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| RESEARCH ARTICLE

Simulating Parametric and Nonparametric Models

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| ABSTRACT

The purpose of this paper was to investigate the performance of the parametric bootstrap data generating processes (DGPs) methods and to compare the parametric and nonparametric bootstrap (DGPs) methods for estimating the standard error of simple linear regression (SLR) under various assessment conditions. When the performance of the parametric bootstrap method was investigated, simple linear models were employed to fit the data. With the consideration of the different bootstrap levels and sample sizes, a total of twelve parametric bootstrap models were examined. Three hypothetical and one real datasets were used as the basis to define the population distributions and the "true" SEEs. A bootstrap paper was conducted on different parametric and nonparametric bootstrap (DGPs) methods reflecting three levels for group proficiency differences, three levels of sample sizes, three test lengths and three bootstrap levels. Bias of the SLR, standard errors of the SLR, root mean square errors of the SLR, were calculated and used to evaluate and compare the bootstrap results. The main findings from this bootstrap paper were as follows: (i) The parametric bootstrap DGP models with larger bootstrap levels generally produced smaller bias likewise a larger sample size. (ii) The parametric bootstrap models with a higher bootstrap level generally yielded more accurate estimates of the standard error than the corresponding models with lower bootstrap level. (iii) The nonparametric bootstrap method generally produced less accurate estimates of the standard error than the parametric bootstrap method. However, as the sample size increased, the differences between the two bootstrap methods became smaller. When the sample size was equal to or larger than 3,000, say 10000, the differences between the nonparametric bootstrap DGP method and the parametric bootstrap DGP model that produced the smallest RMSE were very small. (4) Of all the models considered in this paper, parametric bootstrap DGP models with higher bootstrap performed better under most bootstrap conditions. (5) Aside from method effects, sample size and test length had the most impact on estimating the Simple Linear Regression.

| KEYWORDS

Regression, Bootstrap, Resampling, Replications, Polynomials

| ARTICLE INFORMATION

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1. Introduction

In statistics, resampling is estimating the precision of sample statistics by using subsets of available data or drawing randomly with replacement from a set of data points. The bootstrap method is a resampling method for the purpose of reducing error and providing more reliable statistical inference. The appeal of the bootstrap is that we can use it to make an inference about some experimental results when the statistical theory is uncertain or even unknown. We can also use the bootstrap to assess how well the statistical theory holds: that is, whether an inference we make from a hypothesis test or confidence interval is justified. In other words, bootstrapping is a group of metaphors which refers to a self-sustaining process that proceeds without external help. Historically, bootstrapping also refers to an early technique for computer program development on new hardware and has been replaced by the use of a cross compiler executed by a pre-existing computer. According to Abney (2002), bootstrapping in program development began during the 1950s when each program was constructed on paper in decimal code or in binary code, bit by bit (1s and 0s), because there was no high-level computer language, no compiler, no assembler, and no linker.

There are several forms of the bootstrap methods, but the two major classes of bootstrap methods - parametric and nonparametric will be considered in this paper. In the nonparametric bootstrap (NPB) method, the sample data are regarded as containing all the information about the population, so repeated samples are drawn with replacement from the sample data. While, parametric bootstrap (PB) method uses simulation steps that are similar to those in the nonparametric bootstrap method except that:

i. A parametric model is first fitted to the replications drawn from the populations.
ii. Bootstrap samples are drawn from the fitted replication distributions rather than the original replications.

Note that, after the model fit, the distributions of the replications are still discrete over the same range of possible scores as the original replication distributions. Moreover, in the parametric bootstrap method, a particular mathematical model is established based on theory or experience and is fit to the data, and then repeated samples are drawn from this density or mass function. Furthermore, in the parametric estimation problems, we obtain information about the parameter from a sample of data coming from the underlying probability distribution. It is good to note that there exist semi-parametric bootstrap method which is a combination of parametric and nonparametric methods. They are often used in situations where the fully nonparametric method may not perform well or when the researcher wants to use a parametric model but the functional form with respect to a subset of the regressors or the density of the errors is not known. They rely on parametric assumptions and may be misspecified and inconsistent.

The basic idea of bootstrap testing is that, when a test statistic of interest has an unknown distribution, that distribution can be characterized by using information in the data set that is being analyzed. In fact, the bootstrap idea means that the original sample represents the population from which it was drawn. So resamples from this sample represent what we would get if we took many samples from the population. The bootstrap distribution of a statistic, based on many resamples, represents the sampling distribution of the statistic, based on many samples. Bootstrapping is the practice of estimating properties of an estimator by measuring those properties when sampling from an approximating distribution. One standard choice for an approximating distribution is the empirical distribution of the observed data. The unknown probability distribution F gives the data $\mathbf{X} = (x_1,x_2,\ldots,x_n)$ by random sampling; from **X** we calculate the statistic of interest $\hat{\theta}$ = s(**X**). In the bootstrap world, $\hat{F}$ generates **X*** by random sampling giving $\hat{\theta}^*$ = s(**X***). There is only one observed value of $\hat{\theta}$, but we can generate as many bootstrap replications of $\hat{\theta}^*$ as possible. The crucial step in the bootstrap process is the process by which we construct from **X** an estimation $\hat{\boldsymbol{F}}$ of the unknown population **F**. The approximate distribution obtained by bootstrap can be used as an alternative to inference based on parametric assumptions when those assumptions are in doubt, or where parametric inference is impossible or requires very complicated formulas for the calculation of statistical inference, hypothesis testing and confidence interval, or assessing how well the statistical theory holds. In fact, the bootstrap samples are used to calculate any parameter of the statistics of interest, from which the inference regarding the population can be. Most commonly used parameters of distribution functions can be expressed as functionals of the distribution.

