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| RESEARCH ARTICLE

Boundedness Analysis of the Fractional Maximal Operator in Grand Herz Space on the Hyperplane

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| ABSTRACT

The primary purpose of this work was to prove the boundedness of the Fractional Maximal Operator in Grand Herz Spaces on the Hyperplane. Here, We defined Grand Herz Space in a continuous Case. For Simplicity, We divided our Problem into two theorems by taking two subsets of Hyperplane($\mathbb{U}$) as ($\mathbb{U}(1)$) and its complement ($\mathbb{U}\backslash\mathbb{U}(1)$). We proved the boundedness of the Fractional Maximal Operator in Grand Herz Space on these two subsets of Hyperplane. We also defined the continuous Case of Grand Herz Space. We proved some results to use in our proof. We represented other terms this paper uses, i.e. the Hyperplane and Fractional Maximal operator. Our proof method relied on one of the corollaries we gave in this paper. We proved the condition to apply that corollary, and then by referring to this, we confirmed both of our theorems. This paper is helpful in Harmonic analysis and delivers ways to analyse the solutions of partial differential equations. The Problem of our discussion provides methods to study the properties of very complex functions obtained from different problems from Physics, Engineering and other branches of science. Solutions of nonlinear Partial Differential equations often resulted in such functions which required deep analysis. Our work helps check the boundedness of such types of functions.

| KEYWORDS

Boundedness, Fractional Maximal Operator, Grand Herz Space, Hyperplane

| ARTICLE INFORMATION

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1. Introduction

Grand Herz Spaces (GHS) are in active research these days. Herz spaces (HS) introduced by Herz (Lipschitz 1968, p.14) play a significant role in harmonic analysis and PDEs. These are very helpful in studying functions in every field of Science, i.e. Physics, Chemistry, Biology, Engineering and many more. The boundedness of the maximal operator M_λ was studied for Herz spaces (HS) in Li and Yang (Li 1996, p.14) and Herz spaces (HS) with variable exponents in Almeida and Drihem (Almeida 2012, p.14). See also literature (Izuki 2009, p.14), (Mizuta 2020, p.14), (Ragusa 2009, p.14) and (Samko 2013, p.14). Cruz-Uribe, Fiorenza and Neugebauer (Cruz-Uribe 2012, p.14) proved an extension of the well-known weighted boundedness results for the maximal operator in variable weighted Lebesgue spaces. Capone, Cruz-Uribe and Fiorenza (Capone 2007, p.14) studied the boundedness of the fractional maximal operator M_λ in $L^{p(\cdot)}(\mathbb{R}^n)$. In (Mizuta 2015, p.14) Weighted Morrey spaces of variable exponent and Riesz potentials were studied. The work of Mizuta Y, Shimomura (Mizuta 2021, p.14) shows the boundedness of the fractional maximal operator M_λ in the Herz spaces $H^{p(\cdot),q,\eta}(\mathbb{U})$ on the Hyperplane. Inspired by this work we shall prove the Boundedness of Fractional Maximal Operators in Grand Herz Spaces on the Hyperplane. We proved the result with the help of two theorems by splitting the Hyperplane into two subsets $\mathbb{U}(1)$ and $\mathbb{U}\backslash\mathbb{U}(1)$ and treating them separately. In this paper, C is a constant and is an independent parameter.

2. Preliminaries

2.1 Grand lebesgue space

2.1.1 Definition

The grand lebesgue space is defined as the space of Lebesgue measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\epsilon<p-1} \left(\frac{\epsilon}{|\Omega|}\int_\Omega |f(x)|^{p-\epsilon} dx\right)^{\frac{1}{p-\epsilon}} < \infty$$

Where $|\Omega|$ is Lebesgue measure of Ω.

2.1.2 Hölder inequality for Grand Lebesgue space

By [12] the following Hölder-type inequality holds.

$$\frac{1}{|\Omega|}\int_\Omega fg dx \le \|f\|_{p)}\|g\|_{(p'}\forall f \in L^{p)}(\Omega), \forall g \in L^\infty(\Omega)$$

Where $\|g\|_{(p'}$ is norm of small lebesgue space given by

$$\|g\|_{(p'} = \inf_{g=\sum_{k=1}^\infty g_k}\left\{\sum_{k=1}^\infty \inf_{0<\epsilon<p-1} \epsilon^{\frac{-1}{p-\epsilon}}\left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\epsilon)'} dx\right)^{\frac{1}{(p-\epsilon)'}}\right\}$$

2.2 Fractional Maximal Operator

2.2.1 Definition

For the Euclidean space $\mathbb{R}^n$ $(n \ge 2)$ and $0 \le \lambda \le n$ a measurable function f on $\mathbb{R}^n$, we define the fractional maximal function $M_\lambda f$ as

$$M_\lambda f(x) := \sup_{r>0} \frac{r^\lambda}{|B(x,r)|}\int_{B(x,r)} |f(y)| dy$$

Here $B(x,r)$ is the ball in $\mathbb{R}^n$ of center x and radius $r > 0$ and its Lebesgue measure is $|B(x,r)|$. Fractional maximal operator is the mapping $f \to M_\lambda f$. We can write Mf for $M_\lambda f$ if $\lambda = 0$.

2.3 Hyperplane

2.3.1 Definition

Lets take

$$\mathbb{U} = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : |x'| < 1, x_n > 0\}$$

and

$$\mathbb{U}(r) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : |x'| < 1, 0 < x_n < r\}$$

Where $r > 0$.

2.4 Grand Herz Space

2.4.1 Definition

When $1 \le q < \infty$ and $\Omega \subset \mathbb{R}^n$,a measurable set we represent the Grand Herz space of all measurable functions f on Ω by $H^{p),q,\eta}(\Omega)$

$$\|f\|_{H^{p),q,\eta}(\Omega)} = \left(\int_0^\infty (t^\eta \|f\|_{L^{p)}(\Omega\cap\widetilde{\mathbb{U}}(t))})^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

For $\widetilde{\mathbb{U}}(t) := \{y \in \mathbb{U} : t/2 < y_n < t\}$

Important results: Now we shall modify some main lemmas and corollaries to prove our main theorems. We can write that

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\mathbb{U}} |f(y)|^{p-\epsilon} y_n^{\eta(p-\epsilon)} dy \le C \int_{\mathbb{U}} |f(y)|^{p-\epsilon}\left(C\int_{y_n}^{2y_n} t^{\eta(p-\epsilon)}\frac{dt}{t}\right) dy \quad (1)$$

$$\le C\int_0^\infty t^{\eta(p-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(t)} |f(y)|^{p-\epsilon} dy\right)\frac{dt}{t} \quad (2)$$

For $y_n < t$ and $\widetilde{\mathbb{U}}(t) = \{y \in \mathbb{U} : t/2 < y_n < t\}$

Lemma 2.1 *Suppose* $\mu > 0$

(a) For constants $a \ge 0$ and $B_1 > 0$ satisfying

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx \le B_1 r^{-a} \quad (3)$$

Where $0 < r < 1$,

then a constant $C > 0$ depending on B_1, a, μ exists such that

$$\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))} \le r^\mu + C\left(\int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx\right)^{\frac{1}{p-\epsilon}}$$

for $0 < r < 1$.

