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| RESEARCH ARTICLE

On Modules over G-sets

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| ABSTRACT

Let R be a commutative ring with unity, M a module over R and let S be a G–set for a finite group G. We define a set MS to be the set of elements expressed as the formal finite sum of the form $\sum_{s\in S} m_s\, s$ where $m_s \in M$. The set MS is a module over the group ring RG under the addition and the scalar multiplication similar to the RG–module MG. With this notion, we not only generalize but also unify the theories of both, the group algebra and the group module, and we also establish some significant properties of $(MS)_{RG}$. In particular, we describe a method for decomposing a given RG–module MS as a direct sum of RG–submodules. Furthermore, we prove the semisimplicity problem of $(MS)_{RG}$ with regard to the properties of M_R, S and G.

| KEYWORDS

Group ring, Group module, G–set, Semisimple module, Augmentation map

| ARTICLE INFORMATION

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1. Introduction

Throughout this paper, G is a finite group with identity element e, R is a commutative ring with unity 1, M is an R–module, RG is the group ring, $H \leq G$ denotes that H is a subgroup of G and S is a G–set with a group action of G on S. If N is an R–submodule of M, it is denoted by $N_R \leq M_R$.

MS denote the set of all formal expression of the form $\sum_{s\in S} m_s\, s$ where $m_s \in M$ and $m_s = 0$ for almost every s. For elements $\mu = \sum_{s\in S} m_s\, s$, $\eta = \sum_{s\in S} n_s\, s \in MS$, by writing $\mu = \eta$ we mean $m_s = n_s$ for all $s \in S$.

We define the sum in MS componentwise

$$\mu + \eta = \sum_{s\in S} (m_s + n_s)\, s.$$

It is clear that MS is an R–module with the sum defined above and the scalar product of $\sum_{s\in S} m_s\, s$ by $r \in R$ that is $\sum_{s\in S} (rm_s)\, s$. For $\rho = \sum_{g\in G} r_g\, g \in RG$, the scalar product of $\sum_{s\in S} m_s\, s$ by ρ is

$$\begin{aligned} \rho\mu &= \sum_{s\in S} r_g\, m_s(gs),\ gs = s' \in S, \\ &= \sum_{s'\in S} m_{s'}\, s' \in MS. \end{aligned}$$

It is easy to check that MS is a left module over RG, and also as an R–module, it is denoted by $(MS)_{RG}$ and $(MS)_R$, respectively. The RG–module MS is called G–set module of S by M over RG. It is clear that MS is also a G–set. If S is a G–set and H is a subgroup of G, then S is also an H–set and MS is an RH–module. In addition, if S is a G–set and a group, and $M = R$, then it is easy to verify that RS is a group algebra. On the other hand, if a group acts on itself by multiplication then naturally, we have $(MS)_{RG} = (MG)_{RG}$. Since there is a bijective correspondence between the set of actions of G on a set S and the set of homomorphisms from G to Σ_S (Σ_S is the group of permutations on S), the G–set modules is a large class of RG–modules and we would say that $(MG)_{RG}$ Introduced in (Kosan et al., 2014) considering the group acting itself by multiplication is the first example of the G–set modules. That is why the notion of the RG–module MS presents a generalization of the structure and discussions of RG–module MG and some principal module-theoretic questions arise out of the structure of $(MS)_{RG}$. Therefore, this new concept generalizes not only the group ring (see Anderson & Fuller, 2012; Connell, 1963; Karpilovsky, 1986; Passi, 1979, Passmann, 2011; Shen, 2018) and group algebra (see Alperin & Rowen, 1995; Curtis & Reiner, 1983; Milies & Sehgal, 2002) but also the group module (see Kosan et al. 2014; Kosan & Zemlicka, 2020; Ones et al., 2020; Uc et al., 2016; Uc & Alkan, 2017), and also unifies the theory of these concepts.

The purpose of this paper is to introduce the concept of the RG–module MS, and show the close connection between the properties of $(MS)_{RG}$, M_R, S and G. The semisimplicity of $(MS)_{RG}$ with regard to the properties of M_R, S and G and the decomposition of $(MS)_{RG}$ into RG–submodules will occupy a significant portion of this paper. In Section 1, we present some examples and some properties of $(MS)_{RG}$ to show that an R–module can be extended to RG–modules in various ways via the change of the G–set and the group ring. In Section 2, we give our first major result about the decomposition of a given RG–module MS as a direct sum of RG–submodules. In Section 3, in order to go further into the structure of $(MS)_{RG}$, we first require ε_{MS} that is an extension of the usual augmentation map ε_R and the kernel of ε_{MS} denoted by $\triangle_G\,(MS)$. Then we give the condition for when $\triangle_G\,(MS)$ is an RG–submodule of $(MS)_{RG}$. Finally, we are interested in the semisimplicity of $(MS)_{RG}$ according to the properties of M_R, S and G.

2. Examples of G–set Modules

We start to set out the idea of G–set modules in more detail by considering some examples of G–set modules and establishing some properties of $(MS)_{RG}$. The following examples for $(MS)_{RG}$ show how useful the notion of G–set module for extension of an R–module M to an RG–module. They also point the relations among G–set S, RG–module MS, G and H where $H \leq G$. Example 1 shows that for different group actions on different G–sets of the same finite group we get different extensions of an R–module M to an RG–module. Moreover, we see that these are also RH–modules unsurprisingly in Example 2.