Efron (1979) was the first researcher who discussed bootstrap procedure that can be applied to estimate sampling distributions of estimators for the statistical model called regression model. Apart from establishing the fact that regression improves ability to predict and reduce unexplained variables in the models using bootstrap. Bootstrap can be applied to more general regression models and save time from taking many samples from the population to make statistical inference. Two "Golden Rules" are formulated by Davidson (2007), if observed, help to obtain the best the bootstrap can offer. Also, it is assumed that bootstrap is based on resampling data coming from a variable that is independent and identically distributed (or the observations in the sample are independent and identically distributed). As long as this assumption is satisfied, the bootstrap can be implemented.

Bootstrap methods involve estimating a model many times using simulated data and the simulated data estimates are then used to make inferences from the actual data. Suppose that $\hat{\tau}$ is the realised value of a test statistic τ . If we knew the cumulative distribution function (CDF) of τ under the null hypothesis, say $F(\tau)$, we would reject the null hypothesis whenever $\hat{\tau}$ is abnormal in some sense. For a test that rejects in the upper tail of the distribution, we might choose to calculate a critical value at level α , say $c\alpha$, as defined by the equation

$1-F(c\alpha)=\alpha.$

Then we would reject the null whenever $\hat{\tau} > c\alpha$.

An alternative approach, which is preferable in most circumstances, is to calculate the P value, or marginal significance level,

$p(\hat{\tau})=1-F(\hat{\tau}),$

and reject whenever $p(\hat{\tau})<\alpha$.

According to Mackinnon (2006) and Kim, et.al, (2022), it is easy to see that these two procedures must yield identical inferences, as $\hat{\tau}$ must be greater than $c\alpha$ whenever $p(\hat{\tau})$ is less than α. In most cases of interest to econometricians, we do not know $F(\tau)$. Until recently, the usual approach in such cases has been to replace it by an approximate CDF, say $F_\infty(\tau)$, based on asymptotic theory. The bootstrap provides another way to approximate $F(\tau)$, which may provide a better approximation. It can be used even when τ is complicated to compute. In order to perform a bootstrap test, it is not necessary for τ to have a known asymptotic distribution but we must generate B bootstrap samples, indexed by j, that satisfy the null hypothesis. A bootstrap sample is a simulated data set. The procedure for generating the bootstrap samples, which always involves a random number generator, is called a bootstrap data generating process, or bootstrap DGP.

2. Literature Review

The bootstrap is a statistical technique used more and more widely in econometrics. Generally, it falls in the broader class of resampling methods called the parametric and nonparametric bootstrap. Bootstrapping is the practice of estimating properties of an estimator by measuring those properties when sampling from an approximating distribution. One standard choice for an approximating distribution is the empirical distribution of the observed data. In the case where a set of observations can be assumed to be from an independent and identically distributed population, this can be implemented by constructing a number of resamples of the observed dataset and of equal size to the observed dataset, each of the bootstrap methods is obtained by random sampling with replacement for more details, see, Efron (2000); Efron and Tibshirani (1993). There are many resampling methods like simple, double, weighted, wild, recursive, segmented, residual, parametric, nonparametric and so on and their introductory aspects can be seen in Lahiri (2006), Xu (2008) and Quenouille (1956).

In certain circumstances, such as regression models with independent and identically distributed error terms, appropriately chosen bootstrap methods generally work very well. Bootstrap methods are often used as an alternative to inference based on parametric assumptions, or to examine the stability of the test statistic θ or when those assumptions are in doubt, or where parametric inference is impossible or requires very complicated formulas for the calculation of standard errors, confidence interval, constructing hypothesis tests, etc. There are many bootstrap methods that can be used for econometric analysis especially in regression and the have been discussed extensively by Efron (2000), Efron and Tibshirani (1993), Gonzalez-Manteiga and Crujeiras (2008), Freedman (1981) Good (2004), Hall and Maiti (2006), Hall, Lee and Park (2019), Mahiane and Auvert (2010), Paparoditis, and Politis, (2005). Here the parametric bootstrap and the three basic estimation parameters will be used.

The purpose of this study is to investigate the performance of the external sector statistics in the Nigerian economy. The external sector statistics is very crucial and strong determinants of economic growth. Therefore, it is very necessary to establish the models, the approximate distribution, stability of the test statistics, kernel density and qq plot of the external sector statistics in Nigeria. To the best of my knowledge they have not been established, and they are very important as the nation works hard toward attaining macroeconomic goals.

This study is carried out by using a parametric bootstrap method and by comparing the parametric models and parametric bootstrap method in the regression analysis in terms of their betas and standard errors. Datasets on export, import and gross domestic product (GDP) was used as the basis to define the population and the true standard errors. To buttress the main purposes of bootstrap; suppose we have a set of observations {x1, x2,…xn} and a test statistic θ. The resampling methods are often useful to examine the stability of θ and compute the estimations for the standard error of θ, where the distribution of θ is unknown, or that consistent estimations from the standard error of θ are not available, in this case the resampling (bootstrap) methods are especially useful.

