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| RESEARCH ARTICLE

A New Generalization of the Alternating Harmonic Series

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| ABSTRACT

Kilmer and Zheng (2021) recently introduced a generalized version of the alternating harmonic series. In this paper, we introduce a new generalization of the alternating harmonic series. A special case of our generalization converges to the Kilmer-Zheng series. Then we investigate several interesting and useful properties of this generalized, such as a summation formula related to the Hurwitz -Lerch Zeta function, a duplication formula, an integral representation, derivatives, and the recurrence relationship. Some important special cases of the main results are also discussed.

| KEYWORDS

Alternating Harmonic Series, Dirichlet eta function, Hurwitz -Lerch Zeta function, Polylogarithm function.

| ARTICLE INFORMATION

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1. Introduction

In 2021, Kilmer and Zheng [2021] introduced a generalization of the alternating harmonic series:

$$\boldsymbol{S_k} = \left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)-\left(\frac{1}{k+1}+\cdots+\frac{1}{2k}\right) + \left(\frac{1}{2k+1}+\cdots+\frac{1}{3k}\right)-\left(\frac{1}{3k+1}+\cdots+\frac{1}{4k}\right)+\cdots \tag{1.1}$$

The case $k=1$ reduces immediately to the well-known alternating harmonic series:

$$\boldsymbol{S_1} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\log 2$$

The generalized series $\boldsymbol{S_k}$ converges for each positive integer $\boldsymbol{k}$, and satisfies the following equation ([Kilmer, 2021], corollary 2.3)

$$\boldsymbol{S_k}=\frac{\pi}{2k}\sum_{m=1}^{k-1}\csc\frac{m\pi}{k}+\frac{1}{k}\log 2 \tag{1.2}$$

This series also has an integral representation given by ([Kilmer, 2021], page: 13482)

$$\boldsymbol{S_k}=\int_0^1\frac{1-x^k}{1-x}\frac{1}{x^k+1}dx \tag{1.3}$$

The relationship between Harmonic numbers and the generalized series $\boldsymbol{S_k}$ is given by ([Kilmer, 2021], Theorem 3.2)

$$\lim_{k\to\infty} H_k - S_k = \log\frac{\pi}{2} \tag{1.4}$$

Motivated by the generalized alternating harmonic series $\boldsymbol{S_k}$, we introduce a further generalization of the alternating harmonic series in this paper for which the series $\boldsymbol{S_k}$ defined above is a special case. We study some properties of this series and prove some relationships with other functions.

Before defining the new generalization, we start with some definitions of the family of zeta functions needed for the statement of the results.

Let $\mathbb{Z}^+$, $\mathbb{Z}^-$denote the set of positive integers, negative integers, respectively, also let: $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$ and $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$

And as usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the set of complex numbers.

The Hurwitz -Lerch Zeta function ([Chaudhry, 2021], Eq (1.3)), also ([Nadeem, 2020], Eq (1.1))is an important function in the analytic number theory, and is defined by:

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \qquad \begin{pmatrix} a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \\ \Re(s) > 1 \text{ when } |z| = 1 \end{pmatrix} \tag{1.5}$$

The Hurwitz -Lerch Zeta function generalizes various special functions such as Zeta function $\zeta(s)$, Dirichlet eta function $\eta(s)$, and the Hurwitz (or the generalized) Zeta function $\zeta(s,a)$

$$\Phi(1,s,1) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1 \tag{1.6}$$

$$\Phi(-1,s,1) = \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad \Re(s) > 0 \tag{1.7}$$

$$\Phi(1,s,a) = \zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-, \Re(s) > 1) \tag{1.8}$$

The Polylogarithm function is also related to a special case of The Hurwitz -Lerch Zeta function:

$$z\Phi(z,s,1) = \mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad \begin{pmatrix} s \in \mathbb{C} \text{ when } |z| < 1; \\ \Re(s) > 1 \text{ when } |z| = 1 \end{pmatrix} \tag{1.9}$$

An integral representation of the general Hurwitz -Lerch Zeta function ([2], Eq (1.4)) is given by:

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt \qquad \begin{pmatrix} \Re(a) > 0, \Re(s) > 0 \text{ when } |z| \le 1 \ (z \ne 1); \\ \Re(s) > 1 \text{ when } z = 1 \end{pmatrix} \tag{1.10}$$

Where $\Gamma(s)$ is the Gamma function.