Example 1. *Let M be an R–module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1}\rangle$ and $r = \sum_{g\in D_6} r_g\, g = r_1e + r_2a + r_3a^2 + r_4b + r_5ba + r_6ba^2 \in RD_6$.*

1. *Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an RG–module.*
2. *Let $S = \{D_6, C_3, C_2, Id\}$ and let G act on its set of subgroups $C_3 = \langle a : a^3 = e\rangle \leq D_6$, $C_2 = \langle b : b^2 = e\rangle \leq D_6$, $Id = \{e\} \leq D_6$ by $g * H = gHg^{-1}$ for $H \leq G$, $g \in G$. Then $MS = \{\sum_{s\in S} m_s\, s = m_{Id}Id + m_{C_2}C_2 + m_{C_3}C_3 + m_{D_6}D_6 \mid m_s \in M\}$ and we get*

$$\begin{aligned} r\mu \;&= (r_1m_1 + r_2m_1 + r_3m_1 + r_4m_1 + r_5m_1 + r_6m_1)Id \\ &+\left(r_1m_{C_2} + r_2m_{C_2} + r_3m_{C_2} + r_4m_{C_2} + r_5m_{C_2} + r_6m_{C_2}\right)C_2 \\ &+\left(r_1m_{C_3} + r_2m_{C_3} + r_3m_{C_3} + r_4m_{C_3} + r_5m_{C_3} + r_6m_{C_3}\right)C_3 \\ &+\left(r_1m_{D_6} + r_2m_{D_6} + r_3m_{D_6} + r_4m_{D_6} + r_5m_{D_6} + r_6m_{D_6}\right)D_6. \end{aligned}$$

3. *Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ that is the set of right cosets of a fixed subgroup $H = C_2 = \langle b : b^2 = e\rangle \leq D_6$ and let G act on S by $g * (Hx) = H(gx)$ for $x, g \in G$. Then $MS = \{\sum_{s\in S} m_s\, s = m_{K_1}K_1 + m_{K_2}K_2 + m_{K_3}K_3 \mid m_s \in M\}$ and we have the following relations such that*

$$\begin{array}{ccc} K_1 1 = K_1 & K_2 1 = K_2 & K_3 1 = K_3 \\ K_1 a = K_2 & K_2 a = K_1 & K_3 a = K_1 \\ K_1 a^2 = K_3 & K_2 a^2 = K_3 & K_3 a^2 = K_2 \\ K_1 b = K_1 & K_2 b = K_3 & K_3 b = K_2 \\ K_1 ba = K_2 & K_2 ba = K_1 & K_3 ba = K_3 \\ K_1 ba^2 = K_3 & K_2 ba^2 = K_2 & K_3 ba^2 = K_1. \end{array}$$

So, we get

$$\begin{aligned} r\mu \;&= \left(r_1m_{K_1} + r_4m_{K_1} + r_3m_{K_2} + r_5m_{K_2} + r_2m_{K_3} + r_6m_{K_3}\right)K_1 \\ &+\left(r_2m_{K_1} + r_5m_{K_1} + r_1m_{K_2} + r_6m_{K_2} + r_3m_{K_3} + r_4m_{K_3}\right)K_2 \\ &+\left(r_3m_{K_1} + r_6m_{K_1} + r_2m_{K_2} + r_4m_{K_2} + r_1m_{K_3} + r_5m_{K_3}\right)K_3. \end{aligned}$$

Example 2. *Let M be an R–module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1}\rangle$, $H = C_3 = \langle a : a^3 = e\rangle \leq D_6$ and $k = \sum_{g\in D_6} k_g\, g = k_1e + k_2a + k_3a^2 \in RC_3$.*

1. *Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an RH–module.*
2. *Let $S = \{D_6, C_3, C_2, Id\}$ with the group action defined in* Example 1 *(2). For $\mu = \sum_{s\in S} m_s\, s = m_{Id}Id + m_{C_2}C_2 + m_{C_3}C_3 + m_{D_6}D_6 \in MS$, we get*

$$\begin{aligned} k\mu &= (k_1 m_1 + k_2 m_1 + k_3 m_1) Id + \left(k_1 m_{C_2} + k_2 m_{C_2} + k_3 m_{C_2}\right) C_2 \\ &+ \left(k_1 m_{C_3} + k_2 m_{C_3} + k_3 m_{C_3}\right) C_3 + \left(k_1 m_{D_6} + k_2 m_{D_6} + k_3 m_{D_6}\right) D_6. \end{aligned}$$

3. *Let* $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ *with the group action defined in in* Example 1 *(3). For* $\mu = \sum_{s\in S} m_s\, s = m_{K_1} K_1 + m_{K_2} K_2 + m_{K_3} K_3 \in MS$, *we get*

$$k\mu = \left(k_1 m_{K_1} + k_3 m_{K_2} + k_2 m_{K_3}\right) K_1 + \left(k_2 m_{K_1} + k_1 m_{K_2} + k_3 m_{K_3}\right) K_2 + \left(k_3 m_{K_1} + k_2 m_{K_2} + k_1 m_{K_3}\right) K_3$$

3. Results on G–set Modules

Now, we make a point of some relations between the R–submodules of M and the RG–submodules of MS by the following results.

Lemma 3. *Let* N_1, N_2 *be* R*–submodules of* M. *Then* $N_1 S + N_2 S = MS$ *if and only if* $N_1 + N_2 = M$.

Proof. Let $N_1 S + N_2 S = NS$. Take $m \in M$ and so $ms \in MS$ for any $s \in S$. We write $ms = \sum_{s_i \in S} n_{s_i}\, s_i + \sum_{s_j \in S} n_{s_j}\, s_j$ for $\sum_{s_i \in S} n_{s_i}\, s_i \in N_1 S$ and $\sum_{s_j \in S} n_{s_j}\, s_j \in N_2 S$ where $n_{s_i} \in N_1$, $n_{s_j} \in N_2 S$. So, there exists i, j such that $m = m_{s_i} + m_{s_j}$.
Let $N_1 + N_2 = M$ and $\mu = \sum_{s\in S} m_s\, s \in MS$. For all $s \in S$, we can write $m_s = n_s + n'_s$ where $n_s \in N_1$, $n'_s \in N_2$. Hence, $\mu = \sum_{s\in S} n_s\, s + \sum_{s\in S} n'_s\, s$, and so $N_1 S + N_2 S = NS$. ∎

Lemma 4. *Let* N_1, N_2 *be* R*–submodules of* M. *Then* $N_1 S \cap N_2 S = 0$ *if and only if* $N_1 \cap N_2 = 0$.