(b) For constant $b \ge 0$ and $B_2 > 0$ satisfying

$$\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))} \leq B_2 r^{-b}$$

for $0 < r < 1$,
then a constant $C > 0$ depending on B_2, b, μ satisfying

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx \leq r^{\mu} + C(\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))})^{(p-\epsilon)}$$

for $0 < r < 1$.
proof: Let $\lambda(r) := \|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))}$ then

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} \left(\frac{|f(x)|}{\lambda(r)}\right)^{p-\epsilon} dx = 1 \qquad (4)$$

Assume

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx \leq B_1 r^{-a}$$

Where $0 < r < 1$.
If $r^{\mu} \leq \lambda(r) \leq 1$ then $C^{-1}\lambda(r)^{-(p-\epsilon)} \leq \lambda(r)^{-(p-\epsilon)} \leq C\lambda(r)^{-(p-\epsilon)}$
For all $x \in \widetilde{\mathbb{U}}(r)$
If $\lambda(r) \geq 1$ then $\lambda(r)^{-(p-\epsilon)} \leq \lambda(r)^{-1}$ so that $\lambda(r) \leq B_1 r^{-a}$ by (3) and (4).
Therefore for each $x \in \widetilde{\mathbb{U}}(r)$,$\lambda(r)^{-(p-\epsilon)} \leq C\lambda(r)^{-(p-\epsilon)}$ having a constant $C > 0$ depending on B_1, a. Hence by (4) again

$$\lambda(r) \leq C\left(\int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon}\right)^{\frac{1}{p-\epsilon}}$$

Which proves result (a).
Result (b) follow a similar proof.
From the Lemma 2.1 and relation relation (2) we get following corollary.
Corollary 2.1.1 *Suppose η is a real number*

$$\|f\|_{H^{p),q,\eta}(\mathbb{U}(1))} < \infty$$

Iff

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{1}^{\infty} \left(r^{\eta(p-\epsilon)} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx\right)^{\frac{q-\epsilon}{p-\epsilon}} \frac{dr}{r} < \infty$$

Lemma 2.2 *From* [13] *Suppose $\epsilon < 1$ is a real number*

$$\int_{\{y\in\mathbb{U}:|x-y|>x_n/2\}} |x-y|^{\epsilon-n} dy' \leq C y_n^{\epsilon-1}$$

when $y_n > x_n$ and

$$\int_{y\in\{\mathbb{U}:|x-y|>x_n/2\}} |x-y|^{\epsilon-n} dy' \leq C x_n^{\epsilon-1}$$

when $x_n \geq y_n$
From lemma 2.1 and relation (2) we get the Lemma as given below.
Lemma 2.3 *Suppose $\mu > 0$*
(a) For constants $a \geq 0$ and $B_1 > 0$ such that

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx \leq B_1 r^{a}$$

where $r > 1$,
then a constant $C > 0$ exists which depends on B_1, a, μ so that

$$\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))} \leq r^{-\mu} + C\left(\int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx\right)^{\frac{1}{p-\epsilon}}$$

for $r > 1$.
(b) For constants $b \geq 0$ and $B_2 > 0$

$$\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))} \leq B_2 r^{b}$$

Where $r > 1$,
then a constant $C > 0$ exists which depends on B_2, b, μ so that

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx \leq r^{-\mu} + C(\|f\|_{L^{p)}(\widetilde{\mathbb{U}}(r))})^{(p-\epsilon)}$$

Where $r > 1$.
From Lemma 2.3 and relation (2) we attain following corollary.
Corollary 2.3.1 *Consider that η is a real number*

$$\|f\|_{H^{p),q,\eta}(\mathbb{U}92(\mathbb{U}(1))} < \infty$$

Iff

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_1^\infty (r^{\eta(p-\epsilon)} \int_{\widetilde{\mathbb{U}}(r)} |f(x)|^{p-\epsilon} dx)^{\frac{q-\epsilon}{p-\epsilon}} \frac{dr}{r} < \infty$$

3. Boundedness of FMO in GHS on the Hyperplane

First we shall show boundedness of fractional maximal operator M_λ in $H^{p),q,\eta}(\mathbb{U}(1))$.

Theorem 3.1 *Suppose* $\frac{1}{(p-\epsilon)^*} = \frac{1}{(p-\epsilon)} - \frac{\lambda}{n} > 0$ *on* $\mathbb{U}$ *.Let* $\frac{-1}{(p-\epsilon)*} < \eta < \frac{1}{(p-\epsilon)'}$ *and* $1 \leq (q-\epsilon) \leq (p-\epsilon)*$.

Then $C > 0$ is a constant in such a way that

$$\|M_\lambda f\|_{H^{p*),q,\eta}(\mathbb{U}(1))} \leq C\|f\|_{H^{p),q,\eta}(\mathbb{U}(1))} \quad (5)$$

for $f \in L^1_{loc}(\mathbb{U}(1))$.

proof: For a measurable function f on $\mathbb{U}$ we can start by

$$M_\lambda f(x) \leq M_\lambda[f\chi_{B(x,x_{n/2})}](x) + C\int_{\mathbb{U}\backslash B(x,x_{n/2})} |x-y|^{\lambda-n}|f(y)|dy$$
$$=: A_1(x) + CA_2(x)$$

Now

$$A_1(x) \leq CM_\lambda[f\chi_{\widehat{\mathbb{U}}(t)}](x) \quad (6)$$

Where $x \in \widetilde{\mathbb{U}}(t), \widehat{\mathbb{U}}(t) = \widetilde{\mathbb{U}}(t/2) \cup \widetilde{\mathbb{U}}(t) \cup \widetilde{\mathbb{U}}(2t) = \{y \in \mathbb{U}: |y'| < 1, \frac{t}{4} < y_n < 2t\}$

If $C > 0$ is a constant we shall prove that

$$\|A_1\|_{H^{p*),q,\eta}(\mathbb{U}(1))} \leq C \quad (7)$$

Where $f \in L^1_{loc}(\mathbb{U})$ with $\|f\|_{H^{p),q,\eta}(\mathbb{U}(1))} \leq 1$