2. A new generalization of the alternating harmonic series

Definition 2.1 we consider the following generalizations of the alternating harmonic series:

$$\begin{aligned} AHS(z,s,k) &= \left(z + \frac{z^{\mathbf{2}}}{\mathbf{2}^s} + \dots + \frac{z^{\boldsymbol{k}}}{\boldsymbol{k}^s}\right) - \left(\frac{z^{\boldsymbol{k}+1}}{(\boldsymbol{k}+1)^s} + \dots + \frac{z^{2\boldsymbol{k}}}{(2\boldsymbol{k})^s}\right) \\ &+ \left(\frac{z^{2\boldsymbol{k}+1}}{(2\boldsymbol{k}+1)^s} + \dots + \frac{z^{3\boldsymbol{k}}}{(3\boldsymbol{k})^s}\right) - \left(\frac{z^{3\boldsymbol{k}+1}}{(3\boldsymbol{k}+1)^s} + \dots + \frac{z^{4\boldsymbol{k}}}{(4\boldsymbol{k})^s}\right) + \cdots \end{aligned} \tag{2.1}$$

We will assume that z is a real number, s is a complex, and $\boldsymbol{k}$ is a positive integer.
The convergence of this generalized series will be investigated in the next section.

Observe that for $z = 1$ and $s = 1$ in particular, the generalized $AHS(z,s,k)$ reduces to the series $\boldsymbol{S_k}$ in Eq. (1.1)

Remark 2.1 the generalized $\boldsymbol{AHS}(z,s,k)$ can be written as an alternating series as following:

$$AHS(z,s,k)=\sum_{m=0}^{\infty}\left(\frac{z^{m\boldsymbol{k}+1}}{(m\boldsymbol{k}+1)^s}+\frac{z^{m\boldsymbol{k}+2}}{(m\boldsymbol{k}+2)^s}+\cdots+\frac{z^{mk+k}}{(mk+k)^s}\right)(-1)^m \tag{2.2}$$

This form of alternating series will be used extensively in our main results.

Remark 2.2 For $k=1$,the generalized series $AHS(z,s,k)$ reduces to the polylogarithm function in the following case:

$$AHS(z,s,1)=\frac{z}{1^s}-\frac{z^2}{2^s}+\frac{z^3}{3^s}-\frac{z^4}{4^s}+\cdots=-\mathrm{Li}_s(-z) \qquad \begin{pmatrix} s\in\mathbb{C}\text{ when }|z|<1;\\ \Re(s)>1\text{ when }|z|=1\end{pmatrix} \tag{2.3}$$

3. Convergence and relationships with the family of Zeta functions

In this section, we study the convergence of the generalized series $AHS(z,s,k)$, and show some relationships between this generalized and the family of zeta functions. We also deduce the duplication formula of this generalized.

Lemma 3.1 *the generalized series $AHS(z,s,k)$ converges for all:*
(i) $s\in\mathbb{C}$ when $|z|<1$
(ii) $\Re(s)>0$ when $|z|=1(but\ z\neq -1\ if\ k=odd)$
(iii) $\Re(s)>1$ when $z=-1\ and\ k=odd$