Proof. Let $N_1 S + N_2 S = 0$. Take $n \in N_1 \cap N_2$, and so $ns \in N_1 S \cap N_2 S$. So, $n = 0$ since $ns = 0$.
Conversely, let $N_1 \cap N_2 = 0$. Take $\eta = \sum_{s\in S} n_s\, s \in N_1 S \cap N_2 S$. So $n_s \in N_1 \cap N_2$ and $n_s = 0$ for all $s \in S$. Hence, $N_1 S \cap N_2 S = 0$. ∎

From (Alperin & Rowen, 1995) we recall that if G is a finite group, S and T are G–sets, then $\varphi: S \to T$ is said to be a G–set homomorphism if $\varphi(gs) = g\varphi(s)$ for any $g \in G$, $s \in S$. If φ is bijective, then φ is a G–set isomorphism. Then we say that S and T are isomorphic G–sets, and we write $S \simeq T$.
For $s \in S$, $Gs = \{gs : g \in G\}$ is the orbit of s. It is easy to see that Gs is also a G–set under the action induced from that on S. In addition, a subset S' of S is a G–set under the action induced from S if and only if S' is a union of orbits.

Theorem 5. *Let* M *be an* R*–module,* N *an* R*–submodule of* M, G *a finite group,* S *a* G*–set. Then* $\frac{MS}{NS} \simeq \left(\frac{M}{N}\right) S$.

Proof. We know that NS is an RG–submodule of MS. Define a map θ such that

$$\theta: \quad MS \quad \to \quad \left(\frac{M}{N}\right) S,\ \mu = \sum_{s\in S} m_s\, s \quad \mapsto \quad \theta(\mu) = \sum_{s\in S} (m_s + N)\, s$$

$$\theta(g\mu) = \theta\left(g \sum_{s\in S} m_s\, s\right) = g\theta(\mu)$$

So, θ is a G–set homomorphism. It is clear that θ is a G–set epimomorphism. Furthermore, θ is an RG–epimorphism and we get $\ker\theta = NS$. ∎

Lemma 6. *Any proper subset of an orbit* Gs *of* $s \in S$ *is not a* G*–set under the action induced from* S.

Proof. Suppose that a proper subset T of an orbit Gs of $s \in S$ is a G–set. Then there exist $g \in G$, $gs \notin T$. Take an element hs in T, $h \in G$, and so

$$(gh^{-1})(hs) = g\left(h^{-1}(hs)\right) = gs \notin T.$$

Hence, we call the orbit Gs of $s \in S$ the minimal G–set. Moreover, $S = \bigcup_{i\in I} Gs_i$ where I denotes the index of disjoint orbits of S. Hence, we have

$$MS = M\left(\bigcup_{i\in I} Gs_i\right).$$

∎

Lemma 7. *Let N be an R–submodule of an R–module M, S a G–set. Let I denote the index of disjoint orbits of S, J a subset of I and $S' = \bigcup_{j\in J} Gs_j$ and let Gs_i be an orbit Gs of $s_i \in S$ for $i \in I$. Then we have the following results:*

1. NGs_i is an RG–submodule of MS for $s_i \in S$. Moreover, NGs_i is a minimal RG–submodule of MS containg N under the action induced from that on S.
2. $NS' = N\left(\bigcup_{j\in J} Gs_j\right) = \bigcup_{j\in J}(NGs_j)$.
3. NS' is an RG–submodule of MS.

Proof. 1. It is clear that $NGs_i \subseteq MS$. Let $\eta = \sum_{g\in G} n_g\, gs_i \in NGs_i$, $r \in R$, $h \in G$. Then we have $r\eta \in NGs_i$ and $h\eta = h\left(\sum_{g\in G} n_g\, gs_i\right) = \sum_{g\in G} n_g\, hgs_i = \sum_{hg=g'\in G} n_g\, g's_i \in NGs_i$. Hence, NGs_i is an RG–submodule of MS. Assume that there is an RG–submodule N_1 of MS such that $N_R \le (N_1)_{RG} \le (NGs_i)_{RG}$. Take an element $n \in N$, and so $nhs_i \in N_1$ for some $h \in G$ since $(N_1)_{RG} \le (NGs_i)_{RG}$. Then $h^{-1}(nhs_i) = (nes_i) = ns_i \in N_1$ and $g(ns_i) = ngs_i \in N_1$ for all $g \in G$. This means that $N_1 = NGs_i$.
2, 3. Clear by the definition of MS. ■

Lemma 8. *Let L be an RG–submodule of MS, a fixed $s \in S$. Then,*
1. $L_s = \{x \in M \mid$ there is $y \in L$ such that $y = xs + k$, $k \in MS\}$ is an R–submodule of M.
2. $S_L = \{s \in S \mid$there is $x \in M$, and also $k \in L$ such that $y = xs + k \in L$ $\}$ is a G–set in S under the action induced from that on S.