Now by boundedness of fractional maximal operator [13] We can write that there exists a constant $C > 0$ so that

$$\|M_\lambda f\|_{L^{p*)}(\mathbb{U}(1))} \leq C\|f\|_{L^{p)}(\mathbb{U}(1)))} \quad (8)$$

Using relation (6) and (8) we get

$$\int_0^2 \left(t^\eta \|A_1\|_{L^{p*)}\mathbb{U}(t)}\right)^{(q-\epsilon)} \frac{dt}{t} \leq C\int_0^2 \left(t^\eta \left\|M_\lambda[f\chi_{\widehat{\mathbb{U}}(t)}]\right\|_{L^{p*)}(\mathbb{U})}\right)^{(q-\epsilon)} \frac{dt}{t}$$
$$\leq C\int_0^2 \left(t^\eta \|f\|_{L^{p)}\widehat{\mathbb{U}}(t)}\right)^{(q-\epsilon)} \frac{dt}{t}$$
$$\leq C\int_0^2 \left(t^\eta \|f\|_{L^{p)}\widetilde{\mathbb{U}}(t)}\right)^{(q-\epsilon)} \frac{dt}{t}$$
$$\leq C$$

Relation (7) has proved.

Now we shall prove the following result for a constant $C > 0$

$$\|A_2\|_{H^{p*),q,\eta}(\mathbb{U}(1))} \leq C \quad (9)$$

A measurable function f on $\mathbb{U}$ is in such a way that $\|f\|_{H^{p),q,\eta}(\mathbb{U}(1))} \leq 1$

and $f = 0$ on $(\mathbb{U}\backslash\mathbb{U}(1))$

We consider the case only $1 < (q-\epsilon) \leq (p-\epsilon)*$ for ease

$$\frac{n}{(p-\epsilon)} - \lambda - \frac{1}{(p-\epsilon)*} = \frac{n}{p-\epsilon} - \lambda - \left(\frac{1}{p-\epsilon} - \frac{\lambda}{n}\right)$$
$$= (n-1)\left(\frac{1}{p-\epsilon} - \frac{\lambda}{n}\right) > 0$$

We choose $\epsilon, \mu > 0$ in such a way that $-\frac{1}{(p-\epsilon)'} < \frac{\epsilon}{p-\epsilon} - \lambda < \frac{1}{(p-\epsilon)*}, \eta < \frac{\mu}{q-\epsilon} < \frac{1}{(p-\epsilon)'}$ and

$$G_1 := \{y \in \mathbb{U}\backslash B(x, x_n/2): y_n < x_n\}$$

$$A_{2,1}(x) = \int_{G_1} \left| x-y|^{\lambda-n} f(y)dy \leq \int_{G_1}\right| x-y|^{\lambda-n} f(y) y_n^{-\mu} \left(\int_{y_n}^{2y_n} t^\mu \frac{dt}{t}\right) dy$$
$$= C\int_0^{2x_n} \left(\int_{(\mathbb{U}(t)\cap G_1)} |x-y|^{\lambda-n} f(y) y_n^{-\mu} dy\right) t^\mu \frac{dt}{t}$$
$$= C\int_0^{2x_n} \left(\int_{(\mathbb{U}(t)\cap G_1)} s(y)F(y)dy\right) t^\mu \frac{dt}{t}$$

Here $s(y) = |x-y|^{\lambda - \frac{\epsilon}{p-\epsilon} - \frac{n}{(p-\epsilon)'}} y_n^{-\mu}$ and $F(y) = |x-y|^{\frac{\epsilon-n}{p-\epsilon}} f(y)$. Let $s(y) \in L^{p')}$ and $F(y) \in L^{(p}$ then by using Hölder inequality from [12] we have

$$A_{2,1}(x) = C\int_0^{2x_n}\left(\int_{(\mathbb{U}(t)\cap G_1)} s(y)F(y)dy\right)t^{\mu}\frac{dt}{t}$$
$$\leq C\int_0^{2x_n}\left(\|s\|_{L^{p)}(\widetilde{\mathbb{U}}(t)\cap G_1)}\|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right)t^{\mu}\frac{dt}{t}$$

Here we use generalized Hölder inequality

$$A_{2.1} \leq C\left(\int_0^{2x_n}\left(\|s\|_{L^{p)}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right)^{(q-\epsilon)'}t^{\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}\times\left(\int_0^{2x_n}\left(\|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right)^{(q-\epsilon)}t^{\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)}}$$
$$=: CQ_{1,1}Q_{1,2}$$

Now set

$$Q := \int_{\widetilde{\mathbb{U}}(t)\cap G_1} s(y)^{(p-\epsilon)'}dy$$

Now from Lemma 2.1 for $0 < t < 2x_n$ and $0 < x_n < 1$

$$Q \leq Ct^{-\mu(p-\epsilon)'}\int_{(\widetilde{\mathbb{U}}(t)\cap G_1)}|x-y|^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{n}{(p-\epsilon)'}\right)(p-\epsilon)'}dy$$
$$\leq Ct^{-\mu(p-\epsilon)'}\int_{(\widetilde{\mathbb{U}}(t)\cap G_1)}|x-y|^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-n}dy$$
$$\leq Ct^{-\mu(p-\epsilon)'}\int_{t/2}^{t}x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-1}dy_n$$
$$\leq Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-1}t^{-\mu(p-\epsilon)'+1}$$

and

$$Z_{1,1} = \|s\|_{L^{p)}(\widetilde{\mathbb{U}}(t)\cap G_1)}$$
$$= \sup_{0<\epsilon<p-1}\epsilon^{\frac{1}{(p-\epsilon)'}}\left(\frac{1}{|\widetilde{\mathbb{U}}(t)\cap G_1|}\int |u|^{(p-\epsilon)'}dx\right)^{\frac{1}{(p-\epsilon)'}}$$
$$\leq C\left(x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-1}t^{-\mu(p-\epsilon)'+1}\right)^{\frac{1}{(p-\epsilon)'}}$$
$$\leq Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{1}{(p-\epsilon)'}\right)}t^{-\mu+\frac{1}{(p-\epsilon)'}}$$

Also

$$\int_{\widetilde{\mathbb{U}}(t)\cap G_1}F_k(y)^{(p-\epsilon)}dy = \int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f_k(y)^{(p-\epsilon)}dy$$
$$= \int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy$$
$$\leq Ct^{\epsilon-n-\eta(p-\epsilon)}$$

