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| RESEARCH ARTICLE

An Explanation to the Concept of Actual Infinity and Potential Infinity through Set Theory and Calculus

Ghulam Ali Sabery[1] ✉ Mohammad Naser Mohsini[2] and Marina Dawran[3]

[13]*Junier Teaching Assistant Professor, Department of Mathematics & Physics, Faculty of Engineering, Balkh University, Balkh, Afghanistan*

[2]*Junier Teaching Assistant Professor, Department of Mathematics, Faculty of Science, Balkh University, Balkh, Afghanistan.*

Corresponding Author: Ghulam Ali Sabery, **E-mail**: gh.alisabery@gmail.com

| ABSTRACT

The concept of infinity refers to either an unending process or a limitless quantity. Aristotle introduced two types of infinity: potential infinity and actual infinity. Potential infinity refers to a never-ending process, and actual infinity refers to a collection containing infinitely many elements. This paper presents a descriptive study of the concept of infinity and discusses its properties through set theory and calculus. Infinity plays a central role in the formation and development of mathematics, specifically in limit, derivative, and integral. Moreover, the similarities and differences between potential infinity and actual infinity are explained with the help of set theory and integral differential calculus. The relationship between mathematics and infinity is a vital one. Since infinity is an elusive and contradictory idea without mathematical tools, it is hard to understand it, and there is no other knowledge to explain and make it understandable. By the way, in the absence of infinity, mathematics will never survive. This paper provided some examples to show that without employing mathematics, solving problems involving infinity based on human intuitions or weak induction may provide inaccurate results or lead to contradictions. Therefore, this paper suggested that using mathematical tools is essential for solving problems involving infinity.

| KEYWORDS

Actual Infinity, Calculus, Cardinality, Potential Infinity, Set Theory

| ARTICLE INFORMATION

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1. Introduction

The concept of infinity is one of the most fascinating ideas that the human mind has ever been involved in and has raised amazing paradoxes and controversies in the fields of theology, physics and philosophy. Although the concept of infinity is a non-intuitive idea, it plays a central role in the definition of many mathematical concepts. This powerful concept is so widespread in mathematics that if we decide to cancel it, most of the mathematics content, such as Calculus, differential equations, geometry and set theory, will simply disappear. From the point of view of conceptual analysis and semantics, the study of mathematics and especially the concept of infinity raises several interesting questions. As an example, how can we understand the concept of infinity? Where does infinity come from? What mechanisms make its understanding possible? How does an elusive and contradictory idea like infinite create an objective and precise context like mathematics? Why do different forms of infinity in mathematics have a precise conceptual structure? Of course, these are not new questions; some of them have already attracted the attention of scholars in the fields of philosophy, philosophy of mathematics and logic for centuries. The problem, however, is that, historically, these disciplines have developed quite independently of the natural sciences and of actual human and conceptual reasoning.

The concept of infinity has occupied the human mind for centuries, challenged their understanding of the universe, and expanded the boundaries of mathematics and philosophy. A primitive human, who did not have the ability to count above five or ten, must have thought that he would not be able to count the number of stars by looking at the sky. Popper (1976) writes in his memoir titled "Un ended Quest", "When I was about 8 years old, I couldn't imagine whether the space is finite or infinite (If it is finite, then what is out there?). My father sent me to my uncle to answer my questions, but his explanation was not attractive at all. Popper (1976) says that at that age, it was understandable for me to understand that counting numbers does not end. Educated children who have a little familiarity with numbers are able to understand that counting does not end, but because they do not know the names of large numbers, they say, I do not know numbers greater than one million, for instance (Falk et al., 1986).