Proof. 1. It is obvious that L_s is in M. Let $x_1, x_2 \in L_{s'}$ and $r \in R$. Then, there is $y_1 = x_1s + k_1$, $y_2 = x_2s + k_2 \in L$ and $y_1 + y_2 = (x_1 + x_2)s + k_1 + k_2 \in L$ where $x_1 + x_2 \in MS$. Furthermore, $ry_1 = rx_1s + rk_1 \in L$, and so $rx_1 \in L_s$.
2. Let $s \in S'$ and $g, h \in G$. Then $\exists x \in M, \exists k \in L$ such that $y = xs + k \in L$ and

$$xs + k = y = ey = e(xs + k) = xes + ek = xes + k$$

So, $s = es$. Since s is also an element of S, we have

$$(hg)y = (hg)(xs + k) = (hg)xs + (hg)k.$$

Hence, we get $(hg)s = h(gs)$. ■

Lemma 9. *Let M be an R–module and S a G–set. Let I denote the index of disjoint orbits of S such that $S = \bigcup_{i\in I} Gs_i$ and let Gs_i be an orbit of $s_i \in S$ for $i \in I$. If NGs_i is a simple RG–submodule of MS, then N is a simple R–submodule of M and G is a finite group whose order is invertible in $End_R(M)$ ($|G|^{-1} \in End_R(M)$).*

Proof. Assume that there is an R–submodule L of M such that $L \le N \le M$. Then $(LGs_i)_{RG} \le (NGs_i)_{RG}$, and by Lemma 6 this is a contradiction. So, N is a simple R–submodule of M. ■

Theorem 10. *Let L be a simple RG–submodule of MS. Then there is a unique simple R–submodule N of M and a unique orbit Gs such that $L = NGs$.*

Proof. For some $s \in S$, by Lemma 8 L_s is a non-zero R–module. And so, $L_sGs \neq 0$ is an RG–submodule of L. Since L is simple RG–submodule, we have $L_sGs = L$. Then, by Lemma 9 L_s is a simple R–submodule of M.
Take an element $s' \in S$ such that $L_{s'}$ is non-zero R–submodule of M. Hence, $L_{s'}Gs' = L = L_sGs$. Take an element $x \in L_{s'}Gs'$. And so, we write

$$x = \sum_{i=1}^{n} l_i\, g_i s' = \sum_{i=1}^{n} k_i\, g_i s$$

where $l_i \in L_{s'}$, $k_i \in L_s$, $g_i \in G$ and $n = |G|$. Then, there exists $g_j \in G$ such that $g_1 s = g_j s'$, and $s = g_1^{-1} g_j s'$. So, we get $Gs = Gs'$. That is why we can write

$$Gs = S_L = \{s \in S \mid \text{there is } x \in M, \text{and also } k \in L \text{ such that } y = xs + k \in L\}.$$

Moreover, $N = L_s = L_{s'}$ is unique by the definition of MS. ■

On the other hand, the following example shows that the converse of the theorem does not hold.

Example 11. *Let $R = \mathbb{Z}_3$, $M = \mathbb{Z}_3$, $G = C_2 = \langle a : a^2 = e\rangle$ and $RG = \mathbb{Z}_3C_2$. If $S = G$ and G acts on itself by group multiplication, then $MS = \mathbb{Z}_3C_2$ where $\mathbb{Z}_3C_2$ is semisimple RG–module since $|G| \le \infty$ and characteristic of R does not divide $|G|$ by Maschke's Theorem. Since $\mathbb{Z}_3C_2$ is semisimple there is a unique decomposition of $\mathbb{Z}_3C_2$ by Artin-Weddernburn Theorem. Then, $\mathbb{Z}_3C_2 \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$ as R–module since $|C_2| = 2$. Here, $\mathbb{Z}_3$ is a simple R–submodule of $\mathbb{Z}_3C_2$. Moreover, by (Milies & Sehgal, 2002) we have $\mathbb{Z}_3C_2 \simeq \mathbb{Z}_3C_2\left(\frac{1+a}{2}\right) \oplus$*

$\mathbb{Z}_3C_2\left(\frac{1-a}{2}\right)$ as RG–module where $\mathbb{Z}_3C_2\left(\frac{1+a}{2}\right)$ and $\mathbb{Z}_3C_2\left(\frac{1-a}{2}\right)$ are simple RG–submodules of $\mathbb{Z}_3C_2$. Let $N=\mathbb{Z}_3$ that is a simple R–submodule of M. Hovewer, $NGs=\mathbb{Z}_3C_2$ is not simple RG–module.

Lemma 12. *Let $\{M_i : i \in I\}$ be a family of right R –modules, G a finite group and S a G –set. Then $\left(\left(\bigoplus_{i\in I} M_i\right)S\right)_{RG} = \left(\bigoplus_{i\in I} M_i S\right)_{RG}$*

Proof. Consider the following map

$$\left(\bigoplus_{i\in I} M_i\right)S \quad\to\quad \bigoplus_{i\in I} M_i S, \sum_{s\in S}\left(\dots, m_s^{(i)}, \dots\right)s \quad\mapsto\quad \sum_{s\in S}\left(\dots, m_s^{(i)}s, \dots\right)$$

that is an isomorphism. ■

Theorem 13. *An R–module M_R is projective if and only if $(MS)_{RG}$ is projective.*

Proof. Assume that M_R is projective. Then for an index I, $(R)^{(I)} \simeq M \oplus A$ where A is a right R–module. So, by Lemma 12

$$\begin{aligned}\left((RS)^{(I)}\right)_{RG} &\simeq \left((R)^{(I)}S\right)_{RG}\\ &\simeq \left((M\oplus A)S\right)_{RG}\\ &\simeq (MS)_{RG}\oplus (AS)_{RG}\end{aligned}$$

So, $(MS)_{RG}$ is projective.