As $|x-y| \geq \frac{x_n}{2} > \frac{y_n}{2} > \frac{t}{4}$ and $\epsilon < n$ so from lemma 2.1

$$Z_{1,2} := \|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}$$
$$\leq t^{a} + C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{1}{p-\epsilon}}$$

for $a > 0$ finally we get

$$A_{2,1} \leq C\left(\int_0^{2x_n}\left(x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{1}{(p-\epsilon)'}\right)}t^{-\mu+\frac{1}{(p-\epsilon)'}}\right)^{(q-\epsilon)'}t^{\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\times\left(\int_0^{2x_n}\left(t^{a} + C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{1}{p-\epsilon}}\right)^{(q-\epsilon)}t^{\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$
$$\leq Cx_n^{\lambda-\frac{\epsilon}{p-\epsilon}+a} + Cx_n^{\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}}$$
$$\times\left(\int_0^{2x_n}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$

As $\mu < \frac{q-\epsilon}{(p-\epsilon)'}$ noting that

$$Z_{1,3} := \int_0^{2x_n}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}$$

$$\leq C\int_0^{2x_n}\left(x_n^{\epsilon-n}t^{-\eta(p-\epsilon)}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}$$
$$\leq Cx_n^{(\epsilon-n)\frac{(q-\epsilon)}{(p-\epsilon)}-\eta(q-\epsilon)+\mu}$$

As $(\epsilon-n)\frac{(q-\epsilon)}{(p-\epsilon)}-\eta(q-\epsilon)+\mu<0$ and $(q-\epsilon)\leq(p-\epsilon)*$.
From Minkowski's inequality

$$\left(\int_{\widetilde{\mathbb{U}}(r)}A_{2,1}(x)^{(p-\epsilon)*}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}$$
$$+C\left(\int_{\widetilde{\mathbb{U}}(r)}r^{\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}}(Z_{1,3})^{\frac{(p-\epsilon)*}{(q-\epsilon)}}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}\right)(q-\epsilon)}$$
$$\times\int_0^{2r}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{(p-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(r)\cap G_{1'}}|x-y|\right)^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)}}dx)^{\frac{(p-\epsilon)}{(p-\epsilon)*}}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}$$

Here $G_{1'}=\{\mathbb{U}\backslash B(y,y_n/2)\}$.As

$$\int_{\widetilde{\mathbb{U}}(r)\cap G_{1'}}|x-y|^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)}}dx\leq Cr^{(\epsilon-n)\frac{(p-\epsilon)*}{(p-\epsilon)}+n}$$
$$=Cr^{\left(\frac{\epsilon}{p-\epsilon}-\lambda\right)(p-\epsilon)*}$$

From lemma 2.2 and $\left(\frac{\epsilon}{p-\epsilon}-\lambda\right)(p-\epsilon)*<1$ we have

$$\left(\int_{\widetilde{\mathbb{U}}(r)}A_{2,1}(x)^{(p-\epsilon)*}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{-\mu}\int_0^{2r}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}$$

So

$$\sup_{0<\epsilon<p-1}\epsilon^{\frac{1}{p-\epsilon}}\int_0^1\left(r^{\eta(p-\epsilon)*}\right)\times\left(\int_{\widetilde{\mathbb{U}}(r)}A_{2,1}(x)^{(p-\epsilon)*}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}\frac{dr}{r}\tag{10}$$

$$\leq C\int_0^1 r^{\left(\eta+\lambda-\frac{\epsilon}{p-\epsilon}+a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}\frac{dr}{r}\tag{11}$$
$$+C\int_0^1\left(r^{-\mu+\eta(q-\epsilon)}\int_0^{2r}\left(\int_{\mathbb{U}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{\mu}\frac{dt}{t}\right)\frac{dr}{r}\tag{12}$$
$$\leq C+C\int_0^2\left(\int_{\mathbb{U}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\left(\int_{t/2}^1 r^{-\mu+\eta(q-\epsilon)}\frac{dr}{r}\right)t^{\mu}\frac{dt}{t}\tag{13}$$
$$\leq C+C\int_0^2 t^{\eta(q-\epsilon)}\left(\int_{\mathbb{U}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\frac{dt}{t}\tag{14}$$
$$\leq C\tag{15}$$

When $\eta+\lambda-\frac{\epsilon}{p-\epsilon}+a+\frac{1}{(p-\epsilon)*}>0$ and $-\mu+\eta(q-\epsilon)<0$
In the same way we use $\epsilon,\mu>0$ so that

$$\frac{\epsilon}{p-\epsilon}-\lambda-\left(\eta+\frac{1}{(p-\epsilon)*}\right)<-\eta<\frac{\mu}{q-\epsilon}<\frac{\epsilon}{(p-\epsilon)}-\lambda<\frac{1}{(p-\epsilon)*}$$

and take $G_2:=\{y\in\mathbb{U}\backslash B(x,x_n/2):y_n\geq x_n\}$

$$A_{2,2}(x)=\int_{G_2}|x-y|^{\lambda-n}f(y)dy\leq C\int_{G_2}|x-y|^{\lambda-n}f(y)y_n^{\mu}\left(\int_{y_n}^{2y_n}t^{-\mu}\frac{dt}{t}\right)dy$$
$$\leq C\int_{x_n}^{2}\left(\int_{\mathbb{U}(t)\cap G_2}|x-y|^{\lambda-n}f(y)y_n^{\mu}dy\right)t^{-\mu}\frac{dt}{t}$$
$$\leq C\int_{x_n}^{2}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}w(y)D(y)dy\right)t^{-\mu}\frac{dt}{t}$$

Here $w(y)=|x-y|^{\lambda-\frac{\epsilon}{p-\epsilon}-\frac{n}{(p-\epsilon)'}}y_n^{\mu}$ and $D(y)=|x-y|^{\frac{\epsilon-n}{p-\epsilon}}f(y)$
By Hölder inequality from [12] we get

$$A_{2,2}(x)\leq C\int_{x_n}^{2}\|w\|_{L^{p\prime)}(\widetilde{\mathbb{U}}(t)\cap G_2)}\|D\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_2)}t^{-\mu}\frac{dt}{t}$$

Using generalized Hölder inequality

$$A_{2,2}(x) \le C\left(\int_{x_n}^{2}(\|w\|_{L^{p')}(\widetilde{\mathbb{U}}(t)\cap G_2)})^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\times (\int_{x_n}^{2}\left(\|D\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_2)})^{(q-\epsilon)}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)}}$$
$$=: CQ_{2,1}Q_{2,2}$$

Now take

$$Q := \int_{\widetilde{\mathbb{U}}(t)\cap G_2} w(y)^{(p-\epsilon)'}dy$$

From Lemma 2.2

$$Q \le Ct^{\mu(p-\epsilon)'}\int_{(\widetilde{\mathbb{U}}(t)\cap G_2)}|x-y|^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-n}dy$$
$$\le Ct^{\mu(p-\epsilon)'}\int_{t/2}^{t} y_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-1}dy_n$$
$$\le Ct^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(p-\epsilon)'}$$

