On Modules over $G$-sets

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ABSTRACT

Let $R$ be a commutative ring with unity, $M$ a module over $R$ and let $S$ be a $G$–set for a finite group $G$. We define a set $MS$ to be the set of elements expressed as the formal finite sum of the form $\sum_{s \in S} m_s s$ where $m_s \in M$. The set $MS$ is a module over the group ring $RG$ under the addition and the scalar multiplication similar to the $RG$–module $MG$. With this notion, we not only generalize but also unify the theories of both, the group algebra and the group module, and we also establish some significant properties of $(MS)_{RG}$. In particular, we describe a method for decomposing a given $RG$–module $MS$ as a direct sum of $RG$–submodules. Furthermore, we prove the semisimplicity problem of $(MS)_{RG}$ with regard to the properties of $M_R, S$ and $G$.

KEYWORDS

Group ring, Group module, $G$–set, Semisimple module, Augmentation map

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1. Introduction

Throughout this paper, $G$ is a finite group with identity element $e$, $R$ is a commutative ring with unity $1$, $M$ is an $R$–module, $RG$ is the group ring, $H \leq G$ denotes that $H$ is a subgroup of $G$ and $S$ is a $G$–set with a group action of $G$ on $S$. If $N$ is an $R$–submodule of $M$, it is denoted by $NR \leq MR$.

$MS$ denote the set of all formal expression of the form $\sum_{s \in S} m_s s$ where $m_s \in M$ and $m_s = 0$ for almost every $s$. For elements $\mu = \sum_{s \in S} m_s s$, $\eta = \sum_{s \in S} n_s s \in MS$, by writing $\mu = \eta$ we mean $m_s = n_s$ for all $s \in S$.

We define the sum in $MS$ componentwise

$$\mu + \eta = \sum_{s \in S} (m_s + n_s) s.$$ 

It is clear that $MS$ is an $R$–module with the sum defined above and the scalar product of $\sum_{s \in S} m_s s$ by $r \in R$ that is $\sum_{s \in S} (rm_s) s$. For $\rho = \sum_{g \in G} r_g g \in RG$, the scalar product of $\sum_{s \in S} m_s s$ by $\rho$ is

$$\rho \mu = \sum_{s \in S} r_g m_s (gs), gs = s' \in S,$$

$$= \sum_{s' \in S} m_{s'} s' \in MS.$$ 

It is easy to check that $MS$ is a left module over $RG$, and also as an $R$–module, it is denoted by $(MS)_{RG}$ and $(MS)_{R}$ respectively. The $RG$–module $MS$ is called $G$–set module of $S$ by $M$ over $RG$. It is clear that $MS$ is also a $G$–set. If $S$ is a $G$–set and $H$ is a subgroup of $G$, then $S$ is also an $H$–set and $MS$ is an $RH$–module. In addition, if $S$ is a $G$–set and a group, and $M = R$, then it is easy to verify that $RS$ is a group algebra. On the other hand, if a group acts on itself by multiplication then naturally, we have $(MS)_{RG} = (MG)_{RG}$. Since there is a bijective correspondence between the set of actions of $G$ on a set $S$ and the set of homomorphisms from $G$ to $\Sigma_S$ ($\Sigma_S$ is the group of permutations on $S$), the $G$–set modules is a large class of $RG$–modules and we would say that $(MG)_{RG}$ introduced in (Kosan et al., 2014) considering the group acting itself by multiplication is the first example of the $G$–set modules. That is why
the notion of the $RG$–module $MS$ presents a generalization of the structure and discussions of $RG$–module $MG$ and some principal module-theoretic questions arise out of the structure of $(MS)_{RG}$. Therefore, this new concept generalizes not only the group ring (see Anderson & Fuller, 2012; Connell, 1963; Karplus, 1986; Passi, 1979, Passmann, 2011; Shen, 2018) and group algebra (see Alperin & Rowen, 1995; Curtis & Reiner, 1983; Milas & Sehgal, 2002) but also the group module (see Kosan et al. 2014; Kosan & Zemlicka, 2020, Ones et al., 2020; Uc et al., 2016; Uc & Alkan, 2017), and also unifies the theory of these concepts.

The purpose of this paper is to introduce the concept of the $RG$–module $MS$, and show the close connection between the properties of $(MS)_{RG}$, $M$, $B$, $S$ and $G$. The semisimplicity of $(MS)_{RG}$ with regard to the properties of $M$, $B$, $S$ and $G$ and the decompositions of $(MS)_{RG}$ into $RG$–submodules will occupy a significant portion of this paper. In Section 1, we present some examples and some properties of $(MS)_{RG}$ to show that an $R$–module can be extended into $RG$–modules in various ways via the change of the $G$–set and the ring group. In Section 2, we give our first major result about the decomposition of a given $RG$–module $MS$ as a direct sum of $RG$–submodules. In Section 3, in order to go further into the structure of $(MS)_{RG}$, we first require $\varepsilon_{MS}$ that is an extension of the usual augmentation map $\varepsilon_R$ and the kernel of $\varepsilon_{MS}$ denoted by $\Delta_G (MS)$. Then we give the condition for when $\Delta_G (MS)$ is an $RG$–submodule of $(MS)_{RG}$. Finally, we are interested in the semisimplicity of $(MS)_{RG}$ according to the properties of $M_R$, $S$ and $G$.

2. Examples of $G$–set Modules

We start to set out the idea of $G$–set modules in more detail by considering some examples of $G$–set modules and establishing some properties of $(MS)_{RG}$. The following examples for $(MS)_{RG}$ show how useful the notion of $G$–set module for extension of an $R$–module $M$ to an $RG$–module. They also point the relations among $G$–sets $S$, $RG$–module $MS$, $G$ and $H$ where $H \leq G$. Example 1 shows that for different group actions on different $G$–sets of the same finite group we get different extensions of an $R$–module $M$ to an $RG$–module. Moreover, we see that there are also $RH$–modules unsurprisingly in Example 2.