Proof. we make use of Eq. (2.2):

$$\begin{aligned}AHS(z,s,k)&=\sum_{m=0}^{\infty}\left(\frac{z^{m\boldsymbol{k}+1}}{(m\boldsymbol{k}+1)^s}+\frac{z^{m\boldsymbol{k}+2}}{(m\boldsymbol{k}+2)^s}+\cdots+\frac{z^{mk+k}}{(mk+k)^s}\right)(-1)^m\\&=\boldsymbol{z}\sum_{\boldsymbol{m=0}}^{\infty}\frac{z^{m\boldsymbol{k}}(-1)^m}{(m\boldsymbol{k}+1)^s}+z^2\sum_{\boldsymbol{m=0}}^{\infty}\frac{z^{m\boldsymbol{k}}(-1)^m}{(m\boldsymbol{k}+2)^s}+\cdots+z^k\sum_{\boldsymbol{m=0}}^{\infty}\frac{z^{m\boldsymbol{k}}(-1)^m}{(m\boldsymbol{k}+\boldsymbol{k})^s}\\&=\frac{\boldsymbol{1}}{\boldsymbol{k}^s}\left(\boldsymbol{z}\sum_{\boldsymbol{m=0}}^{\infty}\frac{(-z^k)^m}{\left(m+\frac{1}{k}\right)^s}+z^2\sum_{\boldsymbol{m=0}}^{\infty}\frac{(-z^k)^m}{\left(m+\frac{2}{k}\right)^s}+\cdots+z^k\sum_{\boldsymbol{m=0}}^{\infty}\frac{(-z^k)^m}{\left(m+\frac{k}{k}\right)^s}\right)\end{aligned} \tag{3.1}$$

Observing that each series of the last expression converges for every complex numbers when $|z|<1$
if $z=1$ or $if\ (z=-1\ and\ k\ \boldsymbol{is\ even})$, then each series is alternating series and converges when $\Re(s)>0$
if $(z=-1\ and\ \boldsymbol{k\ is\ odd})$, then each series satisfies the convergence of the Hurwitz-Zeta function (1.8) and converges when $\Re(s)>1$

Corollary 3.1 *using the definition of the Hurwitz -Lerch Zeta function (1.5) to Eq. (3.1), gives:*

$$AHS(z,s,k)=\frac{\boldsymbol{1}}{\boldsymbol{k}^s}\sum_{\boldsymbol{m=1}}^{\boldsymbol{k}}\boldsymbol{\Phi}\left(-z^{\boldsymbol{k}},\boldsymbol{s},\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)z^{\boldsymbol{m}} \tag{3.2}$$

Lemma 3.2 *let $\boldsymbol{k}$ be even, $\Re(s)>0$, The following duplication formula holds:*

$$AHS(1,s,k)+AHS(-1,s,k)=2^{1-s}AHS\left(1,s,\frac{k}{2}\right) \tag{3.3}$$

Proof. Assuming that $\boldsymbol{k}$ **is even** , $\Re(s)>0$, we have $(-1)^{m\boldsymbol{k}}=1$, applying this in Eq. (2.2):

$$\begin{aligned}AHS(1,s,k)+AHS(-1,s,k)=&\sum_{m=0}^{\infty}\left(\frac{1}{(m\boldsymbol{k}+1)^s}+\frac{1}{(m\boldsymbol{k}+2)^s}+\cdots+\frac{1}{(mk+k)^s}\right)(-1)^m\\&+\sum_{m=0}^{\infty}\left(\frac{(-1)^1}{(m\boldsymbol{k}+1)^s}+\frac{(-1)^2}{(m\boldsymbol{k}+2)^s}+\cdots+\frac{(-1)^k}{(mk+k)^s}\right)(-1)^m\end{aligned}$$

$$= \sum_{m=0}^{\infty} \left(\frac{2}{(m\boldsymbol{k}+2)^s} + \frac{2}{(m\boldsymbol{k}+4)^s} + \cdots + \frac{2}{(mk+k)^s} \right) (-1)^m$$

$$= \frac{\mathbf{2}}{\mathbf{2}^{s}} \sum_{m=0}^{\infty} \left(\frac{1}{\left(m\frac{k}{2}+1\right)^s} + \frac{1}{\left(m\frac{k}{2}+2\right)^s} + \cdots + \frac{1}{\left(m\frac{k}{2}+\frac{k}{2}\right)^s} \right) (-1)^m$$

$$= \mathbf{2}^{\mathbf{1}-\boldsymbol{s}} AHS\left(1, s, \frac{k}{2}\right)$$

which completes the proof.