For $y \in (\widetilde{\mathbb{U}}(t) \cap G_2)$.
As $\lambda - \frac{\epsilon}{p-\epsilon} < \frac{1}{(p-\epsilon)'}$ so

$$Q_{2,1} = \left(\int_{x_n}^{2}(\|w\|_{L^{p')}(\widetilde{\mathbb{U}}(t)\cap G_2)})^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$= \left(\int_{x_n}^{2}\left(\sup_{0<\epsilon<p'-1}\epsilon^{\frac{1}{(p-\epsilon)'}}\left(\frac{1}{|\widetilde{\mathbb{U}}(t)\cap G_2|}\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|w(y)|^{(p-\epsilon)'}\right)^{\frac{1}{(p-\epsilon)'}}\right)^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\le C\left(\int_{x_n}^{2}\left(\sup_{0<\epsilon<p'-1}\epsilon^{\frac{1}{(p-\epsilon)'}}\left(\frac{1}{|\widetilde{\mathbb{U}}(t)\cap G_2|}\right)\left(t^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(p-\epsilon)'}\right)^{\frac{1}{(p-\epsilon)'}}\right)^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\le C\left(\int_{x_n}^{2}t^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\le C\left(\int_{x_n}^{2}t^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(q-\epsilon)'-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\le C\left(x_n^{\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}}\right)$$

Since $\frac{\mu}{q-\epsilon} < \frac{\epsilon}{p-\epsilon} - \lambda$ and

$$\int_{\widetilde{\mathbb{U}}(t)\cap G_2} D(y)^{p-\epsilon}dy = \int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy$$
$$\le Ct^{\epsilon-n-\eta(p-\epsilon)}$$

As $|x-y| \ge y_n/3 > t/6$ and $\epsilon < n$ from Lemma 2.1 for $b > 0$

$$\|D\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_2)} \le t^b + C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\lambda(p-\epsilon)-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{1}{p-\epsilon}}$$

for $C > 0$ Finally we obtain

$$A_{2,2}(x) \le C\left(x_n^{\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}}(\int_{x_n}^{2}\left(t^{b(p-\epsilon)}+\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t})^{\frac{1}{q-\epsilon}}\right)$$
$$\le Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}\right)} + Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}\right)}$$
$$\times\left(\int_{x_n}^{2}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$

When $b(q-\epsilon) > \mu$

$$Z_{2,3} = \int_{x_n}^{2}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$
$$\le C\int_{x_n}^{2}\left(t^{\epsilon-n-\eta(p-\epsilon)}\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$

$$\leq Cx_n^{(\epsilon-n)(\frac{q-\epsilon}{p-\epsilon})-\eta(q-\epsilon)-\mu}$$

Since $\frac{(\epsilon-n)}{(p-\epsilon)}-\eta-\frac{\mu}{(q-\epsilon)}<\lambda+\frac{1}{(p-\epsilon)*}-\frac{n}{p-\epsilon}=(1-n)(p-\epsilon)*<0$

Here we apply Minkowski's inequality

$$\left(\int_{\widetilde{\mathbb{U}}(r)} A_{2,2}(x)^{(P-\epsilon)*}dx\right)^{\frac{q-\epsilon}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}$$
$$+C\left(\int_{\widetilde{\mathbb{U}}(r)} r^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}\right)(p-\epsilon)*}(Z_{2,3})^{\frac{(p-\epsilon)*}{q-\epsilon}}dx\right)^{\frac{q-\epsilon}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+$$
$$Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}\right)(q-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(r)}(Z_{2,3})^{\frac{(p-\epsilon)*}{q-\epsilon}}dx\right)^{\frac{q-\epsilon}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}\right)(q-\epsilon)}$$
$$\times\int_{r/2}^{2}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}\left(\int_{(\widetilde{\mathbb{U}}(r)\cap G_2)}|x-y|^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)})}\right)^{\frac{(p-\epsilon)}{(p-\epsilon)*}})dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{\mu}\int_{r/2}^{2}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$

Here $G_2,=\{x\in\mathbb{U}:|x-y|>r/4\}$

So

$$\sup_{0<\epsilon<p-1}\epsilon^{\frac{1}{p-\epsilon}}\int_0^2\left(r^{\eta(p-\epsilon)*}\right)\times\left(\int_{\widetilde{\mathbb{U}}(r)} A_{2,2}(x)^{(p-\epsilon)*}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}\frac{dr}{r} \quad (16)$$

$$\leq C\int_0^2 r^{\left(\eta+\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}\frac{dr}{r} \quad (17)$$
$$+C\int_0^2\left(r^{\mu+\eta(q-\epsilon)}\int_{r/2}^2\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}\right)\frac{dr}{r} \quad (18)$$
$$\leq C+C\int_0^2\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\left(\int_0^{2t} r^{\mu+\eta(q-\epsilon)}\frac{dr}{r}\right)t^{-\mu}\frac{dt}{t} \quad (19)$$

$$\leq C+C\int_0^2 t^{\eta(q-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\frac{dt}{t} \quad (20)$$
$$\leq C \quad (21)$$

Results (15) and (21) shows that by corollary 2.1.1 assertion (9) holds.
By results (7) and (9) result (5) holds.
Hence proof of theorem is complete.

Now in this paper we shall show boundedness of fractional maximal operator M_λ in $H^{p),q,\eta}(\mathbb{U}\backslash\mathbb{U}(1))$.

Theorem 3.2 *Let* $\frac{1}{(p-\epsilon)^*}=\frac{1}{(p-\epsilon)}-\frac{\lambda}{n}>0$ *on Hyperplane and suppose*
$\frac{-n}{(p-\epsilon)*}<\eta<\frac{1}{(p-\epsilon)'}\left(<n-\lambda-\frac{1}{(p-\epsilon)*}\right)$ and $1\leq(q-\epsilon)\leq(p-\epsilon)*$.
Then there is a constant $C>0$ in such a way that

$$\|M_\lambda f\|_{H^{p*),q,\eta}(\mathbb{U}\backslash\mathbb{U}(1))}\leq C\|f\|_{H^{p),q,\eta}(\mathbb{U}\backslash\mathbb{U}(1))} \quad (22)$$

For $f\in L^1_{loc}(\mathbb{U}\backslash\mathbb{U}(1))$

proof: We can write

$$M_\lambda f(x)\leq M_\lambda[f\chi_{B(x,x_{n/2})}](x)+C\int_{\mathbb{U}\backslash B(x,x_{n/2})}|x-y|^{\lambda-n}|f(y)|dy$$
$$=:A_1(x)+CA_2(x)$$