**Example 1.** Let $M$ be an $R$–module, $G = D_6 = \{a, b : a^2 = b^2 = e, b^{-1}ab = a^{-1}\}$ and $r = \sum_{g \in D_6} r_g g = r_1 e + r_2 a + r_3 a^2 + r_4 b + r_5 ba + r_6 ba^2 \in RD_6$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an $RG$–module.
2. Let $S = \{D_6, C_3, C_2, \{1\}\}$ and let $G$ act on its set of subgroups $C_3 = \{a, a^2 = e\} \leq D_6$, $C_2 = \{b, b^2 = e\} \leq D_6$, $\{1\} \leq D_6$ by $g * H = gHg^{-1}$ for $H \leq G, g \in G$. Then $MS = \{\sum_{s \in S} m_s s = m_{id}Id + m_{c_3}C_3 + m_{c_2}C_2 + m_{c_1}C_1 + m_{D_6}D_6 | m_s \in M\}$ and we get

$$r \mu = (r_1 m_{k_1} + r_2 m_{k_2} + r_3 m_{k_3} + r_4 m_{k_4} + r_5 m_{k_5} + r_6 m_{k_6})Id$$
$$+ (r_1 m_{k_1} + r_2 m_{k_2} + r_3 m_{k_3} + r_4 m_{k_4} + r_5 m_{k_5} + r_6 m_{k_6})C_2$$
$$+ (r_1 m_{k_1} + r_2 m_{k_2} + r_3 m_{k_3} + r_4 m_{k_4} + r_5 m_{k_5} + r_6 m_{k_6})C_3$$
$$+ (r_1 m_{k_1} + r_2 m_{k_2} + r_3 m_{k_3} + r_4 m_{k_4} + r_5 m_{k_5} + r_6 m_{k_6})D_6.$$

3. Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ that is the set of right cosets of a fixed subgroup $H = C_2 = \{b, b^2 = e\} \leq D_6$ and let $G$ act on $S$ by $g * (Hx) = H(gx)$ for $x, g \in G$. Then $MS = \{\sum_{s \in S} m_s s = m_{k_1}K_1 + m_{k_2}K_2 + m_{k_3}K_3 | m_s \in M\}$ and we have the following relations such that

$$K_1 1 = K_1, \quad K_2 1 = K_2, \quad K_3 1 = K_3, \quad K_1 a = K_2, \quad K_2 a = K_1, \quad K_3 a = K_1, \quad K_1 a^2 = K_3, \quad K_2 a^2 = K_3, \quad K_3 a^2 = K_2, \quad K_1 b = K_3, \quad K_2 b = K_3, \quad K_3 b = K_2, \quad K_1 ba = K_2, \quad K_2 ba = K_3, \quad K_3 ba = K_3, \quad K_1 ba^2 = K_3, \quad K_2 ba^2 = K_2, \quad K_3 ba^2 = K_1.$$

So, we get

$$r \mu = (r_1 m_{k_1} + r_2 m_{k_1} + r_3 m_{k_1} + r_4 m_{k_1} + r_5 m_{k_1} + r_6 m_{k_1})K_1$$
$$+ (r_2 m_{k_1} + r_3 m_{k_1} + r_4 m_{k_1} + r_5 m_{k_1} + r_6 m_{k_1})K_2$$
$$+ (r_3 m_{k_1} + r_4 m_{k_1} + r_5 m_{k_1} + r_6 m_{k_1})K_3.$$
\[ k\mu = (k_1m_1 + k_2m_1 + k_3m_1)1d + (k_1m_{C_2} + k_2m_{C_2} + k_3m_{C_2})C_2 + (k_1m_{C_3} + k_2m_{C_3} + k_3m_{C_3})C_3 + (k_1m_{D_6} + k_2m_{D_6} + k_3m_{D_6})D_6. \]

3. Let \( S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\} \) with the group action defined in in Example 1 (3). For \( \mu = \sum_{s \in S} m_s s = m_{K_1}K_1 + m_{K_2}K_2 + m_{K_3}K_3 \in MS \), we get

\[ k\mu = (k_1m_{K_1} + k_2m_{K_2} + k_3m_{K_3})K_1 + (k_2m_{K_1} + k_1m_{K_2} + k_3m_{K_2})K_2 + (k_3m_{K_1} + k_2m_{K_2} + k_1m_{K_2})K_3 \]

3. Results on \( G \)-set Modules

Now, we make a point of some relations between the \( R \)-submodules of \( M \) and the \( RG \)-submodules of \( MS \) by the following results.

**Lemma 3.** Let \( N_1, N_2 \) be \( R \)-submodules of \( M \). Then \( N_1S + N_2S = MS \) if and only if \( N_1 + N_2 = M \).

**Proof.** Let \( N_1S + N_2S = NS \). Take \( m \in M \) and so \( ms \in MS \) for any \( s \in S \). We write \( ms = \sum_{i \in ES} n_{s_i} s_i + \sum_{j \in ES} n_{s_j} s_j \) for \( \sum_{i \in ES} n_{s_i} s_i \in N_1S \) and \( \sum_{j \in ES} n_{s_j} s_j \in N_2S \) where \( n_{s_i} \in N_1, n_{s_j} \in N_2 \). So, there exists \( i, j \) such that \( m = n_{s_i} + n_{s_j} \).

Let \( N_1 + N_2 = M \) and \( \mu = \sum_{s \in S} m_s s \in MS \). For all \( s \in S \), we can write \( m_s = n_s + n'_s \) where \( n_s \in N_1, n'_s \in N_2 \). Hence, \( \mu = \sum_{s \in S} n_s + \sum_{s \in S} n'_s \) and so \( N_1S + N_2S = NS \).