3. Methodology

In estimating the bias and the standard error by the bootstrap methods, previous researchers either involved the nonparametric bootstrap method or focused on the two bootstrap methods in the context of the random groups' data collection design or the common-item equivalent groups design and so on; but none has bootstrapped regression model of independent identically distributed errors with mean and variance σ^2 not known. The interest for this paper was ignited by Wang 2011 research work which called for more research on the parametric bootstrap method and for comparative studies of parametric and nonparametric approaches. Also, the fact that parametric and nonparametric models at the sampling stage of the bootstrap methodology lead to procedures which are different from those obtained by applying basic statistical theory. Therefore, this research work is poised to investigate and understand the parametric bootstrap methods and to compare the parametric and nonparametric bootstrap methods in the parameter estimation of the simple linear regression (SLR) under a variety of assessment conditions. Also, to learn about the actual underlying data generating process (DGP) in PB and NPB through the examination of a hypothetical sample data and how to go about choosing the bootstrap DGP. Since the *true* standard errors were unknown that means we need to define the population distribution, a hypothetical and real data sets will be used as an illustration. In addition, hypotheses test, standard errors (precision), bias and bias of the standard error, the sampling distribution, bootstrap distribution of the sample will be established. Moreover, other information criteria; Akaike Information criterion (AIC), Schwart Bayesian Information criterion (SBIC), Hannan-Quinn Information criterion (HQIC) and probabilities of the models will also be established. It is pertinent to note that, all these estimation methods and information criteria together define the performance of an estimator.

Furthermore, the persistent rating of Nigeria among the world's poorest countries despite its enormous natural resources is really disturbing. Though, the positive contribution of the external sector to the economic growth of Nigeria can never be over emphasized. We still need to improve on export, especially, the non-oil products because if adequate care is not taken, the economy will suffer serious setback. Therefore, real data sets will be employed from the Nigeria external sector statistics to define population distributions in this paper and hypotheses test of the external sector statistics and the economic growth in Nigeria, the bootstrap distribution of the sample, their DGP and other information criteria will be established. To achieve this, Secondary quarterly data collected from Central Bank of Nigeria statistical bulletin 04, 2021: Financial statistics from 1991-2021 was analyzed using by S-plus softwares. The S-plus statistical package in which many functions were incorporated by Brennan, Wang, Kim, & Seol, (2009) and Efron and Tibshirani, (1993) will be used since it enables programs written by its S-plus codes to be executed.

Hypotheses

H_A: the distribution of θ is the same for all of the DGPs

H_B: a test statistic is said to be a pivotal

4. Results and Discussion

In the first main section, three subsections are included: (a) examination of the bootstrap DGP models with uncorrelated error term. (b) examination of the bootstrap DGP models, when the error term are not iid. (c) investigation of all bootstrap models under various assessment conditions. For convenience of reference, the following terms and abbreviations are used throughout:. Test 1 is the small bootstrap levels 19 & 99; Tests 2 and 3 refer to the medium and large bootstrap levels 199 & 499 and 999&1999, respectively. Group proficiency level 1 denotes the population θ distribution for Form *X* with the standard normal distribution, which is used as the baseline for comparison; group proficiency level 2 stands for the population θ distribution for Form M with $\theta \sim N(0,\sigma^2)$; and group proficiency level 3 indicates the population θ distribution for Form Z with $\theta \sim N(0,s^2)$. Note again that the θ distribution for Form *Y* is fixed as the standard normal distribution. For comprehensive examination and comparison of the bootstrap models, the bootstrap DGPs of different bootstrap square error of SLR.

The unrestricted residual bootstrap DGP

$$y_t^* = X_t\hat{\beta} + \mu_t^*, \qquad \mu_t^* \sim NID(\hat{\mu}_t) \tag{4.1}$$

Hypothetical Model : SLR Equation Estimated from the Unrestricted Residual:

$$HYPt = bo + b_1A + b_2B + e \tag{4.2}$$

Unrestricted Hypothetical Residual Model (HR311)

$$HYPt = 34.14231687\ b_1 + -0.05696246\ b_2 + e \tag{4.3}$$

Standard error (0.777766946) (0.004003802)

Bias (0.09031) (0.15873)

Real Model: SLR Equation Estimated from the Unrestricted Residual:

$$GDPt = bo + b_1A + b_2B + e \tag{4.4}$$

Unrestricted Real Residual Model (R311)

$$GDPt = 1.450e{+}00IM + 1.730e{+}00EX + e \tag{4.5}$$

Standard error (3.235e-01) (1.841e-01)

Bias (0.03412) (0.06084)

The residual bootstrap DGP using rescaled residuals

$$y_t^* = X_t\hat{\beta} + \mu_t^*, \qquad \mu_t^* \sim EDF(\ddot{\mu}_t) \tag{4.6}$$

where $\ddot{\mu}_t \equiv \left(\frac{n}{n-k}\right)^{1/2} \hat{\mu}_t$

Rescaled Residual Model (HRR311A)

$$HYPt = 34.98142316\ b_1 + 0.06962462\ b_2 + e \tag{4.7}$$

Standard error (0.652269461) (0.000038024)

Bias (0.08355) (0.13463)

The restricted (transformed) residual bootstrap DGP using the diagonals of the 'hat matrix'

$$y_t^* = X_t\hat{\beta} + \mu_t^*, \qquad \mu_t^* \sim EDF(\tilde{\mu}_t) \tag{4.8}$$

where $\tilde{\mu} = X(X^TX)^{-1}X^T$

Transformed Residual Model (HRR311B)

$$HYPt = 34.14231687\ b_1 + -0.05696246\ b_2 + e \tag{4.9}$$

Standard error (0.2267646) (0.000320302)

Bias (0.05789) (0.15854)

Real data set: SLR Equation Estimated from the restricted Residual using (4.4)

Restricted Rescaled Residual Model (RR311) using (4.6)

Rescaled Residual Model (RR311A)

GDPt = 1.445e+00IM + 1.725e+00EX + e (4.10)

Standard error (2.335e-01) (1.671e-01)