Note that

$$A_1(x)\leq CM_\lambda[f\chi_{\widehat{\mathbb{U}}(t)}](x) \quad (23)$$

For $x\in\widetilde{\mathbb{U}}(t),\widehat{\mathbb{U}}(t)=\widetilde{\mathbb{U}}(t/2)\cup\widetilde{\mathbb{U}}(t)\cup\widetilde{\mathbb{U}}(2t)=\{y\in\mathbb{U}:|y'|<1,\frac{t}{4}<y_n<2t\}$

For a constant $C>0$ we shall prove that

$$\|A_1\|_{H^{p*),q,\eta}(\mathbb{U}\backslash\mathbb{U}(1))}\leq C \quad (24)$$

For $f\in L^1_{loc}(\mathbb{U})$ with $\|f\|_{H^{p),q,\eta}(\mathbb{U}92\mathbb{U}(1))}\leq 1$

Now by boundedness of fractional maximal operator [13] We can write that there exists a constant $C > 0$ so that
$\|M_\lambda f\|_{L^{p*)}(\mathbb{U}\setminus\mathbb{U}(1))} \leq C\|f\|_{L^{p)}(\mathbb{U}\setminus\mathbb{U}(1)))}$ (25)

From relations (23) and (25) we get

$$\begin{aligned}\int_1^\infty \left(t^\eta \|A_1\|_{L^{p*)}\mathbb{U}(t)}\right)^{(q-\epsilon)} \frac{dt}{t} &\leq C\int_1^\infty \left(t^\eta \left\|M_\lambda[f\chi_{\mathbb{U}(t)}]\right\|_{L^{p*)}(\mathbb{U})}\right)^{(q-\epsilon)} \frac{dt}{t}\\ &\leq C\int_1^\infty \left(t^\eta \|f\|_{L^{p)}\widehat{\mathbb{U}}(t)}\right)^{(q-\epsilon)} \frac{dt}{t}\\ &\leq C\int_1^\infty \left(t^\eta \|f\|_{L^{p)}\widetilde{\mathbb{U}}(t)}\right)^{(q-\epsilon)} \frac{dt}{t}\\ &\leq C\end{aligned}$$

Where f is a measurable function on $(\mathbb{U}\setminus\mathbb{U}(1))$.
Which proves the result (24).
Now we shall prove that if $C > 0$ is a constant then
$\|A_2\|_{H^{p*),q,\eta}(\mathbb{U}\setminus\mathbb{U}(1))} \leq C$ (26)
Where f is a measurable function on $\mathbb{U}$ so that $\|f\|_{H^{p),q,\eta}(\mathbb{U}\setminus\mathbb{U}(1))} \leq 1$
and $f = 0$ on $(\mathbb{U}(1))$
As $\|A_1\|_{H^{p*),q,\eta}(\mathbb{U}\setminus\mathbb{U}(1))} \leq C$ so we see that

$$\|M_\lambda[f\chi(\mathbb{U}(t_0)]\|_{H^{p*),q,\eta}(U(t_0)\setminus\mathbb{U}(1))} \leq C$$

for $t_0 > 1$.
Hence we treat only $f\chi\mathbb{U}\setminus\mathbb{U}(t_0)$ so that we may suppose that $f = 0$ on $\mathbb{U}(t_0)$.
Take $\epsilon, \mu > 0$ in such a way that $-\frac{1}{(p-\epsilon)'} < \frac{\epsilon}{p-\epsilon} - \lambda < \frac{1}{(p-\epsilon)*}, \eta < \frac{\mu}{q-\epsilon} < \frac{1}{(p-\epsilon)'}$ and
$\frac{1}{(p-\epsilon)*} - \frac{\mu}{q-\epsilon} + \eta < \frac{\epsilon}{p-\epsilon} - \lambda < \frac{1}{(p-\epsilon)*}$.
If $t_0 > 1$ is much larger then

$$-\frac{1}{(p-\epsilon)'} < \frac{\epsilon}{(p-\epsilon)} - \lambda < \frac{1}{(p-\epsilon)*}$$

for $x \in \mathbb{U}\setminus\mathbb{U}(t_0)$
Now considering $G_1 := \{y \in \mathbb{U}\setminus B(x,\ x_n/2) : y_n < x_n\}$
As in proof of theorem 3.1 by Hölder inequality for grand lebesgue space from [12] for

$$x_n > t_0 > 1$$

$$A_{2,1}(x) = \int_{G_1} |x-y|^{\lambda-n} f(y)dy \leq C\int_{t_0}^{2x_n} \left(\|s\|_{L^{p')}(\widetilde{\mathbb{U}}(t)\cap G_1)} \|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right) t^\mu \frac{dt}{t}$$

Here $s(y) = |x-y|^{\lambda - \frac{\epsilon}{p-\epsilon} - \frac{n}{(p-\epsilon)'}} y_n^{-\mu}$ and $F(y) = |x-y|^{\frac{\epsilon-n}{p-\epsilon}} f(y)$
From generalized Hölder inequality

$$\begin{aligned}A_{2.1} &\leq C\left(\int_{t_0}^{2x_n} \left(\|s\|_{L^{p')}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right)^{(q-\epsilon)'} t^\mu \frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}} \times \left(\int_{t_0}^{2x_n} \left(\|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}\right)^{(q-\epsilon)} t^\mu \frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)}}\\ &=: CQ_{2,1}Q_{2,2}\end{aligned}$$

Now take

$$Q := \int_{\widetilde{\mathbb{U}}(t)\cap G_1} s(y)^{(p-\epsilon)'} dy$$

By Lemma 2.2 for $t_0 < t < 2x_n$

$$\begin{aligned}Q &\leq Ct^{-\mu(p-\epsilon)'} \int_{(\widetilde{\mathbb{U}}(t)\cap G_1)} |x-y|^{-\frac{\epsilon(p-\epsilon)'}{p-\epsilon} + \lambda(p-\epsilon)' - n} dy\\ &\leq Ct^{-\mu(p-\epsilon)'} \int_{t/2}^{t} x_n^{-\frac{\epsilon(p-\epsilon)'}{p-\epsilon} + \lambda(p-\epsilon)' - 1} dy_n\\ &\leq Cx_n^{\left(\lambda - \frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)' - 1} t^{-\mu(p-\epsilon)' + 1}\end{aligned}$$

let $t_0 < t < 2x_n$

$$\begin{aligned}Z_{1,1} &= \|s\|_{L^{p')}(\widetilde{\mathbb{U}}(t)\cap G_1)}\\ &= \sup_{0<\epsilon<p'-1} \epsilon^{\frac{1}{(p-\epsilon)'}} \left(\frac{1}{|\widetilde{\mathbb{U}}(t)\cap G_1|} \int_{(\widetilde{\mathbb{U}}(t)\cap G_1)} |s|^{(p-\epsilon)'} dx\right)^{\frac{1}{(p-\epsilon)'}}\end{aligned}$$

$$\leq C\left(x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}\right)(p-\epsilon)'-1} t^{-\mu(p-\epsilon)'+1}\right)^{\frac{1}{(p-\epsilon)'}}$$
$$\leq C x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{1}{(p-\epsilon)'}\right)} t^{-\mu+\frac{1}{(p-\epsilon)'}}$$