**Lemma 4.** Let \( N_1, N_2 \) be \( R \)-submodules of \( M \). Then \( N_1S \cap N_2S = 0 \) if and only if \( N_1 \cap N_2 = 0 \).

**Proof.** Let \( N_1S + N_2S = 0 \). Take \( n \in N_1 \cap N_2 \), and so \( ns \in N_1S \cap N_2S \). So, \( n = 0 \) since \( ns = 0 \).

Conversely, let \( N_1 \cap N_2 = 0 \). Take \( \eta = \sum_{s \in S} n_s s \in N_1S \cap N_2S \). So \( n_s \in N_1 \cap N_2 \) and \( n_s = 0 \) for all \( s \in S \). Hence, \( N_1S \cap N_2S = 0 \).

From (Alperin & Rowen, 1995) we recall that if \( G \) is a finite group, \( S \) and \( T \) are \( G \)-sets, then \( \varphi: S \rightarrow T \) is said to be a \( G \)-set homomorphism if \( \varphi(gs) = g\varphi(s) \) for any \( g \in G, s \in S \). If \( \varphi \) is bijective, then \( \varphi \) is a \( G \)-set isomorphism. Then we say that \( S \) and \( T \) are isomorphic \( G \)-sets, and we write \( S \cong T \).

For \( s \in S, GS = \{gs: g \in G\} \) is the orbit of \( s \). It is easy to see that \( GS \) is also a \( G \)-set under the action induced from that on \( S \). In addition, a subset \( S' \) of \( S \) is a \( G \)-set under the action induced from that on \( S \) if and only if \( S' \) is a union of orbits.

**Theorem 5.** Let \( M \) be an \( R \)-module, \( N \) an \( R \)-submodule of \( M \), \( G \) a finite group, \( S \) a \( G \)-set. Then \( MS = \left( \frac{M}{N} \right) S \).

**Proof.** We know that \( NS \) is an \( RG \)-submodule of \( MS \). Define a map \( \theta \) such that

\[ \theta: MS \rightarrow \left( \frac{M}{N} \right) S, \mu = \sum_{s \in S} m_s s \mapsto \theta(\mu) = \sum_{s \in S} (m_s + N) s \]

\[ \theta(\mu) = \theta \left( g \sum_{s \in S} m_s s \right) = g\theta(\mu) \]

So, \( \theta \) is a \( G \)-set homomorphism. It is clear that \( \theta \) is a \( G \)-set epimorphism. Furthermore, \( \theta \) is an \( RG \)-epimorphism and we get \( \ker \theta = NS \).}

**Lemma 6.** Any proper subset of an orbit \( GS \) of \( s \in S \) is not a \( G \)-set under the action induced from \( S \).

**Proof.** Suppose that a proper subset \( T \) of an orbit \( GS \) of \( s \in S \) is a \( G \)-set. Then there exist \( g \in G, gs \notin T \). Take an element \( hs \) in \( T \), \( h \in G \), and so

\[ (gh^{-1})(hs) = g(h^{-1}(hs)) = gs \notin T. \]

Hence, we call the orbit \( GS \) of \( s \in S \) the minimal \( G \)-set. Moreover, \( S = \bigcup_{I \in I} GS_I \) where \( I \) denotes the index of disjoint orbits of \( S \).

Hence, we have

\[ MS = M \left( \bigcup_{I \in I} GS_I \right) \].
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Lemma 7. Let $N$ be an $R$–submodule of an $R$–module $M$, $S$ a $G$–set. Let $I$ denote the index of disjoint orbits of $S$, $J$ a subset of $I$ and $S^I = \bigcup_{j \in J} G_j$ and let $G_j$ be an orbit $G$ of $s_i \in S$ for $i \in I$. Then we have the following results:

1. $NGS_i$ is an $RG$–submodule of $MS$ for $s_i \in S$. Moreover, $NGS_i$ is a minimal $RG$–submodule of $MS$ containing $N$ under the action induced from that on $S$.
2. $NS^I = N \left( \bigcup_{j \in J} G_j \right) = \bigcup_{j \in J} (NGS_j)$.
3. $NS^I$ is an $RG$–submodule of $MS$.

Proof. 1. It is clear that $NGS_i \subseteq MS$. Let $\eta = \sum_{g \in G} n_g g s_i \in NGS_i$, $r \in R$, $h \in G$. Then we have $r \eta = NGS_i$ and $h \eta = h (\sum_{g \in G} n_g g s_i) = \sum_{g \in G} n_g h g s_i = \sum_{g \in G} g' n_g g' s_i \in NGS_i$. Hence, $NGS_i$ is an $RG$–submodule of $MS$. Assume that there is an $RG$–submodule $N_1$ of $MS$ such that $N_1 \leq (NGS_i)_{RG} \leq (NGS_i)_{RG}$. Let $N$ be an $RG$–submodule of $MS$ for some $G \in G$ since $(N_1)_{RG} \leq (NGS_i)_{RG}$. Then $h^{-1} (h s_i) = (h s_i) = n s_i \in N_1$, and $g (h s_i) = h s_i \in N_1$ for all $g \in G$. This means that $N_1 = NGS_i$.
2. Clear by the definition of $MS$.

Lemma 8. Let $L$ be an $RG$–submodule of $MS$, a fixed $s \in S$. Then,

1. $L_s = \{ x \in M \mid (y = xs + k, k \in MS) \text{ is an } R$–submodule of $M$.

2. $S_L = \{ s \in S \mid \text{ there is } x \in M, \text{ also } k \in L \text{ such that } y = xs + k \} \text{ is a } G$–set in $S$ under the action induced from that on $S$.