Corollary 3.2 *let* $k = 2$ *in* *Eq. (3.3), then applying Eq. (2.3), gives:*

$$\eta(s) = 2^{s-1}\big(AHS(1, s, 2) + AHS(-1, s, 2)\big) \qquad \Re(s) > 0 \qquad (3.4)$$

Corollary 3.3 *on setting* $s = 1$ *in* *Eq. (3.3), gives:*

$$AHS(-1,1,k) = \boldsymbol{S}_{\frac{k}{2}} - \boldsymbol{S}_{\boldsymbol{k}} \qquad \boldsymbol{k} = \boldsymbol{enen} \qquad (3.5)$$

Where $\boldsymbol{S}_{\boldsymbol{k}}$ is the generalized alternating harmonic series defined in Eq. (1.1)

4. An integral representation

In this section, we present a certain integral representation of the generalized series $AHS(z, s, k)$, and discuss some special cases of this integral.

Theorem 4.1 *the series* $AHS(z, s, k)$ *satisfies the following integral representation:*

$$AHS(z, s, k) = \frac{\mathbf{1}}{\Gamma(s)} \int_0^z \frac{x^k - 1}{(x^k+1)(x-1)} \ln\left(\frac{z}{x}\right)^{s-1} dx \qquad \begin{pmatrix} k \in \mathbb{N}, \Re(s) > 0, \\ z \in \,]0,1] \end{pmatrix} \qquad (4.1)$$

Proof.

Using the integral representation of the Hurwitz -Lerch Zeta function in Eq. (1.10):

$$\boldsymbol{\Phi}\left(-z^{\boldsymbol{k}}, \boldsymbol{s}, \frac{\boldsymbol{m}}{\boldsymbol{k}}\right) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-\frac{m}{k}t}}{1 + z^{\boldsymbol{k}} e^{-t}} dt \qquad (4.2)$$

For $k, m \in \mathbb{N}$, observing the domain of Eq. (1.10), it easy to see that the integral in Eq. (4.2) converges for $z \in \,]0,1]$ when $\Re(s) > 0$.

Then we insert Eq. (4.2) in Eq. (3.2) gives:

$$AHS(z, s, k) = \frac{\mathbf{1}}{k^s} \sum_{m=1}^{k} \left(\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-\frac{m}{k}t}}{1 + z^{\boldsymbol{k}} e^{-t}} dt \right) z^{\boldsymbol{m}}$$

$$= \frac{\mathbf{1}}{k^s \Gamma(s)} \int_0^{\infty} \left(\frac{t^{s-1}}{1 + z^{\boldsymbol{k}} e^{-t}} \sum_{m=1}^{k} \left(z e^{-\frac{t}{k}}\right)^m \right) dt$$

$$= \frac{\mathbf{1}}{k^s \Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{1 + z^{\boldsymbol{k}} e^{-t}} \left(z e^{-\frac{t}{k}}\right) \left(\frac{\left(z e^{-\frac{t}{k}}\right)^k - 1}{z e^{-\frac{t}{k}} - 1} \right) dt$$

$$= \frac{\mathbf{1}}{k^s \Gamma(s)} \int_0^{\infty} \left(\frac{z^{\boldsymbol{k}} e^{-t} - 1}{z^{\boldsymbol{k}} e^{-t} + 1} \right) \left(\frac{z e^{-\frac{t}{k}}}{z e^{-\frac{t}{k}} - 1} \right) t^{s-1} dt \qquad (4.3)$$