The ancient Greeks were one of the first to understand and grapple with the concept of infinity. Zeno's paradoxes in the fifth century BC are among the first philosophical discussions about the concept of infinity (Boyer & Merzbach, 2011). Zeno raised the questions of motion and continuity by presenting paradoxical scenarios involving divisibility with a never-ending process. Zeno's paradoxes are based on logical and mathematical arguments that sound rational but lead to strange and illogical results. Here, we can refer to the paradox of Achilles and the tortoise race (Russell, 1970). Aristotle divides infinity into two categories: potential infinity and actual infinity (Moreno & Waldegg, 1991; Bowin, 2007). Potential infinity refers to a concept in mathematics and physics that means a never-ending process. Unfinished processes such as Counting natural numbers is an examples of potential infinity. However, actual infinity refers to a set that contains infinitely many numbers of elements. Infinity is a philosophical concept that challenges the existence of finite quantities and suggests the existence of an unlimited quantity and process. The concept of Actual infinity is a matter of debate between mathematicians and philosophers. Some argue that actual infinity is a useful and necessary concept for some mathematical theories, while others question its logical coherence and argue for a more limited concept of potential infinity, which refers to processes that can continue indefinitely without reaching a finite quantity. In the description of infinity, Russell (1970) says, "There are real difficulties, especially in understanding the concept of infinity, because particular habits of mind are formed by the consideration of finite numbers, and as a result, a misconception is easily extended to infinite values, which sounds like to provide a necessary logic (Russell, 1970). Maor (1987) wrote in his book titled "Infinity and Beyond" that infinity usually induces a sense of terror and futility in the human mind. Moreover, in 1657, Pascal used the concept of infinity in the description of existence; he said, "I feel so immersed in the infinite greatness of space that neither I know anything about it, nor it knows anything about me. The eternal silence of these infinite spaces scares me" (Pascal, 1988).

The cognitive challenges of the concept of infinity, due to its very abstract nature, have been recognized throughout the history of thought and have caused many philosophical debates among philosophers and mathematicians. Although the Greeks accepted almost infinity in their mathematical system, due to the lack of algebraic language and accurate notation system at that time, they failed to define this concept in a mathematical framework (Falk, 1994). In the 17th century, mathematicians such as Galileo and Johan Kepler began to explore infinity within a mathematical framework. Newton and Leibniz laid the foundations of calculus by using the concept of infinitesimals. During the 18th and 19th centuries, other mathematicians worked on the work of Newton and Leibniz to further develop calculus. Notable figures in this period include Leonard Euler, Joseph-Louis Lagrange and Augustin-Louis Cauchy. They expanded the notion of limit, derivative, integral and mathematical series. With the development of mathematics, the concept of potential infinity was explained more in advance, and using mathematical rules, they showed that human intuition is not enough to understand the concept of infinity, and a convincing answer was given to Zeno's paradoxes. In the current context, calculus has become a powerful tool that has revolutionized our understanding of the physical world (Rosenthal, 1951; Boyer and Merzbach, 2011). Galileo understood that the reason for the emergence of paradoxes is due to applying the arithmetic rules of finite numbers to infinite values. He believed that a new calculus should be created to be able to explain infinity in a stable and accurate way. But, Galileo failed to make such a change in the calculus of finite numbers (Falk, 1994). However, in the late 19th century, it was George Cantor who revolutionized our understanding of infinity with his groundbreaking work on set theory. Cantor's approach to infinity was revolutionary at that time because it challenged traditional mathematical thinking. He introduced the different sizes of infinity and showed that all infinities are not necessarily equal. Cantor introduced the concept of transfinite cardinal numbers to represent different sizes of infinity. He defined $\aleph_0$ as the cardinality (number of elements of a set) of the set of natural numbers and showed that there also are infinities greater than $\aleph_0$. Cantor's work on infinity met with considerable opposition from some mathematicians at the time, who considered his ideas counterintuitive and contradictory. However, Centro's contributions laid the foundations of modern set theory and influenced various branches of mathematics and philosophy (Núñez, 2005; Cantor, 1915).