Now, assume that $(MS)_{RG}$ is projective. Then $\left((RS)^{(I)}\right)_{RG} \simeq (MS)_{RG}\oplus B$ where B is a right RG–module for some set I. All these concerning modules are also R–modules and $\left((RS)^{(I)}\right)_R \simeq (MS)_R \oplus B_R$. $\left((RS)^{(I)}\right)_R$ is a free module because $(RS)_R$ is free. Since $(MS)_R$ is direct summand of a free module, it is projective. So, M_R is projective. ■

4. The Decomposition of $(MS)_{RG}$

The theme of this section is the examination of a G–set module $(MS)_{RG}$ through the study of a decomposition of it. The decompositions of RG and $(MG)_{RG}$ obtained from the idempotent defined as $e_H = \frac{\widehat{H}}{|H|}$, where $|H|$ is the order of H and $\widehat{H}=\sum_{h\in H} h$, explained in (Milies & Sehgal, 2002) and (Uc & Alkan, 2017), respectively. A similar method gives a criterion for the decomposition of a G–set module $(MS)_{RG}$. In addition, $End_{RG}MS$ denotes all the RG–endomorphisms of MS.

Lemma 14. *Let M be an R-module and H a normal subgroup of finite group G. If $|H|$, the order of H, is invertible in R then $\tilde{e}_H = \frac{\widehat{H}}{|H|}$ is an idempotent in $End_{RG}(MS)$. Moreover, $\tilde{e}_H$ is central in $End_{RG}(MS)$.*

Proof. Firstly, we will show that $\tilde{e}_H$ is an RG–homomorphism. We start with proving that $\widehat{H}g = g\widehat{H}$for $g \in G$. Since for all $h_i \in H$, there is $h_{ig} \in H$ such that $h_i g = g h_{ig}$, we have that $\widehat{H}g = \sum_{h_i\in H} h_i\, g = \sum_{h_i\in H} g\, h_{ig} = g\widehat{H}$. Therefore, $\frac{\widehat{H}}{|H|}rg = rg\frac{\widehat{H}}{|H|}$ and we have $\tilde{e}_H(rgm) = rg\tilde{e}_H(m)$ for $m \in MS$, $r \in R$ and $g \in G$. It is also clear that $\tilde{e}_H(m+n) = \tilde{e}_H(m) + \tilde{e}_H(n)$ for $m, n \in MS$, $g \in G$.
Secondly, by using the fact that $\widehat{H}.\widehat{H} = |H|.\widehat{H}$, we get

$$\tilde{e}_H\big(\tilde{e}_H(m)\big) = \tilde{e}_H\left(\frac{\widehat{H}}{|H|}m\right) = \tilde{e}_H(m).$$

So, $\tilde{e}_H$ is an idempotent.

Finally, we prove that $\tilde{e}_H$ is a central idempotent in $End_{RG}(MS)$. We will show that $\tilde{e}_H$ commutes with every element of $End_{RG}(MS)$. Let f be in $End_{RG}(MS)$ and so $\widehat{H}f(m) = f\big(\widehat{H}m\big)$ for $m \in MS$. Thus, we have

$$\tilde{e}_H f(m) = \frac{\widehat{H}}{|H|}f(m) = f\left(\frac{\widehat{H}}{|H|}m\right) = f\tilde{e}_H(m).$$

■

For $\mu = \sum_{g\in G} m_g\, g \in MG$ and $s_i \in S$, we write

$$\mu s_i = \sum_{g\in G} m_g\,(gs_i) = \sum_{gs_i\in S} m_{gs_i}\,(gs_i) \in MS$$

Then for $i \in I$ and $\alpha \in M(Gs_i)$, we write $\alpha = \sum_{gs_i\in Gs_i} m_{gs_i}\, gs_i$. Moreover, we write $\beta = \sum_{i\in I}\sum_{gs_i\in Gs_i} m_{gs_i}\, gs_i$ for $\beta = \sum_{s\in S} m_s\, s \in MS$ since $MS = M\left(\bigcup_{i\in I} Gs_i\right)$.

Let H be a normal subgroup of G. It is well known that on G/H we have the group action $g(tH) = gtH$ for $g, t \in G$. Consider $g\big(\sum_{s\in S} m_s\,(Hs)\big) = \big(\sum_{s\in S} m_s\,(gHs)\big)$ for $m_s \in M$.

Let $S' \subset S$ be a G/H–set. Then $S' = \bigcup_{j\in J} G/Hs'_j$ where J denotes the index of disjoint orbits of S' and $MS' = M\left(\bigcup_{j\in J} G/Hs'_j\right)$. Then for $\eta = \sum_{s'\in S'} m_{s'}\, s' \in MS$, we can write $\eta = \sum_{j\in J}\sum_{s'\in G/Hs'_j} m_{s'}\, s'$.

Hence, we have the following result.

Lemma 15. *Let M be an R–module, G a finite group, H a normal subgroup of G, S a G–set and $S' \subset S$ a G/H–set. Then MS' is an RG–module with action defined as* $g\eta = g\left(\sum_{j\in J}\sum_{s'\in G/Hs'_j} m_{s'}\, s'\right) = g(\sum_{j\in J}\sum_{s'\in G/Hs'_j} m_{s'}\left(tHs'_j\right) = \sum_{j\in J}\sum_{s'\in G/Hs'_j} m_{s'}\left(gtHs'_j\right)$ *where* $\eta = \sum_{j\in J}\sum_{s'\in G/Hs'_j} m_{s'}\, s' \in MS'$ *and* $s' = tHs'_j$ *for* $t \in G$.