Bias (0.00027) (0.01622)

Restricted Transformed Residual Model (RR311B) using (4.8)

GDPt = 1.461e+00IM + 1.740e+00EX + e (4.11)

Standard error (1.325e-01) (1.208e-01)

Bias (0.00079) (0.00562)

Parametric bootstrap DGP with nuisance parameter

$y_t^* = X_t\hat{\beta} + \mu_t^*, \qquad \mu_t^* \sim NID(0, s^2)$ (4.12)

Hypothetical Model (HPN311): SLR Equation Estimated from the Hypothetical data set parametric bootstrap DGP with nuisance parameter using (4.2)

HYPt = 34.24231667 b_1 + 0.05356962 b_2 + e (4.13)

Standard error (0.2167646) (0.030320302)

Bias (0.13198) (0.12709)

Real Model (RPN311): SLR Equation Estimated from the Real data set parametric bootstrap DGP with nuisance parameter using (4.4)

GDPt= 1.7100153048b_1+ 1.4012314520 b_2+ e (4.14)

Standard error (0.302130) (0.1785102)

Bias (0.00198) (0.02709)

The wild bootstrap DGP is

$y_t^* = X_t\hat{\beta} + f(\hat{\mu}_t)v_t^*, \qquad \mu_t^* \sim NID(0,1)$ (4.15)

where $f(\hat{\mu}_t) = \frac{\hat{\mu}_t}{(1-h_t)^{1/2}}$

Hypothetical Model (HW311): SLR Equation Estimated from the Hypothetical data set wild bootstrap DGP with nuisance parameter using (4.2)

HYPt = 35.14231687 b_1 + -0.04696246 b_2 + e (4.16)

Standard error (0.3726646) (0.160320302)

Bias (0.19800) (0.13027)

Real Model (HW311): SLR Equation Estimated from the Real data set wild bootstrap DGP with nuisance parameter using (4.4)

GDPt = 1.601e+00EX +1.596e+00IM +e (4.17)

Std. Error (2.0010e-01) (3.7148e-01)

Bias (0.15019) (0.1240)

Pairwise bootstrap DGP

$y_t^* = X_{t\,t}^*\hat{\beta} + \mu_t^*, \qquad [y_{t,}^* x_t^*] \sim NID(\bar{x}, s^2)$ (4.18)

Hypothetical Model (HP311): SLR Equation Estimated from the Hypothetical data set parametric bootstrap DGP with nuisance parameter using (4.2)

HYPt = 34.314751687 b_1 + 0.05216246 b_2 + e (4.19)

Standard error (0.2117646) (0.014320302)

Bias (0.11217) (0.10209)

Real Model (P311): SLR Equation Estimated from the Real data set parametric bootstrap DGP with nuisance parameter using (4.4)

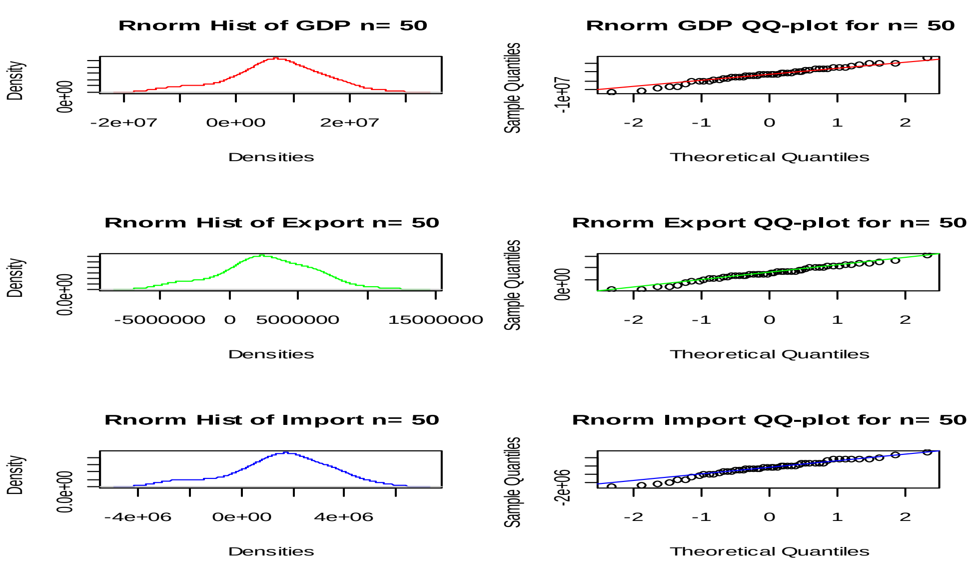
$$GDPt = 1.7302e{+}00\ b_1 + 1.4500e{+}00 b_2 + e \quad (4.20)$$

Std. Error (1.235e-01) (2.129e-01)

Bias (0.01198) (0.01206)