During the last two centuries, more than any other time in history, researchers have been attracted to the concept of infinity and mathematicians and psychologists have paid deep attention to it and published very useful research articles in the field of knowledge, importance and characteristics of this concept. Monaghan (2001) conducted research on young people's views on infinity. In this research, he presents a historical report of studies that examine the ideas of young people about infinity, and the methodological problems in achieving such ideas are the subject of his article. Fennema and Hart (1994) conducted research to investigate the role of gender in understanding mathematical concepts, with the interesting finding that boys provided more appropriate responses to infinity than girls. According to them, the gender difference in understanding the mathematical concept is still one of the most difficult and challenging issues. Hannula et al. (2004) conducted a research project titled "Development of

Understanding and Self-Confidence in Mathematics; Grads 5-8" from 2001 to 2003. This research was a two-year study for fifth to eighth grade school students. The results of their research show that most students do not have a proper and logical view of infinity, but as student's age increases, their understanding of the concept of infinity increases. Another research was conducted by the University of Cyprus to investigate the understanding of school level teachers about infinity. In this research, they considered two aspects of infinity, i.e. as a process or as an object. In addition, teachers' reactions were analyzed when comparing infinite sets or numbers with infinite decimals. Data was collected through a self-assessment questionnaire that was implemented on 40 elementary school teachers. The results showed that most teachers understand infinity as a continuous and endless process; however, they face problems and have misconceptions about this concept (Maria et al., 2010). Conducting studies on abstract concepts such as infinity provides an opportunity for teachers to address misconceptions about the concept of infinity. If these misconceptions are reproduced during teaching, then students' misconceptions about the concept of infinity are reinforced and become very difficult to overcome. The concept of infinity is related to important mathematical concepts, such as the comparison of infinite sets, which are important for arithmetic and algebra. For this reason, teachers should be aware of the difficulties they face in relation to the specific concept in order to avoid "problematic" teaching. Additionally, it is important for teachers to develop a conceptual understanding of infinity (Singer and Vaika, 2003).

The rest of this paper is organized as follows: the second section provides a full description of both actual infinity and potential infinity through set theory and differential integral calculus. Section 3 discusses the importance of mathematical tools in understanding infinity. Finally, section 4 summarizes and concludes the work done in this paper.

2. The Concept of Infinity in the Set Theory and Calculus

This section explains the actual and potential infinity using mathematical concepts. First, we explain the concept of actual infinity from the point of view of set theory; then, we provide a detailed description of the concept of potential infinity with the help of differential integral calculus.

2.1 Infinity in Set Theory

The set theory is a branch of mathematical logic that studies the concept of a set/collection of objects. The set theory provides the basis for mathematics and has a significant influence on various fields of mathematics. The formation of sets can be attributed to ancient civilizations such as Babylonia and Egypt, where sets were used to count and organize objects. However, it was in ancient Greece that the foundations of the theory of sets began to form, resulting in philosophers such as Zeno and Aristotle also proposing interesting discussions about the concept of infinity and its consequences (Beth, 1959; Kunen, 2014). Cantor is often referred to as the father of set theory because he introduced many fundamental concepts of a set's cardinality. Cantor's work on sets led to groundbreaking discoveries, including his famous argument that there are different (transfinite cardinal numbers) infinity. He also developed the concept of countable and uncountable sets, which revolutionized our understanding of infinity (Cantor, 1915). Counting is undoubtedly one of the oldest human activities. Even a child who is unable to count can tell if there is a chair in the room for each guest. To understand this, he/she only needs each guest to sit on a chair. With this process, a one-to-one relationship will be formed between the chair and the guest. It can also be said that seats and people are related to each other one-to-one, that is, in such a way that there is exactly one person for each seat and vice versa. If we represent the guests and chair as two sets, we can see that the number of elements of the two sets is equal. Therefore, this simple and basic idea can also be extended to arbitrary sets. For example, consider the following sets.

$$\mathbb{N} = \{1, 2, 3, 4, \cdots, n, \cdots\}, S = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}.$$

By creating a one-to-one relationship between the elements of two sets, $\mathbb{N}$ and S, we will see that for any element, n of N, there will be an element, $\frac{1}{n}$ for S. Hence, the set N will be having as many elements as S.