By writing $ze^{-\frac{t}{k}} = x \leftrightarrow t = k \ln\left(\frac{z}{x}\right)$, then the integral in Eq. (4.3) becomes:

$$AHS(z,s,k)=\frac{\mathbf{1}}{k^s\Gamma(s)}\int_{ze^0}^{ze^{-\infty}}\frac{x^k-1}{(x^k+1)}\frac{x}{(x-1)}\left(k\ln\left(\frac{z}{x}\right)\right)^{s-1}\left(-\frac{k}{x}\right)dx$$
$$=\frac{k^s}{k^s\Gamma(s)}\times-\int_z^0\frac{x^k-1}{(x^k+1)(x-1)}\ln\left(\frac{z}{x}\right)^{s-1}dx$$
$$=\frac{\mathbf{1}}{\Gamma(s)}\int_0^z\frac{x^k-1}{(x^k+1)(x-1)}\ln\left(\frac{z}{x}\right)^{s-1}dx$$

this proves Eq. (4.1)

Corollary 4.1 *The special case* $\boldsymbol{s}=\mathbf{1}$ *of Eq. (4.1) gives:*

$$AHS(z,1,k)=\int_{\mathbf{0}}^{z}\frac{x^k-1}{(x^k+1)(x-1)}dx \qquad \boldsymbol{k}\in\mathbb{N}, z\in\,]0,1] \tag{4.4}$$

Observe that for z =1, the integral in Eq. (4.4) reduces to that in Eq. (1.3)

Corollary 4.2 *The special case* $\boldsymbol{k}=\mathbf{2}$ *of Eq. (4.4) is:*

$$AHS(z,1,2)=\int_{\mathbf{0}}^{z}\frac{x^2-1}{(x^2+1)(x-1)}dx=\frac{\mathbf{1}}{\mathbf{2}}\int_{\mathbf{0}}^{\boldsymbol{z}}\frac{\mathbf{2}\boldsymbol{x}}{x^2+1}\boldsymbol{dx}+\int_{\mathbf{0}}^{\boldsymbol{z}}\frac{\mathbf{1}}{x^2+1}\boldsymbol{dx}$$
$$=\frac{\mathbf{1}}{\mathbf{2}}\ln(\boldsymbol{z}^2+\mathbf{1})+\tan^{-1}\boldsymbol{z}\quad \boldsymbol{for}\qquad z\in\,]0,1] \tag{4.5}$$

5. Derivatives and the recurrence relationship

In this section, we present the recurrence relationship of the generalized $AHS(z,s,k)$, and show that for nonpositive integer s, the generalized series $AHS(z,s,k)$ converges to a rational function.

Lemma 5.1 *for* $|z|<\mathbf{1},\boldsymbol{k}\in\mathbb{N}$*, we have the following derivative:*

$$\frac{d}{dz}AHS(z,1,k)=\frac{z^{\boldsymbol{k}}-\mathbf{1}}{(z^{\boldsymbol{k}}+\mathbf{1})(z-\mathbf{1})} \tag{5.1}$$

Proof**.** set $\boldsymbol{s}=\mathbf{1}$ in Eq. (2.2), and then applying $\frac{d}{dz}$ both sides:

$$\frac{d}{dz}AHS(z,1,k)=\frac{d}{dz}\sum_{m=0}^{\infty}\left(\frac{z^{m\boldsymbol{k}+1}}{m\boldsymbol{k}+1}+\frac{z^{m\boldsymbol{k}+2}}{m\boldsymbol{k}+2}+\cdots+\frac{z^{mk+k}}{mk+k}\right)(-1)^m$$
$$=\sum_{m=0}^{\infty}\left(z^{m\boldsymbol{k}}+z^{m\boldsymbol{k}+1}+\cdots+z^{mk+k-1}\right)(-1)^m$$
$$=\left(\mathbf{1}+z+\cdots+z^{\boldsymbol{k}-\mathbf{1}}\right)-\left(z^{\boldsymbol{k}}+\cdots+z^{\mathbf{2}\boldsymbol{k}-\mathbf{1}}\right)+\left(z^{\mathbf{2}\boldsymbol{k}}+\cdots+z^{\mathbf{3}\boldsymbol{k}-\mathbf{1}}\right)-\cdots$$
$$=\left(\mathbf{1}+z+\cdots+z^{\boldsymbol{k}-\mathbf{1}}\right)\times\left(1-z^{\boldsymbol{k}}+z^{2\boldsymbol{k}}-z^{3\boldsymbol{k}}+\cdots\right.$$
$$=\left(\sum_{\boldsymbol{n}=\mathbf{0}}^{\boldsymbol{k}-\mathbf{1}}\boldsymbol{z}^{\boldsymbol{n}}\right)\left(\sum_{\boldsymbol{n}=\mathbf{0}}^{\infty}z^{\boldsymbol{nk}}(-\mathbf{1})^{\boldsymbol{n}}\right)$$
$$=\frac{z^{\boldsymbol{k}}-\mathbf{1}}{z-\mathbf{1}}\times\frac{1}{\mathbf{1}+z^{\boldsymbol{k}}}\qquad |z|<\mathbf{1}$$