Theorem 16. *Let H be a normal subgroup of G, $|H|$ invertible in R and $\tilde{e}_H$, defined above, then we have* $MS = \tilde{e}_H.MS \oplus (1-\tilde{e}_H).MS$ *and there exists a G/H–set $S' \subset S$ such that $\tilde{e}_H.MS \simeq MS'$. More precisely,* $\tilde{e}_H.MS = \tilde{e}_H\left(M\left(\bigcup_{i\in I} Gs_i\right)\right) \simeq M\left(\bigcup_{i\in I} \tilde{e}_H Gs_i\right)$

Proof. Firstly, we know that $MG = \tilde{e}_H.MG \oplus (1-\tilde{e}_H).MG$ and $\tilde{e}_H.MG \simeq M(G/H)$ by the theorem in (Uc & Alkan, 2017). Since $\tilde{e}_H$ is a central idempotent by Lemma 14, we get $MS = \tilde{e}_H.MS \oplus (1-\tilde{e}_H).MS$. Now, consider $\theta: G \to G.\tilde{e}_H$ where $g \mapsto g\tilde{e}_H$. This is a group homomorphism since $\theta(gh) = gh\tilde{e}_H = gh\tilde{e}_H^2 = g\tilde{e}_H h\tilde{e}_H = \theta(g)\theta(h)$. It is clear that θ is a group epimorphism. We have $ker\theta = \{g \in G \mid g\tilde{e}_H = \tilde{e}_H\} = \{g \in G \mid (g-1)\tilde{e}_H = 0\} = H$ since $(g-1)\frac{\hat{H}}{|H|} = 0$ and $g\hat{H} = \hat{H}$ for $g \in H$. Moreover, we get $\frac{G}{er\theta} = \frac{G}{H} \simeq \mathrm{Im}\theta = G\tilde{e}_H$. So,

$$\tilde{e}_H.MS = \tilde{e}_H\left(M\left(\bigcup_{i\in I} Gs_i\right)\right) = M\left(\bigcup_{i\in I} G\tilde{e}_H s_i\right) \simeq M\left(\bigcup_{i\in I} (G/H)s_i\right)$$

Since $gHs_i = gHs_l$ for $s_i, s_l \in S$, $i, l \in I$, we get a G/H–set $S' \subset S$ where $\bigcup_{j\in J}(G/H)s_j = S' \subseteq S$. Hence

$$\tilde{e}_H.MS \simeq M\left(\bigcup_{i\in I}(G/H)s_i\right) = M\left(\bigcup_{j\in J}(G/H)s_j\right) = MS'$$

So, $\tilde{e}_H.MS \simeq MS'$. ■

Theorem 17. *Let M be an R–module and G a finite group. For a G–set $S = \bigcup_{i\in I} Gs_i$ (I denotes the index of disjoint orbits of S), $MS \simeq \bigoplus_{i\in I} MG \setminus \ker\theta_i$ where $\theta_i: MG \to MGs_i$ are RG–epimorphisms.*

Proof. Since $MGs_i \cap MGs_j = \emptyset$ for $i \neq j \in I$ where $S = \bigcup_{i\in I} Gs_i$ and I denotes the index of disjoint orbits of S, we have $MS = M\left(\bigcup_{i\in I} Gs_i\right) = \bigoplus_{i\in I} MGs_i$.

Consider

$$\theta_i:\quad MG \quad\to\quad MGs_i, \sum_{g\in G} m_g\, g \quad\mapsto\quad \sum_{g\in G} m_g\, gs_i$$

For $\mu = \sum_{g\in G} m_g\, g \in MG$, $r \in R$, $h \in G$, we have

$$\theta_i(r\mu) = \theta_i\left(r\sum_{g\in G} m_g\, g\right) = \theta_i\left(\sum_{g\in G} r\, m_g g\right) = \sum_{g\in G} r\, m_g g s_i = r\sum_{g\in G} m_g\, gs_i = r\theta_i\left(\sum_{g\in G} m_g\, g\right) = r\theta_i(\mu).$$

$$\theta_i(h\mu) = \theta_i\left(h\sum_{g\in G} m_g\, g\right) = \theta_i\left(\sum_{g\in G} m_g\, hg\right) = \sum_{g\in G} m_g\, hgs_i = h\left(\sum_{g\in G} m_g\, gs_i\right) = h\theta_i\left(\sum_{g\in G} m_g\, g\right) = h\theta_i(\mu).$$

Hence, θ_i is an RG–homomorphism. It is clear that θ_i is an epimorphism. Moreover, $MG \setminus \ker\theta_i \simeq \mathrm{Im}\theta_i = MGs_i$. Then,

$$MS = M\left(\bigcup_{i\in I} Gs_i\right) = \bigoplus_{i\in I} MGs_i \simeq \bigoplus_{i\in I} MG \setminus \ker\theta_i.$$

■

5. Augmentation Map on MS

In the theory of the group ring, the augmentation ideal denoted by $\triangle(RG)$ is the kernel of the usual augmentation map ε_R such that

$$\varepsilon_R:\quad RG \quad\to\quad R, \sum_{g\in G} r_g\, g \quad\mapsto\quad \sum_{g\in G} r_g.$$

The augmentation ideal is always the nontrivial two-sided ideal of the group ring and we have $\triangle(RG) = \{\sum_{g\in G} r_g\,(g-1): r_g \in R, g \in G\}$. The augmentation ideal $\triangle(RG)$ is of use for studying not only the relationship between the subgroups of G and the ideals of RG but also the decomposition of RG as direct sum of subrings.

In (Kosan et al., 2014), ε_R is extended to the following homomorphism of R–modules

$$\varepsilon_M:\quad MG \quad\to\quad M, \sum_{g\in G} m_g\, g \quad\mapsto\quad \sum_{g\in G} m_g.$$

The kernel of ε_M is denoted by $\triangle(MG)$ and

$$\triangle(MG)=\left\{\sum_{g\in G}m_g\,(g-1): m_g\in M, g\in G\right\}.$$

We devote this section to ε_{MS} that is an extension of ε_M, and to the kernel of ε_{MS} denoted by $\triangle_G\,(MS)$.