Definition 1. Two sets A and B are called equivalent sets; if $f: A \to B$ is a bijective function, we write $A \sim B$.
Definition 2. An infinite set A is called countable if it is equivalent to the set of natural numbers, N.
Theorem 1. The following propositions are true (Jain and Gupta, 1986).

- The set of rational numbers and every subset of natural numbers is countable.
- Each real number interval is equivalent to the set of real numbers.
- Real numbers and any real intervals are not countable.
- The union, intersection, Cartesian product of two or more countable sets is countable.

Definition 3. The cardinality of a set A is the size or number of elements of it. The cardinality of the set A is denoted by $Card(A)$ or $|A|$.

For finite sets, cardinality is simply the number of elements in the set. For example, if we have $A = \{a, b, c\}$, then $|A| = 3$. But for infinite sets, determining cardinality can be more complicated.

Definition 4. If α and β are two cardinal numbers, and there exist sets A and B such that their cardinalities are α and β, respectively. We say $\alpha \leq \beta$ if the set A is equivalent to one of the subsets of B. Or in other words, $\alpha < \beta$, whenever the set A is equivalent to one of the subsets of B and $A \nsim B$.

As the results of definition 4, for two non-empty sets A and B, we will have:

- If $A \sim B$, then $Card(A) = Card(B)$
- If $A \subseteq B$, then $Card(A) \leq Card(B)$
- If A is a non-empty set and $\wp(A)$ is its power set, then $Card(\wp(A)) \leq Card(A)$.

2.1.1 How many Actual infinities are there?

The question of how many infinities are there seems to be very illogical. When we talk about infinity, it means that there is nothing beyond. However, George Cantor provides the opportunity to think more about this question. Of course, the word infinity in this section means the transfinite cardinal numbers. A detailed description of these questions is provided as follows:

The first infinity: George Cantor denotes the smallest infinity by $\aleph_0$, which is called countable infinity. The smallest infinity (countable infinity) is the cardinality of natural numbers. Since every infinite countable set is equivalent to the set of natural numbers, therefore, all infinite countable sets have the cardinality, $\aleph_0$ (Jain and Gupta, 1986).

The second infinity: The second infinity is denoted by $\aleph_1$ and it is the cardinality of the set of real numbers. Based on theorem 1, every open or closed interval of real numbers is equivalent to the whole set of real numbers. Therefore, the cardinality of every open or closed interval of real numbers will be $\aleph_1$. We know that the set of natural numbers is a subset of real numbers; therefore, according to the definition 4, $\aleph_0 < \aleph_1$. Moreover, George Cantor proved that $\aleph_1 = 2^{\aleph_0}$ (Niñones, 2005, Jain and Gupta, 1986).

The third infinity: We show the third infinity by $\aleph_2$. Undoubtedly, $\aleph_2$ must be greater than the previous two infinities. If we make natural and real numbers the basis of thinking about infinite numbers, our intuition tells us that we don't have a set of numbers that has a greater cardinality than real numbers. But by constructing new sets, we will show that there is a third infinity, and we will generalize this process later. Assume that $\aleph_2$ is the cardinality of the set F, so that F is the set of real functions defined on the interval [0,1]. Núñez (2005) explains in a logical way that $\aleph_1 < \aleph_2$. As a result, this process can be generalized as follows.

$$1 < 2 < 3 < \cdots < \aleph_0 < \aleph_1 < \aleph_2 < \cdots \qquad (1)$$

So that $\aleph_i = 2^{\aleph_{i-1}}$ for $i = 0,1,2,\cdots$. Then we have the set of cardinal numbers $C = \{1,2,3,\cdots,\aleph_0,\aleph_1,\aleph_2,\cdots\}$. Now, the question raised is whether the set of cardinal numbers is countable or not. Although there is no well-founded answer to this question in the literature, according to George Cantor's continuum hypothesis, they seem to be countable (Jain and Gupta, 1986).