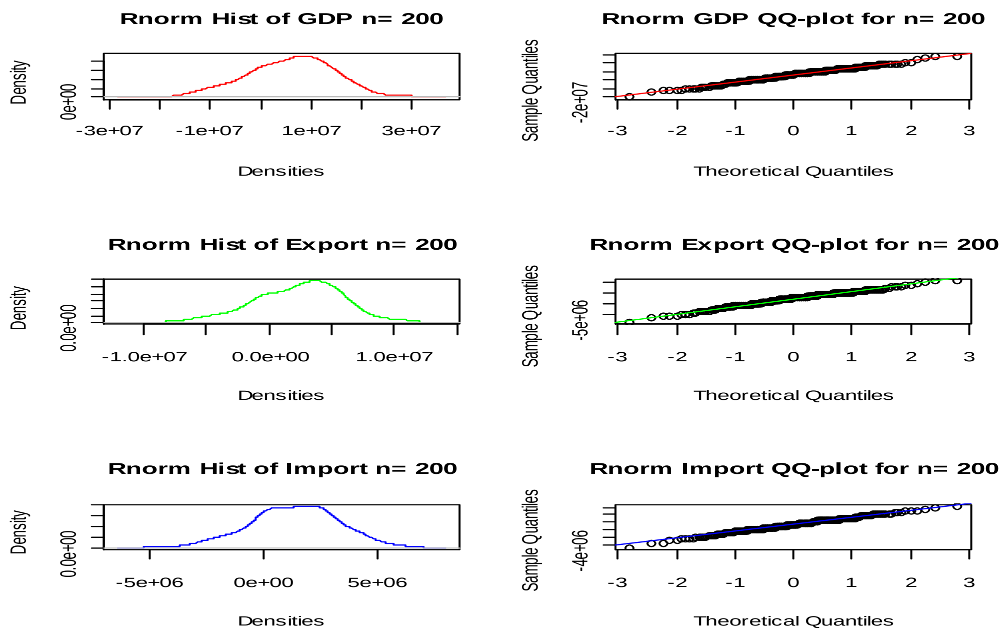
Here, the three regression approaches (OLS, MLE and MOM)on the original data set when simulated. compare the three models in terms of their betas, standard errors and bias to ascertain the best under several conditions, then the kernel density and qq plots of the original data set when simulated, using the following codes

```
>fix(step3.sim.run)
function (n=50)
{X<-cbind(gdp.dat[,"gdp"], gdp.dat[,"export"],gdp.dat[,"import"])
# The Ordinary Least Squares (OLS) Approach
Residuals:
    Min       1Q   Median       3Q      Max
-1997916  -505551   -12114   515768  1898213
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 5.145e+04  1.585e+05   0.325    0.747
export      1.601e+00  1.235e-01  12.968  < 2e-16 ***
import      1.596e+00  2.129e-01   7.499 1.44e-09 ***
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 856700 on 47 degrees of freedom
Multiple R-squared: 0.9895,     Adjusted R-squared: 0.989
F-statistic:  2213 on 2 and 47 DF,  p-value: < 2.2e-16
$counts function gradient
    478       92
$hessian
             [,1]          [,2]          [,3]          [,4]
[1,] 2.328306e-04  2.378563e+01  1.326040e+01  0.0000000000
[2,] 2.378563e+01  1.556862e+08  8.634348e+07 -0.0001164153
[3,] 1.326040e+01  8.634348e+07  5.072171e+07 -0.0001164153
[4,] 0.000000e+00 -1.164153e-04 -1.164153e-04  0.0000000000
```

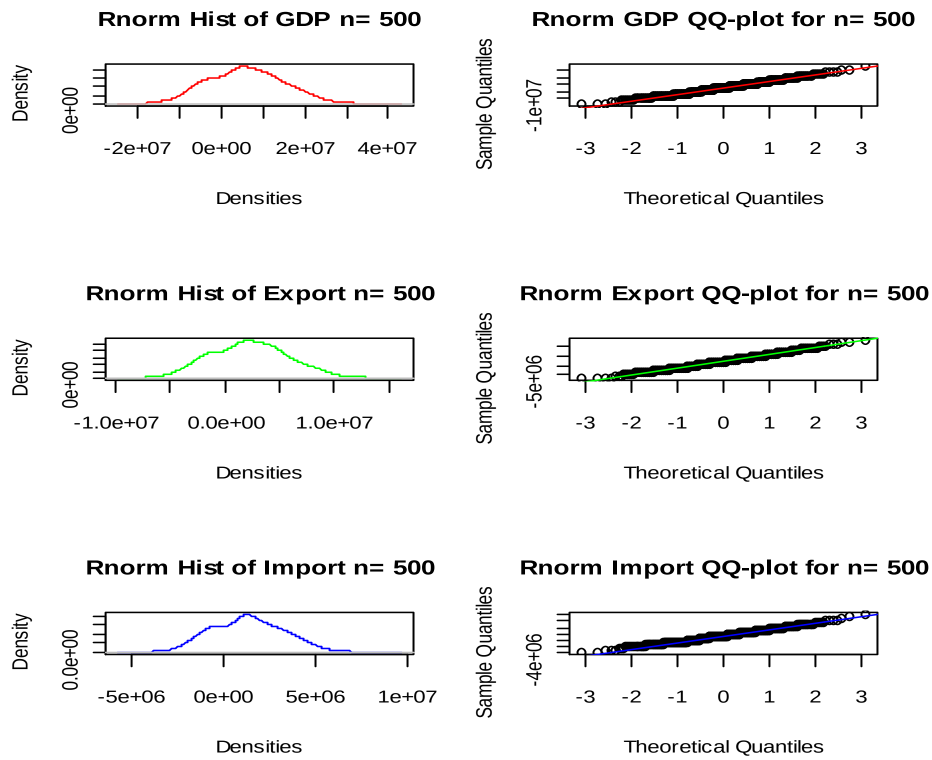



```
> step2.sim.run(n=200)
Coefficients:
              Estimate     Std. Error    t value       Pr(>|t|)
(Intercept)   -2.8719e+04   8.2063e+04   -3.4997e-01   7.2636e-01
```

```
zm1        1.6959e+00   6.5449e-02   2.5912e+01   4.8855e-148
zm2        1.4579e+00   1.1821e-01   1.2333e+01   5.9882e-35
J-Test: degrees of freedom is 0
            J-test              P-value
Test E(g)=0:    0.000563562913780727   *******
```