Proof. 1. It is obvious that $L_s$ is a submodule of $L$. Indeed, $x_1, x_2 \in L_s$ and $r \in R$. Then, there is $y_1 = x_1 s + k_1, y_2 = x_2 s + k_2 \in L$ and $y_1 + y_2 = (x_1 + x_2) s + k_1 + k_2 \in L$ where $x_1 + x_2 \in MS$. Furthermore, $r y_1 = r x_1 s + r k_1 \in L$, and so $r x_1 \in L_s$.
2. Let $s \in S'$ and $g, h \in H$. Then $\exists x \in M$, $\exists k \in L$ such that $y = xs + k \in L$ and $x + k = y = ey = e (xs + k) = xes + ek = xes + k$.

So, $s = es$. Since $s$ is also an element of $S$, we have

$$(hg)y = (hg)(xs + k) = (hg)xs + (hg)k.$$ 

Hence, we get $(hg)s = h(gs)$.

Lemma 9. Let $M$ be an $R$–module and $S$ a $G$–set. Let $I$ denote the index of disjoint orbits of $S$ such that $S = \bigcup_{i \in I} G_i$ and let $G_i$ be an orbit of $s_i \in S$ for $i \in I$. If $NGS_i$ is a simple $RG$–submodule of $MS$, then $N$ is a simple $R$–submodule of $M$ and $G$ is a finite group whose order is invertible in $End_R(M)$ $(|G|^{-1} \in End_R(M))$.

Proof. Assume that there is an $R$–submodule $L$ of $M$ such that $L \leq N \leq M$. Then $(LGS_i)_{RG} \leq (NGS_i)_{RG}$, and by Lemma 6 this is a contradiction. So, $N$ is a simple $R$–submodule of $M$.

Theorem 10. Let $L$ be a simple $RG$–submodule of $MS$. Then there is a unique simple $R$–submodule $N$ of $M$ and a unique orbit $GS$ such that $L = NGS$.

Proof. For some $s \in S$, by Lemma 8 $L_s$ is a non-zero $R$–module. And so, $L_sG_s \neq 0$ is an $RG$–submodule of $L$. Since $L$ is simple $RG$–submodule, we have $L_sG_s = L$. Then, by Lemma 9 $L_s$ is a simple $R$–submodule of $M$.

Take an element $s' \in S$ such that $L_{s'}$ is non-zero $R$–submodule of $M$. Hence, $L_sG_{s'} = L = L_sG_s$. Take an element $x \in L_sG_{s'}$. And so, we write

$$x = \sum_{i = 1}^n l_i g_i s' = \sum_{i = 1}^n k_i g_i s$$

where $l_i \in L_{s'}, k_i \in L_s, g_i \in G$ and $n = |G|$. Then, there exists $g_j \in G$ such that $g_j s = g_j s'$, and $s = g_j^{-1} g_j s'$. So, we get $G_s = G_{s'}$.

That is why we can write

$G_s = S_L = \{ s \in S \mid \text{ there is } x \in M, \text{ also } k \in L \text{ such that } y = xs + k \in L \}$.

Moreover, $N_L = L_s = L_{s'}$ is unique by the definition of $MS$.

On the other hand, the following example shows that the converse of the theorem does not hold.

Example 11. Let $R = \mathbb{Z}_2, M = \mathbb{Z}_2, G = C_2 = (a : a^2 = e)$ and $RG = \mathbb{Z}_2 C_2$. If $S = G$ and $G$ acts on itself by group multiplication, then $MS = \mathbb{Z}_2 C_2$ where $\mathbb{Z}_2 C_2$ is semisimple $RG$–module since $|G| \leq \infty$ and characteristic of $R$ does not divide $|G|$ by Maschke’s Theorem.

Since $\mathbb{Z}_2 C_2$ is semisimple there is a unique decomposition of $\mathbb{Z}_2 C_2$ by Artin-Wedderburn Theorem. Then, $\mathbb{Z}_2 C_2 = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ as $R$–module since $|C_2| = 2$. Here, $\mathbb{Z}_2$ is a simple $R$–submodule of $\mathbb{Z}_2 C_2$. Moreover, by (Milne & Sehgal, 2002) we have $\mathbb{Z}_2 C_2 \cong \mathbb{Z}_2 C_2 \left( \frac{1 + i}{2} \right)$.
Let $M_i$ be a family of right $R$-modules, $G$ a finite group and $S$ a $G$-set. Then
\[
\left( \bigoplus_{i \in I} M_i S \right)_{RG} = \left( \bigoplus_{i \in I} S \right)_{RG}
\]

**Proof.** Consider the following map
\[
\left( \bigoplus_{i \in I} M_i S \right) \rightarrow \bigoplus_{i \in I} M_i S \sum_{S \in \tilde{G}} \left( \ldots, m_s^{(i)} \ldots \right) S \rightarrow \sum_{S \in \tilde{G}} \left( \ldots, m_s^{(i)} \ldots \right)
\]
that is an isomorphism. $\blacksquare$

**Theorem 13.** An $R$-module $M_R$ is projective if and only if $(MS)_{RG}$ is projective.

**Proof.** Assume that $M_R$ is projective. Then for an index $I$, $(R)^{(i)} = M \oplus A$ where $A$ is a right $R$-module. So, by Lemma 12
\[
\left( (RS)^{(i)} \right)_{RG} = \left( (R)^{(i)} S \right)_{RG} = \left( (M \oplus A)S \right)_{RG} = \left( MS \right)_{RG} \oplus \left( AS \right)_{RG}
\]
So, $(MS)_{RG}$ is projective.