2.1.2 Algebraic operations

When we perform four mean arithmetic operations and power rules on numbers in elementary mathematics, usually, they are considered completely as a single concept, and the operation takes place regardless of their relationship with objects and other concepts. But, when we perform arithmetic operations on infinities, each desired number is related to a set, and any operation that takes place on it, the desired set should be in the focus of our attention.

Addition: If α and β are two cardinal numbers, and there are separate sets A and B such that their cardinality is α and β, respectively. So $\alpha + \beta = Card(A \cup B)$. Now, the addition of cardinal numbers can be summarized as follows:

- $\aleph_0 + \aleph_0 = \aleph_0 = \aleph_0 + 0$
- $n + \aleph_0 = \aleph_0,\ n \in \mathbb{N}$
- $\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0$
- $\aleph_0 + \aleph_1 = \aleph_1$
- $\aleph_1 + \aleph_2 = \aleph_2$
- $\aleph_1 + \aleph_1 + \cdots + \aleph_1 = \aleph_1$
- $\aleph_1 + \aleph_2 + \cdots + \aleph_n = \aleph_n,\ n \in \mathbb{N}$

(Niñones, 2005, Jain and Gupta, 1986).

Subtraction: The operation of subtracting infinities can become one of the most complex logical, philosophical and mathematical issues. The basic rules that apply to real numbers may not apply to infinity, and this issue causes ambiguity. We explain the issue with an example. Assume that the operation of subtraction on infinity also applies, so the equation $\aleph_0 - \aleph_0 = 0$ must be true. On the other hand, we know that $\aleph_0 + 1 = \aleph_0$, so by removing $\aleph_0$ from both sides of the equation, we will have $0 = 1$, and this is a contradiction. Therefore, when performing the subtraction operation on infinity, we should be careful because the basic mathematical rules on infinity are not true. It can be said that the subtracting of infinity causes ambiguity (Niñones, 2005).

Multiplication: If α and β are two cardinal numbers, and there are separate sets A and B such that their cardinality is α and β, respectively. Then, $\alpha\beta = Card(A \times B)$. Rules such as transformative, unitary and distributive properties above infinity also apply (Jain and Gupta, 1986).

- $\alpha \cdot \beta = \beta \cdot \alpha$
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$
- $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
- $n \cdot \aleph_0 = \aleph_0,\ n \in$
- $\aleph_0 \cdot \aleph_0 \cdots \aleph_0 = \aleph_0$
- $\aleph_0 \cdot \aleph_1 = \aleph_1$
- $\aleph_0 \cdot \aleph_1 \cdot \aleph_2 = \aleph_2$
- $\aleph_1 \cdot \aleph_1 \cdots \aleph_1 = \aleph_1$

Division: The division operation on infinity, like the subtraction operation, leads to ambiguity. Therefore, there is no defined rule for dividing infinity from another infinity. In other words, the result of infinite division leads to ambiguity.

Power: If α and β are two cardinal numbers, and there are two separate sets A and B such that their cardinality is α and β, respectively. Therefore, α^β is the number of functions defined from the set A on B; $\varphi_j: B \to A$, where j is a characteristic set which depends on the number of elements of two sets A and B. The rules of power can be generally summarized as follows (Jain and Gupta, 1986; Niñones, 2005):

- $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$
- $\left(\alpha^\beta\right)^\gamma = \alpha^{\beta \cdot \gamma}$
- $(\alpha \cdot \beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$
- $\alpha^\gamma \leq \beta^\gamma$, *if* $\alpha \leq \beta$
- $\alpha^\gamma \leq \alpha^\beta$, *if* $\gamma \leq \beta$