As

$$\int_{\widetilde{\mathbb{U}}(t)\cap G_1} F(y)^{(p-\epsilon)} dy \leq C t^{\epsilon-n-\eta(p-\epsilon)}$$

From lemma 2.3

$$Z_{1,2} := \|F\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_1)}$$
$$\leq t^{-a} + C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1} |x-y|^{\epsilon-n} f(y)^{(p-\epsilon)} dy\right)^{\frac{1}{p-\epsilon}}$$

for $C > 0$. Finally we get

$$A_{2,1} \leq C\left(\int_{t_0}^{2x_n} (x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{1}{(p-\epsilon)'}\right)} t^{-\mu+\frac{1}{(p-\epsilon)'}})^{(q-\epsilon)'} t^{\mu} \frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\times\left(\int_{t_0}^{2x_n}\left(t^{-a} + C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1} |x-y|^{\epsilon-n} f(y)^{(p-\epsilon)} dy\right)^{\frac{1}{p-\epsilon}}\right)^{(q-\epsilon)} t^{\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$
$$\leq C x_n^{\lambda-\frac{\epsilon}{p-\epsilon}-a} + C x_n^{\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}}$$
$$\times\left(\int_{t_0}^{2x_n}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1} |x-y|^{\epsilon-n} f(y)^{(p-\epsilon)} dy\right)^{\frac{q-\epsilon}{p-\epsilon}} t^{\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$

For $-\lambda(q-\epsilon)+\mu > 0$ noting that

$$Z_{1,3} := \int_{t_0}^{2x_n}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_1} |x-y|^{\epsilon-n} f(y)^{(p-\epsilon)} dy\right)^{\frac{q-\epsilon}{p-\epsilon}} t^{\mu}\frac{dt}{t}$$
$$\leq C x_n^{(\epsilon-n)\frac{(q-\epsilon)}{(p-\epsilon)}-\eta(q-\epsilon)+\mu}$$

As $\eta(q-\epsilon) < \mu$ so by applying Minkowski's inequality we get

$$\left(\int_{\widetilde{\mathbb{U}}(r)} A_{2,1}(x)^{(p-\epsilon)*} dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}$$
$$\leq C x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}$$
$$+C\left(\int_{\widetilde{\mathbb{U}}(r)} r^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}\right)(p-\epsilon)*} (Z_{1,3})^{\frac{(p-\epsilon)*}{(q-\epsilon)}} dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}}$$
$$\leq C x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)} + C r^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-\frac{\mu}{q-\epsilon}\right)(q-\epsilon)}$$
$$\times\left(\int_{t_0}^{2r}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{(p-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(r)\cap G_{1'}} |x-y|\right)^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)})} dx\right)^{\frac{(p-\epsilon)}{(p-\epsilon)*}} dy\right)^{\frac{q-\epsilon}{p-\epsilon}} t^{\mu}\frac{dt}{t}$$

Here $G_{1'} = \mathbb{U}\backslash B(y, y_n/2)$.Since

$$\int_{\widetilde{\mathbb{U}}(r)\cap G_{1'}} |x-y|^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)}} dx \leq C r^{(\epsilon-n)(\frac{(p-\epsilon)*}{(p-\epsilon)}+n}$$
$$= C r^{\left(\frac{\epsilon}{p-\epsilon}-\lambda\right)(p-\epsilon)*}$$

From lemma 2.2 and $(\frac{\epsilon}{p-\epsilon}-\lambda)(p-\epsilon)* < 1$ we attain

$$\left(\int_{\widetilde{\mathbb{U}}(r)} A_{2,1}(x)^{(p-\epsilon)*} dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)*}} \leq C r^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)} + C r^{-\mu}\int_{t_0}^{2r}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon} dy\right)^{\frac{q-\epsilon}{p-\epsilon}} t^{\mu}\frac{dt}{t}$$

So

$$\sup_{0<\epsilon<p-1} \epsilon^{\frac{1}{p-\epsilon}} \int_{t_0}^{\infty} \left(r^{\eta(p-\epsilon)}\right) \times \left(\int_{\widetilde{\mathbb{U}}(r)} A_{2,1}(x)^{(p-\epsilon)*} dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)}} \frac{dr}{r} \tag{27}$$
$$\leq C\int_{t_0}^{\infty} r^{\left(\eta+\lambda-\frac{\epsilon}{p-\epsilon}-a+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)} \frac{dr}{r} \tag{28}$$

$+C\int_{t_0}^{\infty}\left(r^{(-\mu+\eta(q-\epsilon))}\int_{t_0}^{2r}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}} t^{\mu}\frac{dt}{t}\right)\frac{dr}{r}$ (29)

$\leq C + C\int_{t_0}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\left(\int_{\frac{t}{2}}^{\infty} r^{(-\mu+\eta(q-\epsilon))}\frac{dr}{r}\right)t^{\mu}\frac{dt}{t}$ (30)

$\leq C + C\int_{t_0}^{\infty} t^{\eta(q-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(t)} f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\frac{dt}{t}$ (31)

$\leq C$ (32)

Where $\eta+\lambda-\frac{\epsilon}{p-\epsilon}+\frac{1}{(p-\epsilon)_*}<\lambda<\frac{\mu}{q-\epsilon}$ and $-\mu+\eta(q-\epsilon)<0$

In the same way we will take $\epsilon,\mu>0$ so that $-\eta<\frac{\mu}{q-\epsilon}<\frac{\epsilon}{p-\epsilon}-\lambda<\frac{n}{(p-\epsilon)_*}$.

Now considering $G_2:=\{y\in\mathbb{U}\backslash B(x,x_n/2): y_n\geq x_n\}$

$$A_{2,2}(x)=\int_{G_2}|x-y|^{\lambda-n}f(y)dy$$

By Hölder inequality from [12] and then by generalized Hölder inequality we get

$$A_{2,2}(x)\leq C\left(\int_{x_n}^{\infty}\left(\|w\|_{L^{p)}(\widetilde{\mathbb{U}}(t)\cap G_2)}\right)^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\times\left(\int_{x_n}^{\infty}\left(\|D\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_2)}\right)^{(q-\epsilon)}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)}}$$
$$=:CQ_{2,1}Q_{2,2}$$

Here $w(y)=|x-y|^{\lambda-\frac{\epsilon}{p-\epsilon}-\frac{n}{(p-\epsilon)'}}y_n^{\mu}$ and $D(y)=|x-y|^{\frac{\epsilon-n}{p-\epsilon}}f(y)$

Now let

$$Q:=\int_{\widetilde{\mathbb{U}}(t)\cap G_2} w(y)^{(p-\epsilon)'}dy$$

From Lemma 2.2

$$Q\leq Ct^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(p-\epsilon)'}$$

Where $y\in(\widetilde{\mathbb{U}}(t)\cap G_2)$.