Now, assume that $(MS)_{RG}$ is projective. Then $(RS)^{(i)}_{RG} = (MS)_{RG} \oplus B$ where $B$ is a right $R$-module for some set $I$. All these concerning modules are also $R$-modules and $(RS)^{(i)}_R = (MS)_R \oplus B_R$. $(RS)^{(i)}_R$ is a free module because $(RS)_R$ is free. Since $(MS)_R$ is direct summand of a free module, it is projective. So, $M_R$ is projective. $\blacksquare$

4. The Decomposition of $(MS)_{RG}$

The theme of this section is the examination of a $G$-set module $(MS)_{RG}$ through the study of a decomposition of it. The decompositions of $RG$ and $(MG)_{RG}$ obtained from the idempotent defined as $e_H = \frac{H}{|H|}$, where $|H|$ is the order of $H$ and $\tilde{H} = \sum_{h \in H} h$, explained in (Milies & Sehgal, 2002) and (Uc & Alkan, 2017), respectively. A similar method gives a criterion for the decomposition of a $G$-set module $(MS)_{RG}$. In addition, $End_{RG}MS$ denotes all the $RG$-endomorphisms of $MS$.

**Lemma 14.** Let $M$ be an $R$-module and $H$ a normal subgroup of finite group $G$. If $|H|$, the order of $H$, is invertible in $R$ then $\hat{e}_H = \frac{\tilde{H}}{|H|}$ is an idempotent in $End_{RG}(MS)$. Moreover, $\hat{e}_H$ is central in $End_{RG}(MS)$.

**Proof.** Firstly, we will show that $\hat{e}_H$ is an $RG$-homomorphism. We start with proving that $\hat{H}g = g\hat{H}$ for $g \in G$. Since for all $h_i \in H$, there is $h_{ig} \in H$ such that $h_{ig}h = gh_{ig}$, we have that $\hat{H}g = \sum_{i \in H} h_{ig}h = \sum_{h \in H} h \hat{H}g = g\hat{H}$. Therefore, $\hat{H}r_g = r_g\frac{\hat{H}}{|H|}$ and we have $\hat{e}_H(rgm) = rge_H(\tilde{H})m$ for $m \in MS$, $r \in R$ and $g \in G$. It is also clear that $\hat{e}_H(m + n) = \hat{e}_H(m) + \hat{e}_H(n)$ for $m, n \in MS$, $g \in G$.

Secondly, by using the fact that $\hat{H}\tilde{H} = |H|^2$, we get
\[
\hat{e}_H(\hat{e}_H(m)) = \hat{e}_H\left( \frac{\tilde{H}}{|H|}m \right) = \hat{e}_H(m).
\]
So, $\hat{e}_H$ is an idempotent.

Finally, we prove that $\hat{e}_H$ is a central idempotent in $End_{RG}(MS)$. We will show that $\hat{e}_H$ commutes with every element of $End_{RG}(MS)$. Let $f$ be in $End_{RG}(MS)$ and so $\hat{H}f(m) = f(\tilde{H}m)$ for $m \in MS$. Thus, we have
\[
\hat{e}_Hf(m) = \hat{H}\frac{\tilde{H}}{|H|}f(m) = f\left(\frac{\tilde{H}}{|H|}m\right) = f\hat{e}_H(m).
\]

For $\mu = \sum_{g \in G} m_g g \in MG$ and $s_i \in S$, we write
\[
\mu S_i = \sum_{g \in G} m_g (gs_i) = \sum_{g \in G} m_{gs_i} (gs_i) \in MS
\]
Then for $i \in I$ and $a \in M(GS_i)$, we write $\alpha = \sum_{g \in G} m_{gs_i} g s_i$. Moreover, we write $\beta = \sum_{i \in I} \sum_{g \in G} m_{gs_i} g s_i$ for $\beta = \sum_{s \in S} m_s s \in MS$ since $MS = M(\bigcup_{i \in I} G S_i)$.

Let $H$ be a normal subgroup of $G$. It is well known that on $G/H$ we have the group action $g(tH) = gtH$ for $g, t \in G$. Consider $g(\sum_{s \in S} m_s s) = (\sum_{s \in S} m_s (g H s))$ for $m_s \in M$.

Let $S' \subseteq S$ be a $G/H$-set. Then $S' = \bigcup_{j \in J} G/Hs'_j$ where $J$ denotes the index of disjoint orbits of $S'$ and $MS' = M\left(\bigcup_{j \in J} G/Hs'_j\right)$. Then for $\eta = \sum_{s' \in S'} m_{s'} s'$, we can write $\eta = \sum_{j \in J} \sum_{s' \in Hs'_j} m_{s'} s'$.

Hence, we have the following result.
Lemma 15. Let \( M \) be an \( R \)--module, \( G \) a finite group, \( H \) a normal subgroup of \( G \), \( S \) a \( G \)--set and \( S' \subset S \) a \( G/H \)--set. Then \( MS' \) is an \( RG \)--module with action defined as 
\[
\eta g = g \left( \sum_{t \in T} s_t^{\ast} \big( h s'_t \big) \right) = g \left( \sum_{t \in T} s_t^{\ast} m_{s'_t} \right) = \sum_{t \in T} s_t^{\ast} m_{s'_t} (g h s'_t)
\]
where 
\[
\eta = \sum_{t \in T} s_t^{\ast} m_{s'_t} s' \in MS' \text{ and } s' = h s'_t \text{ for } t \in G.
\]

Theorem 16. Let \( H \) be a normal subgroup of \( G \), \( |H| \) invertible in \( R \) and \( \tilde{e}_H \), defined above, then we have \( MS = \tilde{e}_H \cdot MS \oplus (1 - \tilde{e}_H) \cdot MS \) and there exists a \( G/H \)--set \( S' \subset S \) such that \( \tilde{e}_H \cdot MS = MS' \). More precisely, \( \tilde{e}_H \cdot MS = \tilde{e}_H \left( \bigcup_{i \in I} G S_i \right) \approx M \left( \bigcup_{i \in I} \tilde{e}_H G S_i \right) \).