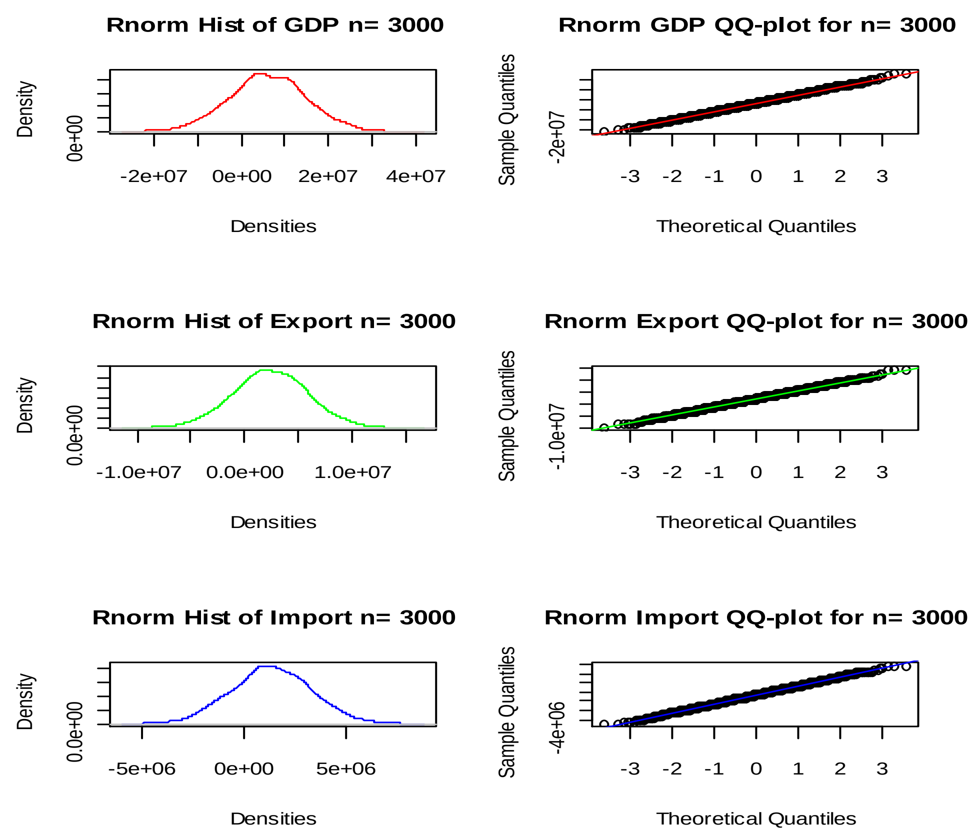



```
> step2.sim.run(n=500)
Call:
lm(formula = gdp ~ export + import, data = as.data.frame(gdp.dat))
Residuals:
    Min       1Q   Median       3Q      Max
-2857570  -578952   -32482   573406  2466749
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 2.224e+03  4.695e+04   0.047    0.962
export      1.697e+00  4.280e-02  39.646   <2e-16 ***
import      1.532e+00  7.668e-02  19.982   <2e-16 ***
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 878700 on 497 degrees of freedom
Multiple R-squared: 0.991,      Adjusted R-squared: 0.9909
F-statistic: 2.722e+04 on 2 and 497 DF,  p-value: < 2.2e-16
```




```
> step2.sim.run(n=3000)
lm(formula = gdp ~ export + import, data = as.data.frame(gdp.dat))
Residuals:
    Min       1Q    Median     3Q     Max
-3098758  -606585   -12611   583386  2808667
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 6.158e+04  1.987e+04    3.10  0.00195 **
export      1.710e+00  1.753e-02   97.53  < 2e-16 ***
import      1.481e+00  3.094e-02   47.88  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 891900 on 2997 degrees of freedom
Multiple R-squared: 0.9895,     Adjusted R-squared: 0.9895
F-statistic: 1.413e+05 on 2 and 2997 DF,  p-value: < 2.2e-16
```




```
> step2.sim.run(n=10000)

Call:
lm(formula = gdp ~ export + import, data = as.data.frame(gdp.dat))

Residuals:
     Min        1Q     Median     3Q      Max
-3555259  -610011  -11089   613374  3571528

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 3.865e+04  1.085e+04   3.562  0.00037 ***
export      1.715e+00  9.525e-03 180.016  < 2e-16 ***
import      1.478e+00  1.677e-02  88.173  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 884900 on 9997 degrees of freedom
Multiple R-squared: 0.9896,     Adjusted R-squared: 0.9896
F-statistic: 4.755e+05 on 2 and 9997 DF,  p-value: < 2.2e-16
```