As $\lambda-\frac{\epsilon}{p-\epsilon}<\frac{1}{(p-\epsilon)'}$ so that

$$Z_{2,1}=\left(\int_{x_n}^{\infty}\left(\|w\|_{L^{p)}(\widetilde{\mathbb{U}}(t)\cap G_2)}\right)^{(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\leq C\left(\int_{x_n}^{\infty}t^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\mu\right)(q-\epsilon)'}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{(q-\epsilon)'}}$$
$$\leq C\left(x_n^{\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}}\right)$$

As $\frac{\mu}{q-\epsilon}<\frac{\epsilon}{p-\epsilon}-\lambda$ and

$$\int_{\widetilde{\mathbb{U}}(t)\cap G_2} D(y)^{p-\epsilon}dy\leq Ct^{(\epsilon-n-\eta(p-\epsilon))}$$

From lemma 2.3 for $b>0$

$$\|D\|_{L^{(p}(\widetilde{\mathbb{U}}(t)\cap G_2)}\leq t^{-b}+C\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{(p-\epsilon)}dy\right)^{\frac{1}{p-\epsilon}}$$

for $C>0$. Finally we get

$$A_{2,2}(x)\leq C\left(x_n^{\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}}\left(\int_{x_n}^{\infty}\left(t^{-b(p-\epsilon)}+\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}\right)$$
$$\leq Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-b\right)}+Cx_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{(q-\epsilon)}\right)}$$
$$\times\left(\int_{x_n}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}\right)^{\frac{1}{q-\epsilon}}$$

If $-b(q-\epsilon)-\mu<0$ then

$$Z_{2,3}=\int_{x_n}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)\cap G_2}|x-y|^{\epsilon-n}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$

$$\leq C\int_{x_n}^{\infty}\left(t^{\epsilon-n-\eta(p-\epsilon)}\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$
$$\leq Cx_n^{(\epsilon-n)\left(\frac{q-\epsilon}{p-\epsilon}\right)-\eta(q-\epsilon)-\mu}$$

As $\frac{(\epsilon-n)(q-\epsilon)}{(p-\epsilon)}-\eta(q-\epsilon)-\mu<0$

so from Minkowski's inequality we get

$$\left(\int_{\widetilde{\mathbb{U}}(r)}A_{2,2}(x)^{(P-\epsilon)*}dx\right)^{\frac{q-\epsilon}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+$$
$$C\left(\int_{\widetilde{\mathbb{U}}(r)}x_n^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}\right)(p-\epsilon)*}(Z_{2,3})^{\frac{(p-\epsilon)*}{q-\epsilon}}dx\right)^{\frac{q-\epsilon}{(p-\epsilon)*}}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}\right)(q-\epsilon)}$$
$$\times\left(\int_{r/2}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}\left(\int_{(\widetilde{\mathbb{U}}(r)\cap G_2)}|x-y|^{(\epsilon-n)\left(\frac{(p-\epsilon)*}{(p-\epsilon)}\right)}dx\right)^{\frac{(p-\epsilon)}{(p-\epsilon)*}}\right)dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$
$$\leq Cr^{\left(\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}+Cr^{\mu}\int_{r/2}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}$$

Here $G_{2'}=\{x\in\mathbb{U}:|x-y|>\frac{r}{4}\}$ and

$$\lambda-\frac{\epsilon}{p-\epsilon}+\frac{\mu}{q-\epsilon}-\frac{C}{(p-\epsilon)*}\leq\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)*}$$

Therefore

$$\sup_{0<\epsilon<p-1}\epsilon^{\frac{1}{p-\epsilon}}\int_{t_0}^{\infty}\left(r^{\eta(p-\epsilon)}\right)\times\left(\int_{\widetilde{\mathbb{U}}(r)}A_{2,2}(x)^{(p-\epsilon)*}dx\right)^{\frac{(q-\epsilon)}{(p-\epsilon)}}\frac{dr}{r}\qquad(33)$$
$$\leq C\int_{t_0}^{\infty}(r^{\left(\eta+\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)*}\right)(q-\epsilon)}\frac{dr}{r}\qquad(34)$$
$$+C\int_{t_0}^{\infty}\left(r^{\mu+\eta(q-\epsilon)}\int_{r/2}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}t^{-\mu}\frac{dt}{t}\right)\frac{dr}{r}\qquad(35)$$
$$\leq C+C\int_{t_0}^{\infty}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\left(\int_{t_0}^{2t}r^{\mu+\eta(q-\epsilon)}\frac{dr}{r}\right)t^{-\mu}\frac{dt}{t}\qquad(36)$$
$$\leq C+C\int_{t_0}^{\infty}t^{\eta(q-\epsilon)}\left(\int_{\widetilde{\mathbb{U}}(t)}f(y)^{p-\epsilon}dy\right)^{\frac{q-\epsilon}{p-\epsilon}}\frac{dt}{t}\qquad(37)$$
$$\leq C\qquad(38)$$

When $\eta+\lambda-\frac{\epsilon}{p-\epsilon}-b+\frac{1}{(p-\epsilon)}*<0$ and $\mu+\eta(q-\epsilon)>0$

Results (32) and (38) shows that by corollary (2.3.1) assertion (26) holds.

By results (26) and (24) result (22) holds. Hence proof of theorem is complete.

4. Conclusion

Here, we took continuous cases for Grand Herz Space. We studied the boundedness of the FMO in GHS on Hyperplane. To prove our required result, we split our work into two theorems. We successfully proved both theorems. Hence, theorem 3.1 and theorem 3.2 prove the boundedness FMO in GHS on Hyperplane. Grand Herz Space is a newly defined space. It is advantageous in studies and properties of solutions of many differential equations arising from Physics and other fields of Science. It was challenging to prove boundedness on the entire Hyperplane, and due to this difficulty, we had to divide the Hyperplane into two subsets. This way, we confirmed our required claim but had to follow a lengthy procedure. The continuous case of Grand Herz Space is a newly defined space, and still, there are many operators whose boundedness can be checked through this space. Our work in this paper has many applications in solutions of nonlinear partial differential equations.

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