Proof. Firstly, we know that \( MG = \tilde{e}_H \cdot MG \oplus (1 - \tilde{e}_H) \cdot MG = M(G/H) \) by the theorem in (Uc & Alkan, 2017). Since \( \tilde{e}_H \) is a central idempotent by Lemma 14, we get \( MS = \tilde{e}_H \cdot MS \oplus (1 - \tilde{e}_H) \cdot MS \). Now, consider \( \theta: G \rightarrow \tilde{e}_H \) where \( g \mapsto g \tilde{e}_H \). This is a group homomorphism since \( \theta(gh) = g h \tilde{e}_H = gh \tilde{e}_H = gh \tilde{e}_H = \theta(g) \theta(h) \). It is clear that \( \theta \) is a group epimorphism. We have \( \ker \theta = \{ g \in G \mid g \tilde{e}_H = \tilde{e}_H \} = \{ g \in G \mid (g - 1) \tilde{e}_H = 0 \} = H \) since \( (g - 1) \tilde{e}_H = 0 \) and \( g H = H \) for \( g \in H \). Moreover, we get \( \frac{g}{\ker \theta} \approx \frac{g}{H} \). So, 
\[
\tilde{e}_H \cdot MS = \tilde{e}_H \left( \bigcup_{i \in I} G S_i \right) = M \left( \bigcup_{i \in I} \tilde{e}_H G S_i \right).
\]
Since \( g H s_i = g H s_i \) for \( s_i \in S \), \( i \in I \), we get a \( G/H \)--set \( S' \subset S \) where \( \bigcup_{j \in J} G(G/H) s_j = S' \subseteq S \). Hence
\[
\tilde{e}_H \cdot MS \approx M \left( \bigcup_{i \in I} (G/H) s_i \right) = M \left( \bigcup_{j \in J} (G/G) s_j \right) = MS'.
\]
So, \( \tilde{e}_H \cdot MS \approx MS' \).

Theorem 17. Let \( M \) be an \( R \)--module and \( G \) a finite group. For a \( G \)--set \( S = \bigcup_{i \in I} G S_i \) (\( I \) denotes the index of disjoint orbits of \( S \)), \( MS = \bigoplus_{i \in I} MG \setminus \ker \theta_i \) where \( \theta_i: MG \rightarrow MG S_i \) are \( RG \)--epimorphisms.

Proof. Since \( MG S_i \cap MG S_j = \emptyset \) for \( i \neq j \in I \) where \( S = \bigcup_{i \in I} G S_i \) and \( I \) denotes the index of disjoint orbits of \( S \), we have \( MS = \bigcup_{i \in I} MG S_i \).

Consider \( \theta_i: MG \rightarrow MG S_i, \sum_{g \in G} m_g g \rightarrow \sum_{g \in G} m_g g S_i \). For \( \mu = \sum_{g \in G} m_g g \in MG, r \in R, h \in G \), we have
\[
\theta_i(r \mu) = \theta_i \left( \sum_{g \in G} r m_g g \right) = \sum_{g \in G} r m_g g S_i = r \sum_{g \in G} m_g g S_i = r \theta_i \left( \sum_{g \in G} m_g g \right) = r \theta_i(\mu).
\]
\[
\theta_i(h \mu) = \theta_i \left( \sum_{g \in G} m_g h g \right) = \sum_{g \in G} m_g h g S_i = h \left( \sum_{g \in G} m_g g S_i \right) = h \theta_i \left( \sum_{g \in G} m_g g \right) = h \theta_i(\mu).
\]
Hence, \( \theta_i \) is an \( RG \)--homomorphism. It is clear that \( \theta_i \) is an epimorphism. Moreover, \( MG \setminus \ker \theta_i \approx \mathrm{Im} \theta_i = MG S_i \). Then, \( MS = M \left( \bigcup_{i \in I} G S_i \right) = \bigoplus_{i \in I} MG S_i \approx \bigoplus_{i \in I} MG \setminus \ker \theta_i \).

5. Augmentation Map on \( MS \)

In the theory of the group ring, the augmentation ideal denoted by \( \triangle (RG) \) is the kernel of the usual augmentation map \( \varepsilon_R \) such that
\[
\varepsilon_R: RG \rightarrow R, \sum_{g \in G} r_g g \rightarrow \sum_{g \in G} r_g.
\]
The augmentation ideal is always the nontrivial two-sided ideal of the group ring and we have \( \triangle (RG) = \{ \sum_{g \in G} r_g (g - 1); r_g \in R, g \in G \} \). The augmentation ideal \( \triangle (RG) \) is of use for studying not only the relationship between the subgroups of \( G \) and the ideals of \( RG \) but also the decomposition of \( RG \) as direct sum of subrings.

In (Kosan et al., 2014), \( \varepsilon_R \) is extended to the following homomorphism of \( R \)--modules
\[
\varepsilon_M: MG \rightarrow M, \sum_{g \in G} m_g g \rightarrow \sum_{g \in G} m_g.
\]
The kernel of \( \varepsilon_M \) is denoted by \( \triangle (MG) \) and
\[ \Delta (MG) = \left\{ \sum_{g \in G} m_g (g-1); m_g \in M, g \in G \right\}. \]

We devote this section to \( \varepsilon_{MS} \) that is an extension of \( \varepsilon_M \), and to the kernel of \( \varepsilon_{MS} \) denoted by \( \Delta_G (MS) \).