2.1.3 George Cantor's Continuum Hypothesis

The continuum hypothesis stated that there is no set A whose cardinality is μ such that $\aleph_0 < \mu < \aleph_1$. Cantor's hypothesis still stands, and no one has found a contraction example for this hypothesis (Jain and Gupta, 1986). If we extend this hypothesis, one can say that there is no set A whose cardinality is μ such that $\aleph_i < \mu < \aleph_{i+1}$, for $i \in \mathbb{N}$. Therefore, the set of transfinite cardinal number C_{TR} can be defined as $C_{TR} = \{\aleph_0, \aleph_1, \aleph_2, \cdots\}$. Hence, one can say that there will be as many transfinite cardinal numbers as natural numbers.

2.2 Infinity in Calculus

Differential and integral calculus is based on the assumption that there are infinitely many uncountable real numbers. That is, if we consider any two real numbers, there will be a real number between them. This property is called the integrity property of real numbers. If real numbers didn't have the property of integrity, calculus and many other branches of mathematics would not exist. In this section, we try to explain the infinity in limit, derivative, integral and mathematical series. The infinity that is discussed in calculus is potential infinity, and actual infinity has no meaning in this part of mathematics. It's better to say that, contrary to its name, it has no reality in calculus and all mathematical concepts are based on the potential infinity. Of course, it must be admitted that the concept of infinity is more abstract than any other concept in mathematical knowledge, but it would not be an exaggeration to say that actual infinity is the most abstract abstraction in mathematical concepts.

The notion of a limit is one of the basic concepts in mathematical knowledge that describes the behavior of a function around a point. The limit is developed based on the concept of the infinitesimal. Therefore, the notion of limit can be considered the most important step in the formation of modern mathematics. The concept of limit is the only tool in mathematics that is able to describe the concept of infinity, and wherever there is infinity, the presence of a limit is certain. Now, we describe the limit of function and explain the concept of infinity using limit. Let the function $y = f(x)$ is defined on the interval $c \in (a, b)$ (it is not necessary that it is defined at c). If the independent variable x is sufficiently approached to c, a real number L is called the limit of the function $y = f(x)$, if for every positive number ε (epsilon) there exists a positive number δ (delta) such that

$$|f(x) - L| < \varepsilon \leftrightarrow 0 < |x - c| < \delta. \qquad (2)$$

Usually, in mathematics, (2) is written as $\lim_{x \to c} f(x) = L$ (Thomas et. al, 2010). Now, using (2), we explain the following limit in order to describe the concept of potential.

$$1.\ \lim_{x \to \infty} f(x) = L \qquad 2. \lim_{x \to c} f(x) = \infty \qquad 3. \lim_{x \to \infty} f(x) = \infty$$

1. This means that when the variable x becomes as large as possible, the function approaches the number L. If N is an arbitrary large positive number, then $x \to \infty$ means that $|x| > N$. That is, the variable x becomes larger than any large number you can imagine, and the function approaches the number L accordingly.
2. This means that when the variable x approaches a number c, the function is bigger than any big number you can imagine. If M is an arbitrary large positive number, then $|f(x)| > M$. That is, the function is greater than any arbitrary positive number.
3. This means that when the variable x gets as big as you want, the function gets bigger than any big number you can imagine. If N and M are arbitrary large positive numbers. That is, when the variable x becomes larger than any arbitrary positive number such as N ($|x| > N$), then the function $f(x)$ will become greater than any arbitrary positive number such as M ($|f(x)| > M$).