For other values of $\boldsymbol{s}$, the derivative of $AHS(z,s,k)$ results the series itself but with a lower order:

$$z\frac{d}{dz}AHS(z,s,k)=AHS(z,s-1,k) \tag{5.2}$$

Using lemma 5.1 in Eq. (5.2), we can express the generalized series $AHS(z, s, k)$ as rational function for nonpositive integer order $\boldsymbol{s}$

Corollary 5.1 *for* $|z| < \mathbf{1}$, $\boldsymbol{k} \in \mathbb{N}$*, the following holds by applying Eq. (5.1) to Eq. (5.2):*

$$AHS(z, 0, k) = \frac{z(z^{\boldsymbol{k}} - 1)}{(z^{\boldsymbol{k}} + 1)(z - 1)} \tag{5.3}$$

Corollary 5.2 *for* $|z| < \mathbf{1}$, $\boldsymbol{k} \in \mathbb{N}$*, setting* $\boldsymbol{s} = \mathbf{0}$ *in Eq. (5.2), and applying Eq. (5.3) gives:*

$$AHS(z, -1, k) = \frac{z(-z^{\mathbf{2}\boldsymbol{k}} + \mathbf{2}\boldsymbol{k}(z - \mathbf{1})z^{k} + \mathbf{1})}{(z^{k} + \mathbf{1})^{\mathbf{2}}(z - \mathbf{1})^{\mathbf{2}}} \tag{5.4}$$

More generally, for each $\boldsymbol{m}$ positive integer repeated application of $\left(z\frac{d}{dz}\right)$ to $AHS(z, 0, k)$, gives:

Corollary 5.3 *for* $|z| < \mathbf{1}$, $\boldsymbol{k}, \boldsymbol{n} \in \mathbb{N}$*, the following holds:*

$$AHS(z, -n, k) = \left(z\frac{d}{dz}\right)^{\boldsymbol{n}} \left(\frac{z(z^{\boldsymbol{k}} - 1)}{(z^{\boldsymbol{k}} + 1)(z - 1)}\right) \tag{5.5}$$

6. Conclusion

In this paper, we introduced a new generalization of the alternating harmonic series given by Eq. (2.1), a special case of this generalized reduces to the generalized alternating Harmonic series $\boldsymbol{S_k}$ defined in Eq. (1.1), we studied the convergence of this generalized, and demonstrated relationships with the family of zeta functions such as Eq. (3.2), Eq. (3.4). We also obtained the duplication formula in Eq. (3.3) and presented an integral representation of this generalized series in Eq. (4.1). Finally, we deduced a recurrence relationship for the generalized and showed that for nonpositive integer order s the generalized series $AHS(z, s, k)$ is a rational function.

However, it is important to expand future studies to gain a more comprehensive understanding of the new series and its applications. By exploring a wider range of values and delving deeper into the intricacies of this new generalization, researchers can uncover additional features and properties. Additionally, researchers can improve upon the new version and further generalize the series, discovering even more interesting and exciting properties. This approach will enhance the overall knowledge and applications of the new series in mathematics.

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