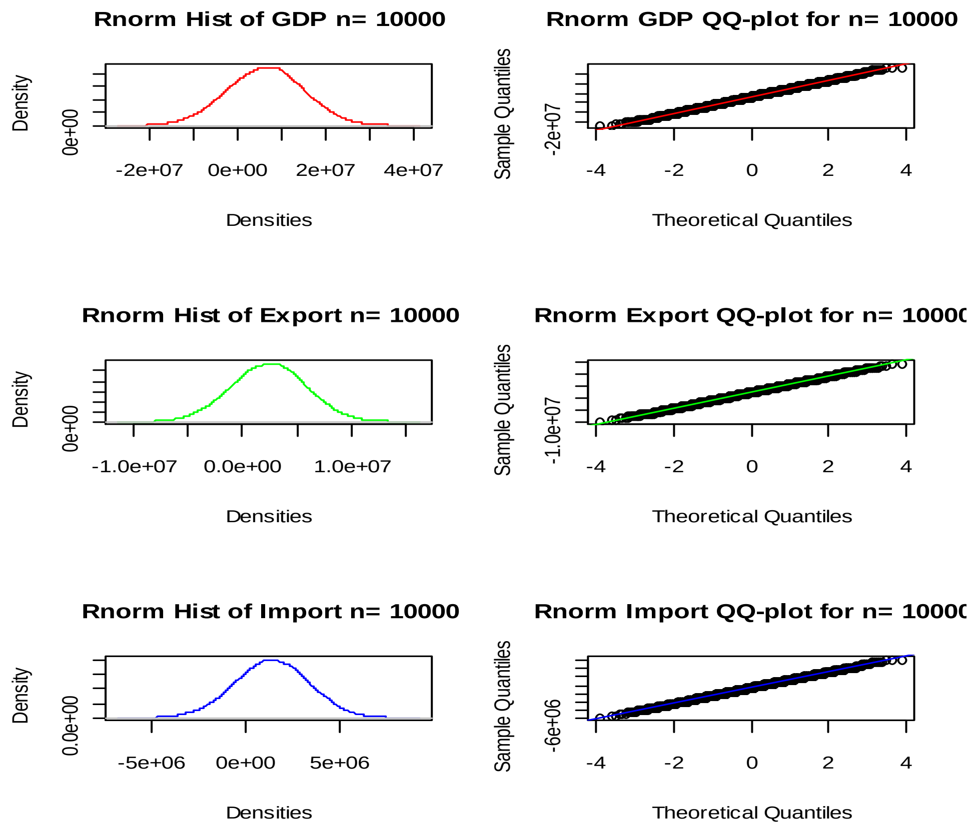


Table 1. Selection of the Best Model Based on other Evaluation Criteria

	R^2	R^2_{adj}	AIC	HQIC	SBIC	Convergence
$Parametric_{OLS}$	0.989564	0.989099	85.8622	92.8659	92.9692	1
$Parametric_{MLE}$	0.986853	0.987091	87.2397	86.2802	87.3139	1
$Parametric_{MOM}$	0.979855	0.988096	92.8241	87.4069	87.8204	1
$Parametric_{BOOT}$	0.989851*	0.999899*	84.9*	85.4435*	86.7872*	0

4.1 Interpretation of Results and Discussion

In terms of the bias, the nonparametric bootstrap models yielded much larger bias than the bootstrap models across most estimated values. The smallest and the second smallest bias were associated with the models of parametric bootstrap. Therefore, it seems clear that nonparametric bootstrap models were associated with larger bias while bootstrapped restricted parametric models were related to lower bias. Within the condition of equal sample sizes for both tests, these parametric bootstrap models could be generally divided into two groups based on the magnitude of the conditional bias in the plots (1) models *RW311, RPN311, and RP311, which produced larger bias; (2) models R311, RR311A and RR311B, which yielded smaller bias. It was observed that the differences among the models RR311A and RR311B were small, indicating that the parametric bootstrap models generated from restricted and transformed residual in this paper were very similar in terms of the bias of the SLR.

All the bootstrap models, particularly, the parametric bootstrap models with different sample sizes were similar in bias estimates across most of the estimated values, especially when the sample size was equal to or larger than 1,000. Generally the larger the sample size, the smaller the bias was. Across all combinations of the factors, the standard error from model RR311B was almost always smaller than those from the other parametric bootstrap models in all the score ranges. Thus, model RR311B also produced the smallest standard error. This is not surprising, because SLR models with low bootstrap level would produce less smoothed distributions, and thus larger standard errors. However, lower standard errors associated with models of high sample size and bootstrap level came at the cost of having lower bias and also less power loss in the estimates.

It should be noted that, when the sample size was 3,000, the standard error from all the parametric bootstrap models were so close that they were indistinguishable in the middle of the estimated values. The RMSE is an evaluation index which reflects both bias and standard error. As indicated above, a higher bootstrap level was associated with the smallest standard error and smallest bias, while a lower bootstrap level and sample sizes related to the smallest bias and standard error. Thus, in Test 1, model RR311B yielded the smallest conditional bias and RMSE. The same conclusion was drawn for Tests 2 and 3 when 2% of the scores at the lower end were excluded. When all the score points were taken into account, the results for Test 1 did not change, but for Tests 2 and 3, the impact of the estimated values of very low frequencies at the lower end was large. When the sample size was 200, model RR311B produced the smallest RMSEs; when the sample size was 1,000, model RR311A and RR311B produced the smallest RMSEs; when the sample size increased from (3,000 to 10000), it was model RR311B that produced the smallest total errors.

- **Bias and RMSE**: When the sample size was small (< 20) in almost all the bootstrap conditions, the nonparametric bootstrap method performed better than all of the parametric bootstrap models by showing the smallest conditional bias and RMSE. If the parametric bootstrap models with the high bootstrap level had not been explored in this paper, the nonparametric bootstrap model would have been exclusively ranked first in producing the smallest bias in the three tests. But for large sample (≥ 1,000), the reverse is the case.
- **Standard Error and RMSE**: Under all the bootstrap conditions, the nonparametric bootstrap method produced the largest standard error and RMSE.
- **Standard Error and Bias**: Almost the same finding was observed as with except that the nonparametric bootstrap method produced the largest standard error and bias under most, but not all, of the bootstrap conditions.
- **Other Information Criteria:** Across all the bootstrap conditions, it was obvious that all the models that worked well have very high AIC and adjusted R^2, confirming that the models are good model for further studies and predictions in the economic sectors.