We use the word "approach" to explain the notion of limit. That is, when the independent variable, x gets close enough to a point. x_0, and the function, $f(x)$, gets close enough to a number, L, but do not reach the desired values exactly. Therefore, there will be a distance between the variables (independent or dependent) and the desired values, which are represented by ε, δ, etc., and these distances are called infinitesimal. Infinitesimal refers to a positive number that is smaller than any positive real number you can imagine. Hence, infinitesimals don't take any specific numerical value because there is no smallest positive real number at all. As we know, the derivative and integral of a function are also explained with the help of the limit notion. If the function $y = f(x)$ is continuous on the interval $x \in (a, b)$, then the derivative of the function, $f(x)$ is defined as follows:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (3)$$

Since Δx is an infinitesimal, the numerator and denominator of the fraction (3) are two infinitesimals. Therefore, it can be said that the derivative of a function is the instantaneous changes of the function. On the other hand, if $f(x) \geq 0$ for $x \in (a, b)$, in order to obtain the area surrounded by function curve x axis in the given interval, we need to use the limit Riemann sum. The limit of a Riemann sum is the value that the sum approaches as the width of the subintervals approaches zero, and the number of subintervals approaches infinity. This limit represents the exact area under a curve and is known as the definite integral of the function over the given interval. Let us denote the area surrounded by the function curve and x axis by S; hence, we will have:

$$S = \lim_{n \to \infty} \sum_{k=0}^{n} f(x_k) \Delta x = \int_a^b f(x) dx \qquad (4)$$

where $\Delta x = (b - a)/n$ and $x_k = a + k\Delta x$. It is easy to see that calculus is formed by assuming the existence of the potential infinity. If this concept is abolished, many mathematical concepts will suffer structural problems. Eqs (4) shows that if the function $y = f(x)$ is a piecewise continuous on the interval $[a, b]$, the series on the right side of Eqs (4) is convergent. Therefore, S is a finite numerical value. By introducing numerical and functional series in mathematics, it can be seen that presenting finite numbers as an infinite sum of real numbers is one of the forms of presenting real numbers. In addition to this, with the help of this branch of mathematics, we can understand Zeno's paradoxes (Huggett, 2010a,b).

3. Discussion

In this section, we explain why we need to employ mathematical methods to solve problems involving infinity and, if we don't use them, how a contradictory result comes out. Moreover, we will explain that the potential infinity is not a number but a symbol in mathematics to show an unending process, and there is no largest number at all.

3.2 Intuitive mistake in mathematical series

The sum of the limits of an infinite synonym is called a series. The topic of series in mathematics is one of the unique concepts; here, we want to explain the error of intuitive understanding with the help of the following series.

$$1, -1, 1, -1, 1, -1, \cdots \qquad (5)$$

Suppose that we are not familiar with the rules of mathematical series, and we are asked to find the sum of the series of numbers given in Eqs. (5), so we will arrive at two different answers, that's mean;

$$(1-1)+(1-1)+\cdots=0+0+\cdots=0 \qquad (6)$$
$$(1+1+\cdots)-(1+1+\cdots)=\infty-\infty \qquad (7)$$

The sum of the results in relation (6)-(7) seems correct and logical from the point of view of pure logic and weak induction based on our intuition, but this answer will not be acceptable for a wise person, and meanwhile, You cannot explain the reason for your non-acceptance because both answers were received based on a so called logical arguments. We know that calculus, if $|x|<1$ we will have:

$$P(x)=\frac{1}{x+1}=1-x+x^2-x^3+\cdots \qquad (8)$$

Now, if we substitute $x=1$ in (8), according to (6) and (7), we will get

$$\frac{1}{2}=0 \text{ و } \frac{1}{2}=\infty-\infty \qquad (9)$$

Eqs. (9) presents a strange contradiction. The reason for this contradiction is the non-observance of a simple mathematical rule and that in (8) $|x|<1$. Therefore, it can be deduced from (9) that when we deal with infinite concepts, we have to resort to mathematical laws, our intuition is not able to understand the problem correctly, and the answer resulting from weak induction is full of contradictions. If we get back to (5), now we can say that the series is not convergent. Zeno's paradoxes puzzled the people of that time because they were not familiar with calculus, and without using mathematical tools, one could not have a proper understanding of the quantities that somehow deal with infinity. Presenting numbers in the form of a numerical series is a form of presenting a number that can only be understood with the help of mathematical rules. For example, the understanding of the equation $1=0.\overline{9}$ is very simple for university and even school level students, while it is difficult and even impossible to understand it through philosophical and logical arguments. As a result, employing mathematics makes it easier to understand any problems related to infinity, and the answer will be accurate and reliable.