Definition 18. *The map*

$$\varepsilon_{MS}:\quad MS\ \to\ M,\ \sum_{s\in S}m_s\,s\ \mapsto\ \sum_{s\in S}m_s$$

is called augmentation map on MS.

In addition, $\varepsilon_{MS}(m_s s_1)=\varepsilon_{MS}(m_s s_2)=m_s$ for $m_s s_1$, $m_s s_2\in MS$ where $m_s\in M$, $s_1,s_2\in S$, however $m_s s_1\neq m_s s_2$. Hence, ε_{MS} is not one-to-one.

Lemma 19. *Let M be an R–module, G a group and S a G–set. Then $\varepsilon_{MS}(r\mu)=\varepsilon(r)\,\varepsilon_{MS}(\mu)$ for $\mu=\sum_{s\in S}m_s\,s\in MS$, $r=\sum_{g\in G}r_g\,g\in RG$. In particular, ε_{MS} is an R–homomorphism.*

Proof. Let $\mu=\sum_{s\in S}m_s\,s\in MS$, $r=\sum_{g\in G}r_g\,g\in RG$, then

$$\varepsilon_{MS}(r\mu)=\varepsilon_{MS}\left(\sum_{gs\in S}(r_g m_s)\,(gs)\right)=\varepsilon_{MS}\left(\sum_{s'\in S}m_{s'}\,s'\right),\ m_{s'}=r_g m_s, gs=s'\in S,$$

$$=\left(\sum_{g\in G}r_g\right)\left(\sum_{s\in S}m_s\right)=\varepsilon(r)\varepsilon_{MS}(\mu).$$

In addition, for $\mu=\sum_{s\in S}m_s\,s$, $\eta=\sum_{s\in S}n_s\,s\in MS$, $t\in R$,

$$\varepsilon_{MS}(\mu+\eta)=\varepsilon_{MS}\left(\sum_{s\in S}(m_s+n_s)\,s\right)=\sum_{s\in S}m_s+\sum_{s\in S}n_s$$

$$\varepsilon_{MS}(t\mu)=\varepsilon_{MS}\left(\sum_{s\in S}(tm_s)\,s\right)=t\sum_{s\in S}m_s$$

■

Furhermore,

$$ker(\varepsilon_{MS})=\{\mu=\sum_{s\in S}m_s\,s\in MS\mid\varepsilon_{MS}(\mu)=\varepsilon_{MS}\left(\sum_{s\in S}m_s\,s\right)=\sum_{s\in S}m_s=0\}.$$

It is clear that $ker(\varepsilon_{MS})\neq 0$ because for $m_s s_1+(-m_s s_2)\in MS$, where $m\in M$, $s_1\neq s_2\in S$, we have

$$\varepsilon_{MS}\big(m_s s_1+(-m_s s_2)\big)=\varepsilon_{MS}(m_s s_1)+\varepsilon_{MS}(-m_s s_2)=0$$

Thus, $m_s s_1+(-m_s s_2)\in ker(\varepsilon_{MS})$. Moreover, we will characterize the elements of the kernel of ε_{MS} in detail. For this purpose, we define $\triangle_{G,H}\,(MS)=\{\sum_{h\in H}(h-1)\,\mu_h\mid\mu_h\in MS\}$ where H is a subgroup of finite group G.

Theorem 20. *Let M be an R–module, H a subgroup of G, $|H|$ invertible in R, S a G–set and $\tilde{e}_H$, defined in Lemma 14. Then, $\triangle_{G,H}\,(MS)$ is an RG–module and $\triangle_{G,H}\,(MS)=(1-\tilde{e}_H).MS$.*

Proof. $\triangle_{G,H}\,(MS)$ is obviously an RG–module. Now, take any element $\alpha\in\triangle_{G,H}\,(MS)$. Then we get

$$\alpha=\sum_{h\in H}(h-1)\,\mu_h=\sum_{h\in H}(h-1)\left(\sum_{s\in S}m_s\,s\right)=\sum_{h\in H}\left(\sum_{s\in S}m_s\,(h-1)s\right)$$

$$=\sum_{h\in H}\left(\sum_{s\in S}m_s\,(hs-s)\right)=\sum_{h\in H}\left(\sum_{s\in S}m_s\,(hs-1)-(s-1)\right)$$

On the other hand, for any element $\beta\in(1-\tilde{e}_H).MS$

$$\beta=(1-\tilde{e}_H)\eta=(1-\tilde{e}_H)\left(\sum_{s\in S}n_s\,s\right)=\left(1-\frac{\widehat{H}}{|H|}\right)\left(\sum_{s\in S}n_s\,s\right)=-\frac{1}{|H|}\left(\sum_{h\in H}(h-1)\right)\left(\sum_{s\in S}n_s\,s\right)$$

$$= \left(\sum_{h\in H}(h-1)\right)\left(\sum_{s\in S}n'_s\, s\right) = \sum_{h\in H}(h-1)\left(\sum_{s\in S}n'_s\, s\right) = \sum_{h\in H}\left(\sum_{s\in S}n'_s\,(hs-1)-(s-1)\right)$$

where $\eta \in MS$, $n'_s = -\frac{1}{|H|}n_s$. Hence, $\beta \in \triangle_{G,H}(MS)$. Similarly, $\alpha \in MS.(1-\tilde{e}_H)$. ■

Furthermore, we write $\triangle_{G,G}(MS) = \triangle_G(MS)$. It is clear that $ker(\varepsilon_{MS}) = \triangle_G(MS)$ and we have $ker(\varepsilon_{MS}) = \triangle_G(MS) = (1-\tilde{e}_G).MS$. Recall that $\triangle_R(G)$ is the augmetation ideal of RG and for a normal subgroup N of G, $\triangle_R(G,N)$ denote the kernel of the natural epimorphism $RG \to R(G/N)$ induced by $G \to G/N$. Moreover, $\triangle_R(G,N)$ is a two-sided ideal of RG generated by $\triangle_R(N)$.