**Definition 18.** The map

\[ \varepsilon_{MS} : MS \rightarrow M, \sum_{s \in S} m_s s \mapsto \sum_{s \in S} m_s \]

is called augmentation map on \( MS \).

In addition, \( \varepsilon_{MS}(ms_1) = \varepsilon_{MS}(ms_2) = m_s \) for \( m_s s_1, m_s s_2 \in MS \) where \( m_s \in M, s_1, s_2 \in S \), however \( m_s s_1 \neq m_s s_2 \). Hence, \( \varepsilon_{MS} \) is not one-to-one.

**Lemma 19.** Let \( M \) be an \( R \)-module, \( G \) a group and \( S \) a \( G \)-set. Then \( \varepsilon_{MS}(r \mu) = \varepsilon(r) \varepsilon_{MS}(\mu) \) for \( \mu = \sum_{s \in S} m_s s \in MS, r = \sum_{g \in G} r g g \in RG \). In particular, \( \varepsilon_{MS} \) is an \( R \)-homomorphism.

**Proof.** Let \( \mu = \sum_{s \in S} m_s s \in MS, \eta = \sum_{s \in S} n_s s \in MS, t \in R, \)

\[ \varepsilon_{MS}(r(\mu + \eta)) = \varepsilon_{MS}\left( \sum_{s \in S} (r m_s + r n_s) s \right) = \sum_{s \in S} (r m_s + r n_s) s = \sum_{s \in S} r m_s s = r \sum_{s \in S} m_s s = t \sum_{s \in S} m_s s. \]

Furthermore,

\[ \ker(\varepsilon_{MS}) = \{ \mu = \sum_{s \in S} m_s s \in MS | \varepsilon_{MS}(\mu) = \varepsilon_{MS}\left( \sum_{s \in S} m_s s \right) = \sum_{s \in S} m_s = 0 \}. \]

It is clear that \( \ker(\varepsilon_{MS}) \neq 0 \) because for \( m s_1 + (-m s_2) \in MS \), where \( m \in M, s_1, s_2 \in S \), we have

\[ \varepsilon_{MS}(m s_1 + (-m s_2)) = \varepsilon_{MS}(m s_1) + \varepsilon_{MS}(-m s_2) = 0 \]

Thus, \( m s_1 + (-m s_2) \in \ker(\varepsilon_{MS}) \). Moreover, we will characterize the elements of the kernel of \( \varepsilon_{MS} \) in detail. For this purpose, we define \( \Delta_G, H (MS) = \{ \sum_{h \in H} (h-1) \mu_h | \mu_h \in MS \} \) where \( H \) is a subgroup of finite group \( G \).

**Theorem 20.** Let \( M \) be an \( R \)-module, \( H \) a subgroup of \( G, |H| \) invertible in \( R, S \) a \( G \)-set and \( \varepsilon_H \), defined in Lemma 14. Then, \( \Delta_G, H (MS) \) is an \( RG \)-module and \( \Delta_G, H (MS) = (1 - \varepsilon_H).MS \).

**Proof.** \( \Delta_G, H (MS) \) is obviously an \( RG \)-module. Now, take any element \( \alpha \in \Delta_G, H (MS) \). Then we get

\[ \alpha = \sum_{h \in H} (h-1) \mu_h = \sum_{h \in H} (h-1) \sum_{s \in S} m_s s = \sum_{h \in H} \left( \sum_{s \in S} m_s (h-1)s \right) = \sum_{h \in H} \left( \sum_{s \in S} m_s h s - s \right) = \sum_{h \in H} \left( \sum_{s \in S} m_s (hs - 1) - (s - 1) \right). \]

On the other hand, for any element \( \beta \in (1 - \varepsilon_H).MS \)

\[ \beta = (1 - \varepsilon_H) \eta = (1 - \varepsilon_H) \left( \sum_{s \in S} n_s s \right) = \left( 1 - \frac{R}{|H|} \right) \left( \sum_{s \in S} n_s s \right) = \frac{1}{|H|} \left( \sum_{h \in H} (h-1) \right) \left( \sum_{s \in S} n_s s \right). \]
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\[= \left( \sum_{k \in H} (h-1) \right) \left( \sum_{s \in S} n^s \right) = \sum_{k \in H} \left( \sum_{s \in S} n^s \right) = \sum_{k \in H} \left( \sum_{s \in S} (h-1)(hs-1)-(s-1) \right)\]

where \( \eta \in MS, n^s = -\frac{1}{|H|} n^s_s. \) Hence, \( \beta \in \Delta_{G,H}(MS). \) Similarly, \( \alpha \in MS(1-\beta). \)

Furthermore, we write \( \Delta_{G,G}(MS) = \Delta_G(MS). \) It is clear that \( ker(\epsilon_{MS}) = \Delta_G(MS) \) and we have \( ker(\epsilon_{MS}) = \Delta_G(MS) = (1-\beta_G)MS. \) Recall that \( \Delta_G(G,N) \) is the augmentation ideal of \( RG \) and for a normal subgroup \( N \) of \( G, \Delta_G(G,N) \) denote the kernel of the natural epimorphism \( RG \to R(G/N) \) induced by \( G \to G/N. \) Moreover, \( \Delta_R(G,N) \) is a two-sided ideal of \( RG \) generated by \( \Delta_R(N). \)