In general, the nonparametric bootstrap method produced larger total error than the parametric bootstrap method. However, as the sample size increased, the differences between the two bootstrap methods became smaller. With a relatively short test and a not-too-skewed distribution, such as Test 1, when the sample size was equal to or larger than 3,000, the differences between the nonparametric bootstrap method and the parametric bootstrap model that produced the smallest RMSE (i.e., model RR311B) were very small (<0.01). For a longer test and a left-skewed distribution, then for Test 3, when the 2% of the scores at the low end were excluded, the same relationship between the two bootstrap methods existed as in Test 1.

Considering the complexity of the parametric bootstrap models, the model data fit issue, and the possible convergence problem, if the accuracy of the standard error was not required to be very high (no more than the third or fourth decimal point), the nonparametric bootstrap method may turn out to be a more feasible option for estimating the standard error SEE when the sample size is 3,000 or larger. The parametric bootstrap models investigated can be divided into two groups based on the magnitude of the RMSE: (1) unrestricted models produced less accurate estimates of the standard errors under all bootstrap conditions; (2) restricted models produced more accurate estimates of the standard errors. Thus, as the sample size and the bootstrap level increases, it should be noted that the results yielded by these models were very similar. An important finding of this paper is that restricted and transformed parametric bootstrap DGP models produced more accurate estimates.

In practice, if SLR models are employed as the parametric bootstrap method to estimate the standard error. We are advice to use the restricted and transformed parametric bootstrap DGP models and check the kernel density of the empirical distributions that are close to normal (at least not too skewed). In fact, models with high sample size and high bootstrap level are a good choice. Even though in the situation the nonparametric did not work well, there are situations where it can work perfectly well. It is pertinent to note that in a situation where the distribution of a test is skewed, and all the scores need to be taken into account, no matter how small the sample size and the bootstrap level is. It can be used to fit the model but if the sample size is large (>=1,000), the parametric bootstrap models with higher bootstrap level can be considered.

5. Summary of Findings/Conclusion

The findings from this paper regarding the performance of the parametric bootstrap method to estimate the SLR are summarized as follows and discussed with reference to the related literature.

First, it was found that the parametric bootstrap models with larger sample size and bootstrap level generally produced smaller bias than those with lower polynomial degrees. This is expected because the fitted distribution with SLR models is more similar to the distribution of the original data. Given the range of the bootstrap DGP methods investigated, the parametric bootstrap models can be divided into two groups based on parametric and nonparametric bootstrap DGPs with the magnitude of the conditional bias, standard error and RMSE: (1) bootstrap DGP models that produced larger bias, standard error and RMSE; (2) bootstrap DGP models that produced yielded smaller bias, standard error and RMSE. It was observed that the differences among the models with higher sample sizes were small, indicating that the parametric bootstrap models were very similar in terms of the yielded smaller bias, especially when the sample size was very large.

The finding on standard error in SLR was consistent with what was found in MacKinnon and Davidson (2006) but not in Cui and Kolen (2008) and Wang and Zhang (2009). In Cui and Kolen (2008) and Wang and Zhang (2009), the parametric bootstrap models with higher polynomial degrees tended to result in larger SE than the models with lower degrees. This is not surprising, because polynomial log-linear models with higher polynomial degrees would produce less smoothed distributions, and thus larger standard errors. It was also found that when the sample size was large (n = 3,000), the SE produced by all these parametric bootstrap models was small and similar at all the score points except for the two ends.

Comparing the parametric and nonparametric bootstrap methods in estimating the standard error and bias was another main purpose of this paper. The finding on this topic was consistent with the previous research done by Cui & Kolen, (2007) and Wang & Zhang, (2009): in the sense that, in most bootstrap conditions, the nonparametric bootstrap method generally produced less accurate estimates of the standard error than the parametric bootstrap method. The nonparametric bootstrap method was inferior to the parametric bootstrap method in that it produced the largest standard error across all bootstrap conditions. The result on bias shows that the parametric bootstrap models produced smaller bias than the nonparametric bootstrap method. This result on bias was not consistent with Cui and Kolen's (2007) because in his paper, he discovered that the nonparametric bootstrap method generally produced smaller conditional bias than the parametric bootstrap method. But the result was consistent with Davidson (2008). As to the comparison of the two bootstrap methods, the bootstrap paper also showed that, as the sample size increased, the differences between the two bootstrap methods became smaller. When the sample size was equal to or larger than 3,000, the differences between the nonparametric bootstrap method and the parametric bootstrap model that produced the smallest RMSE (i.e., model RPN311) were very small. When the bootstrap distributions are skewed and all the estimates are taken into account, any of the restricted or transformed bootstrap DGPs can be used to fit the model if the sample size is large ($n \geq 200$). In most bootstrap conditions, the nonparametric bootstrap DGP method generally produced less accurate estimates of the standard error than the parametric bootstrap DGP method.

The purpose of the work is to provide a good conceptual understanding of the bootstrap DGP especially the parametric method aspect with hypothetical and concrete example. This paper found that the bootstrap distribution created by resampling matches the properties of the sampling distribution. The heavy computation needed to produce the bootstrap distribution replaces the heavy theory (Central limit theorem, mean and standard deviation of x) that tells us about the sampling distribution and its accurate estimates. The great advantage of the bootstrap idea is that apart from it often works even when theory fails; it ascertains whether a particular theory holds. The models from this research work will be useful to government in predicting and forecasting trends in the Nigeria economy especially, in the external sector statistics, since the stability of the test statistic θ of ESS, its distribution and models will be established. It will also yield results that will be useful for future research and point a direction for such future studies.

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