Theorem 21. *If N is a normal subgroup of G, then $\triangle_{G,N}(MS) = \triangle_R(N).MS$.*

Proof. We know that $\triangle_R(N) = \{\sum_{n\in N} r_n(n-1) \mid r_n \in R\}$ and $\triangle_{G,H}(MS) = \{\sum_{h\in H}(h-1)\mu_h \mid \mu_h \in MS\}$. For $\alpha = \sum_{n\in N} r_n(n-1) \in \triangle_R(N)$, $\mu = \sum_{s\in S} m_s\, s \in MS$,

$$\alpha\mu = \left(\sum_{n\in N} r_n(n-1)\right)\left(\sum_{s\in S} m_s\, s\right) = \sum_{n\in N} r_n(n-1)\left(\sum_{s\in S} m_s\, s\right) = \sum_{n\in N}(n-1)\left(\sum_{s\in S}(r_n m_s)\, s\right) = \sum_{n\in N}(n-1)\mu_n$$

where $\mu_n = \sum_{s\in S}(r_n m_s)\, s \in MS$. ■

6. Semisimple G–set Modules

In examination of the studies in group rings which make use of the theory of group modules (see Kosan et al., 2014; Kosan & Zemlicka, 2020; Uc & Alkan, 2017), the semisimplicity problem of the G–set module arises. In (Connell, 1963; Milies & Seghal, 2002; Passmann, 2011), the generalized Maschke's Theorem states that a group ring RG is a semisimple Artinian ring if and only if R is a semisimple Artinian ring, G is finite and $|G|^{-1} \in R$. A module theoretic version of the Maschke's Theorem is proven in (Kosan et al., 2014) for group modules. This version states that for a nonzero R–module M and a group G, MG is a semisimple module over RG if and only if M is a semisimple module and G is a finite group whose order is invertible in $End_R(M)$ that is all the R–endomorphisms of M. The purpose of this section is giving a criterion for the semisimplicity of a G–set module to generalize the Maschke's Theorem via the G–set modules.

Theorem 22. *Let M be a nonzero R–module, G a group, S a G–set. If $X \cap \triangle_G(MS) = 0$ for some nonzero RG–submodule X of $(MS)_{RG}$, then each orbit Gs of S for $s \in S$ is a finite set.*

Proof. Firstly, we know that $\triangle_G(MS)$ is an RG–submodule of $(MS)_{RG}$. Assume that Gs is an infinite orbit for some $s \in S$. Then for any $0 \neq x = m_1 s_1 + \ldots + m_k s_k \in X$ where $s_1, \ldots, s_k \in Gs$ are distinct and $m_i s_i \neq 0$, there is an element g of G such that $s_1 g \neq s_j$ for $1 \le j \le k$. Hence, $(1-g)x = \sum_{s_i\in S} m_i\, s_i - \sum_{s_i\in S} m_i\, gs_i \neq 0$, and also $(1-g)x \in Y$. On the other hand, $0 \neq (1-g)x = \sum_{s_i\in S} m_i\,(s_i-1) - \sum_{s_i\in S} m_i\,(gs_i-1) \in \triangle_G(MS)$. Then, $X \cap \triangle_G(MS) \neq 0$ and this is a contradiction. ■

We recall the following lemma in (Lam, 2001), and also in (Kosan et al., 2014).

Lemma 23. (Kosan et al., 2014; Lam, 2001) *Let $X \le Y$ be right RG–modules and G be a finite group whose order is invertible in $End_R(V)$. If X is a direct summand of Y as R–modules, then X is a direct summand of Y as RG–modules.*

Theorem 24. *If M is a semisimple R–module, G is a finite group whose order is invertible in $End_R(M)$ ($|G|^{-1} \in End_R(M)$), and S is a finite G–set, then $(MS)_{RG}$ is semisimple.*

Proof. Assume that M is a semisimple R–module, G is a finite group whose order is invertible in $End_R(M)$, and S is a finite G–set. Let Y be an RG–submodule of MS. Firstly, $(MS)_R$ is semisimple since M_R is semisimple. Hence, Y_R is a direct summand of $(MS)_R$. Moreover, $|G|^{-1} \in End_R(MS)$ since G is finite and $|G|^{-1} \in End_R(M)$. So, Y_{RG} is a direct summand of $(MS)_{RG}$ by Lemma 23 that means $(MS)_{RG}$ is semisimple. ■

7. Conclusion

In the context of this study, we establish the set denoted as MS, which encompasses elements represented as a formal finite sum in the format $\sum_{s\in S} m_s\, s$ where m_s belongs to the set M and S is a G –set. It is noteworthy that the set MS exhibits module-like properties with respect to the group ring RG, supporting both addition and scalar multiplication, akin to the RG –module MG. Therefore, incorporating G –set modules enable us to extend and consolidate the theories pertaining to both group algebra and group modules. Additionally, we identify crucial properties of $(MS)_{RG}$, elucidating a technique for decomposing the RG –module

MS into a direct sum of RG −submodules. Moreover, we substantiate the semisimplicity issue of $(MS)_{RG}$ concerning the characteristics of M_R, S and G. On the other hand, if the properties of M_R, S and G can be determined when the semi-simplicity of $(MS)_{RG}$ is given, a quite strong result related to the semisimplicity of G −set modules is obtained bilaterally. In addition, the regularity of $(MS)_{RG}$, such as the examination of the semisimplicity of $(MS)_{RG}$, can be characterized according to the properties of M_R, S and G and other necessary parameters.

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