**Theorem 21.** If \( N \) is a normal subgroup of \( G, \) then \( \Delta_{G,N}(MS) = \Delta_R(N).MS. \)

**Proof.** We know that \( \Delta_R(N) = \{ \sum_{n \in N} r_n (n-1) | r_n \in R \} \) and \( \Delta_G(MS) = \{ \sum_{k \in H} (h-1)\mu_k | \mu_k \in MS \}. \) For \( \alpha = \sum_{n \in N} r_n (n-1) \in \Delta_R(N), \mu = \sum_{s \in S} m_s s \in MS, \)

\[a\mu = \left( \sum_{n \in N} r_n (n-1) \right) \left( \sum_{s \in S} m_s s \right) = \sum_{n \in N} r_n (n-1) \left( \sum_{s \in S} m_s s \right) = \sum_{s \in S} \left( \sum_{n \in N} (r_n m_s) s \right) = \sum_{s \in S} (n-1) \mu_n \]

where \( \mu_n = \sum_{s \in S} (r_n m_s) s \in MS. \)

**6. Semisimple G-Set Modules**

In examination of the studies in group rings which make use of the theory of group modules (see Kosan et al., 2014; Kosan & Zemlicka, 2020; Uc & Alkan, 2017), the semisimplicity problem of the \( G \)-set module arises. In (Connell, 1963; Milies & Seghal, 2002; Passmann, 2011), the generalized Maschke’s Theorem states that a group ring \( RG \) is a semisimple Artinian ring and if only if \( R \) is a semisimple Artinian ring. \( G \) is finite and \( |G|^{-1} \in R. \) A module theoretic version of the Maschke’s Theorem is proven in (Kosan et al., 2014) for group modules. This version states that for a nonzero \( R \)-module \( M \) and a group \( G, MG \) is a semisimple module over \( RG \) if and only if \( M \) is a semisimple module and \( G \) is a finite group whose order is invertible in \( End_R(M) \) that is all the \( R \)-endomorphisms of \( M. \) The purpose of this section is giving a criterion for the semisimplicity of a \( G \)-set module to generalize the Maschke’s Theorem via the \( G \)-set modules.

**Theorem 22.** Let \( M \) be a nonzero \( R \)-module, \( G \) a group, \( S \) a \( G \)-set. If \( X \cap \Delta_G(MS) = 0 \) for some nonzero \( RG \)-submodule \( X \) of \( (MS)_{RG} \), then each orbit \( GS \) of \( S \) for \( s \in S \) is a finite set.

**Proof.** Firstly, we know that \( \Delta_G(MS) \) is an \( RG \)-submodule of \( (MS)_{RG}. \) Assume that \( GS \) is an infinite orbit for some \( s \in S. \) Then for any \( 0 \neq x = m_1 s_1 + \ldots + m_k s_k \in X \) where \( s_1, \ldots, s_k \in GS \) are distinct and \( m_i s_i \neq 0, \) there is an element \( g \) of \( G \) such that \( s_j g \neq s_j \) for \( 1 \leq j \leq k. \) Hence, \( (1-g)x = \sum_{i=1}^{k} m_i s_i - \sum_{i=1}^{k} m_i g s_i \neq 0, \) and also \( (1-g)x \in X. \) On the other hand, \( 0 \neq (1-g)x = \sum_{i=1}^{k} m_i (s_i) - 1 \in \Delta_G(MS). \) Then, \( X \cap \Delta_G(MS) = 0 \) and this is a contradiction.

We recall the following lemma in (Lam, 2001), and also in (Kosan et al., 2014).

**Lemma 23.** (Kosan et al., 2014; Lam, 2001) Let \( X \leq Y \) be right \( RG \)-modules and \( G \) be a finite group whose order is invertible in \( End_R(Y). \) If \( X \) is a direct summand of \( Y \) as \( R \)-modules, then \( X \) is a direct summand of \( Y \) as \( RG \)-modules.

**Theorem 24.** If \( M \) is a semisimple \( R \)-module, \( G \) is a finite group whose order is invertible in \( End_R(M) \) \( (|G|^{-1} \in End_R(M)) \), and \( S \) is a finite \( G \)-set, then \( (MS)_{RG} \) is semisimple.

**Proof.** Assume that \( M \) is a semisimple \( R \)-module, \( G \) is a finite group whose order is invertible in \( End_R(M) \), and \( S \) is a finite \( G \)-set. Let \( Y \) be an \( RG \)-submodule of \( MS. \) Firstly, \( (MS)_{RG} \) is semisimple since \( M_R \) is semisimple. Hence, \( Y_{RG} \) is a direct summand of \( (MS)_{RG}. \) Moreover, \( |G|^{-1} \in End_R(MS) \) since \( G \) is finite and \( |G|^{-1} \in End_R(M). \) So, \( Y_{RG} \) is a direct summand of \( (MS)_{RG} \) by Lemma 23 that means \( (MS)_{RG} \) is semisimple.

**7. Conclusion**

In the context of this study, we establish the set denoted as \( MS, \) which encompasses elements represented as a formal finite sum in the format \( \sum_{s \in S} m_s s \) where \( m_s \) belongs to the set \( M \) and \( S \) is a \( G \)-set. It is noteworthy that the set \( MS \) exhibits module-like properties with respect to the group ring \( RG, \) supporting both addition and scalar multiplication, akin to the \( RG \)-module \( MG. \) Therefore, incorporating \( G \)-set modules enable us to extend and consolidate the theories pertaining to both group algebra and group modules. Additionally, we identify crucial properties of \( (MS)_{RG}, \) elucidating a technique for decomposing the \( RG \)-module
into a direct sum of $RG$-submodules. Moreover, we substantiate the semisimplicity issue of $(MS)_{RG}$ concerning the characteristics of $M_R$, $S$, and $G$. On the other hand, if the properties of $M_R$, $S$, and $G$ can be determined when the semi-simplicity of $(MS)_{RG}$ is given, a quite strong result related to the semisimplicity of $G$-set modules is obtained bilaterally. In addition, the regularity of $(MS)_{RG}$, such as the examination of the semisimplicity of $(MS)_{RG}$, can be characterized according to the properties of $M_R$, $S$, and $G$ and other necessary parameters.

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