3.3 Infinity, zero and the greatest number

If the question is raised whether, infinity is the greatest number? The answer is that there is no greatest number, and infinity is not a number but a never-ending process. Consider the number $a_n=1/n$. If n becomes large enough, then a_n will approach zero. If n becomes small enough, then a_n will become infinitely large. Hence, we have the following.

$$.\lim_{n\to\infty}\frac{1}{n}=0, \quad \lim_{n\to 0^+}\frac{1}{n}=+\infty, \ \lim_{n\to 0^-}\frac{1}{n}=-\infty$$

It is wrong to present the above limits as $\frac{1}{\infty}=0$ and $\frac{1}{0}=\infty$, because ∞ is not a number but a symbol to represent the potential infinity. While dividing a number by zero is an undefined operation. We must be careful when the variable n approaches zero; it never reaches the desired destination, zero, and there will be a small positive distance, say ε, between n and zero. That means; $|n-0|=|0^{\pm}-0|<\varepsilon$. Where the symbol $0^{\pm}$ is a positive/negative number very close to zero. When a variable approaches infinity, it means that the variable is larger than any positive number we can imagine. Therefore, we should not use the infinity as a real number in the mathematical computations. Moreover, as we know that, the negative and the positive numbers in calculus means moving in the opposite direction; likewise, negative infinity is in the opposite direction of positive infinity.

4. Conclusion

Mathematics distinguishes itself from other branches of knowledge and conceptual systems because it is highly idealized and essentially abstract. No empirical method of mere observation can lead to an accurate understanding of mathematics. For example, consider a point, the simplest concept in Euclidean geometry, which has only location but no dimension. How can you test a conjecture about Euclidean points by experimenting if they are dimensionless? And how can you empirically observe a line if it only has length but no width? The discovery (invention) of mathematics is the most amazing discovery of mankind. Mathematics has a very extraordinary ability to make these abstract concepts understandable and explain them to humans. Moreover, mathematics plays a fundamental role in explaining the behavior of natural phenomena. As Galileo says that, “God has designed the universe in mathematical language and by understanding mathematics, you can understand God's mind”. Generalization and abstraction are the most obvious properties of mathematics, and they are developed through them. Furthermore, mathematics has become a unique language in the science’s world, and almost every scientific conclusion takes place based on it and. also employing mathematics will increase the accuracy and reliability of any computational activity or experiments. The concept of infinity is the most abstract among mathematical concepts, which has a very mysterious and ambiguous nature. Infinity can never be experienced in the physical world, so understanding it without a powerful and accurate tool like mathematics would be very vague. Aristotle introduced two types of infinity: potential infinity and actual infinity. Potential infinity refers to a never-ending process, and actual infinity refers to a collection containing infinitely many elements. The relationships between infinity and mathematics are existential in understanding, developing, and expressing both. The concept of infinity and other amazing concepts that have come in the form of extraordinary phenomena have formed a complex mathematical knowledge. The concept of infinity is one of the most fascinating ideas that the human mind has ever been involved in and has raised amazing paradoxes and controversies in the fields of theology, physics and philosophy. Although the concept of infinity is a non-intuitive idea, it plays a central role in the definition of many mathematical concepts. Moreover, we have seen that solving problems involving infinity without employing mathematics may lead to inaccurate and contradictory results. Therefore, this paper suggested that mathematical tools must be used for solving the problems involving infinity.

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