{A}\|, c = \|\ominus \boldsymbol{A} \oplus \boldsymbol{B}\|, \boldsymbol{m} = \|\ominus \boldsymbol{E} \oplus \boldsymbol{B}\|, \boldsymbol{k} = \|\ominus \boldsymbol{E} \oplus \boldsymbol{A}\|$.

Proof : For simplicity, let

$$c_1 = \|\ominus A \oplus D\|,\ c_2 = \|\ominus B \oplus D\|,\ x \oplus y = \|\ominus C \oplus D\|, y = \|\ominus E \oplus D\|$$

be gyrolenghts of gyrovectors$\ominus \boldsymbol{A} \oplus \boldsymbol{D}$,$\ominus \boldsymbol{B} \oplus \boldsymbol{D}$,$\ominus \boldsymbol{C} \oplus \boldsymbol{D}$and $\ominus \boldsymbol{E} \oplus \boldsymbol{D}$. For the right gyrotriangles $\boldsymbol{ADC}$and $\boldsymbol{ADE}$**,** by the (3.6), we have

$$\left(\frac{\gamma_{x\oplus y}}{\gamma_b}\right)^2 \otimes (x \oplus y)^2 \oplus c_1^2 = b^2 \quad (4.1)$$

and

$$\left(\frac{\gamma_y}{\gamma_k}\right)^2 \otimes y^2 \oplus c_1^2 = k^2. \quad (4.2)$$

From (4.1) and (4.2), we obtain that

$$\left(\frac{\gamma_{x\oplus y}}{\gamma_b}\right)^2 \otimes (x \oplus y)^2 \ominus \left(\frac{\gamma_y}{\gamma_k}\right)^2 \otimes y^2 = b^2 \ominus k^2. \quad (4.3)$$

Similary, for the right gyrotriangles $\boldsymbol{BDC}$and $\boldsymbol{BDE}$**,** by the (3.6), we have

$$\left(\frac{\gamma_{x\oplus y}}{\gamma_a}\right)^2 \otimes (x\oplus y)^2 \oplus c_2^2 = a^2 \quad (4.4)$$

and

$$\left(\frac{\gamma_y}{\gamma_m}\right)^2 \otimes y^2 \oplus c_2^2 = m^2. \quad (4.5)$$

We have by (4.4), (4.5)

$$\left(\frac{\gamma_{x\oplus y}}{\gamma_a}\right)^2 \otimes (x\oplus y)^2 \ominus \left(\frac{\gamma_y}{\gamma_m}\right)^2 y^2 = a^2 \ominus m^2 \quad (4.6)$$

Hence by, (4.3) and (4.6), we have

$$\ominus \frac{\gamma_y^2\gamma_b^2}{\gamma_k^2} \otimes y^2 \oplus \frac{\gamma_y^2\gamma_a^2}{\gamma_m^2} y^2 = (b^2 \ominus k^2)\gamma_b^2 \ominus (a^2 \ominus m^2)\gamma_a^2 \quad (4.7)$$

On the other hand, from Theorem 3.4., for the right gyrotriangles $\boldsymbol{BDC}$, $\boldsymbol{BDE}$, $\boldsymbol{ADC}$, $\boldsymbol{ADE}$, we get

$$\frac{\gamma_y^2}{\gamma_m^2} = \frac{1}{\gamma_{c_2}^2} \qquad \frac{\gamma_a^2}{\gamma_{c_2}^2} = \frac{1}{\gamma_{x\oplus y}^2}, \qquad \frac{\gamma_y^2}{\gamma_k^2} = \frac{1}{\gamma_{c_1}^2}, \qquad \frac{\gamma_b^2}{\gamma_{c_1}^2} = \gamma_{x\oplus y}^2. \quad (4.8)$$

These equations imply that

$$\ominus \left(\frac{1}{\gamma_{c_1}^2}\gamma_b^2\right) \otimes y^2 \oplus \left(\frac{1}{\gamma_{c_2}^2}\gamma_a^2\right) \otimes y^2 = (b^2 \ominus k^2) \otimes \gamma_b^2 \ominus (a^2 \ominus m^2) \otimes \gamma_a^2$$

and

$$\ominus \gamma_{x\oplus y}^2 \otimes y^2 \oplus \gamma_{x\oplus y}^2 \otimes y^2 = (b^2 \ominus k^2) \otimes \gamma_b^2 \ominus (a^2 \ominus m^2) \otimes \gamma_a^2.$$

Then we obtain

$$(a^2 \ominus m^2) \otimes \gamma_a^2 = (b^2 \ominus k^2) \otimes \gamma_b^2.$$

5. Conclusion

The Einstein relativisticvelocitymodel is a model of hyperbolicgeometry. Many of theorem of Euclideangeometry are relativelysimilarform in the Einsteinrelativisticvelocity model is a model of hyperbolicgeometry. In Euclidean geometry, the sum of the squares of the lengths of opposite sides of convex or concave quadrilaterals whose diagonals intersect perpendicularly is equal to each other, that is,

$$a^2 - m^2 = b^2 - k^2 \quad (5.1)$$

In an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ for agyrotriangle, a $\boldsymbol{ABC}$ with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$,and sides$\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}and\boldsymbol{b} = \ominus \boldsymbol{C} \oplus \boldsymbol{A}$. (5.1) turns to

$$(a^2 \ominus m^2) \otimes \gamma_a^2 = (b^2 \ominus k^2) \otimes \gamma_b^2.$$

where $a = \|\ominus \boldsymbol{B} \oplus \boldsymbol{C}\|$, $b = \|\ominus \boldsymbol{C} \oplus \boldsymbol{A}\|$, $c = \|\ominus \boldsymbol{A} \oplus \boldsymbol{B}\|$, $\boldsymbol{m} = \|\ominus \boldsymbol{E} \oplus \boldsymbol{B}\|$, $\boldsymbol{k} = \|\ominus \boldsymbol{E} \oplus \boldsymbol{A}\|$. In the Euclideanlimit, of large s, $s \to \infty$, gamma factor $\gamma_{\boldsymbol{u}}$ reduces to 1, gyroequalty in Theorem 4.1 reduces to the

$$a^2 - m^2 = b^2 - k^2$$

inEuclideangeometry.

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