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| RESEARCH ARTICLE

Finite Dimensional Labeled Graph Algebras

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| ABSTRACT
Given a directed graph E and a labeling L, one forms the labeled graph algebra by taking a weakly left-resolving labeled space (E, L, B) and considering a generating family of partial isometries and projections. In this paper, we discuss details in the formulation of the algebras, provide examples, and formulate a process that describes the algebra given the graph and a labelling.

| KEYWORDS

Labeled graph, Directed graph, Cuntz–Krieger algebra, Graph algebra.

| ARTICLE INFORMATION

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1. Introduction

A directed graph $E = (E^0, E^1, s, r)$ consists of a countable set E^0 of vertices and E^1 of edges, and maps $s,r : E^1 \rightarrow E^0$ identifying the source (origin) and the range (terminus) of each edge. The graph is row-finite if each vertex emits at most finitely many edges. A vertex is a sink if it is not a source of any edge. A path is a sequence of edges $e_1e_2 \dots e_n$ with $r(e_i) = s(e_{i+1})$ for each $i = 1, 2, \dots, n-1$. An infinite path is a sequence $e_1 e_2 \dots$ of edges with $r(e_i)=s(e_{i+1})$ for each i.

For a finite path $p=e_1e_2 \dots e_n$, we define $s(p):=s(e_1)$ and $r(p):= r(e_n)$.

We use the following notation

$$E^* := \bigcup_{n=0}^{\infty} E^n, \text{ where } E^n := \{p : p \text{ is a path of length } n\}$$

Let $E = (E^0, E^1, s, r)$ be a directed graph and let A be a set of alphabets (colors). A labeling is a function $L : E^1 \rightarrow A$. Without loss of generality, we will assume that $A = L(E^1)$. The pair (E, L) is called a labeled graph.

Look at the graph in Example 1.1 below and the next for examples of labeled graphs.

Given a labeled graph (E, L), we extend the labeling function L canonically to the sets E^* as follows.
Using A^n for the set of words of size n, L is defined from E^n into A^n as

$$\mathcal{L}(e_1e_2 \dots e_n) = \mathcal{L}(e_1)\mathcal{L}(e_2)\dots\mathcal{L}(e_n).$$

Following tradition, we use

$$\mathcal{L}^*(E) := \bigcup_{n=1}^{\infty} \mathcal{L}(E^n).\mathcal{L}^*(E) := \bigcup_{n=1}^{\infty} \mathcal{L}(E^n).$$

This is the set of all finite ``legal" words in the labeled graph. Thus, *abc* ϵ L*(E) if there is a path of three edges labeled *"a*", *"b*", and *"c*", in that order.

For a word $\alpha = a_1a_2 \dots a_n \in \mathcal{L}^n(E)$
we write

$$s(\alpha) := \{s(p) : p \in E^n,\ \mathcal{L}(p) = \alpha\}$$

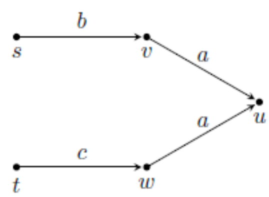
And

$$r(\alpha) := \{r(p) : p \in E^n,\ \mathcal{L}(p) = \alpha\}.$$

Each of these sets is a subset of E^0. The use of s and r for an edge/path versus a label/word should be clear from the context.

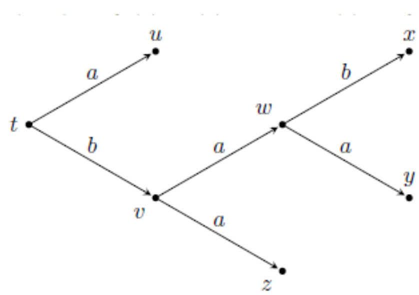
A labeled graph (E, L) is said to be left-resolving if for each vϵ E^0 the function L : $r^{-1}(v)$ → A is injective. In other words, no two edges pointing to the same vertex are labeled the same.

Example 1.1
In the labeled graph below, we have two edges, both labeled a, with a range of u, thus our graph is not left-resolving.



Let B be a non-empty subset of P(E^0), where P(E^0) is the power set of E^0, the set of all subsets of E^0.
Given a set A ϵ B we write L(AE^1) for the set {L(e): e ϵ E^1 and s(e) ϵ A}.
For a set A ϵ B and a word α ϵ L^n(E) we define the relative range of α with respect to A as

$$r(A, \alpha) := \{r(p) : \mathcal{L}(p) = \alpha \text{ and } s(p) \in A\}.$$



In the above graph

$$r(\{t, v, z\}, a) = \{u, w, z\},\ r(\{t, v, w\}, b) = \{v, x\}$$

and

$$r(\{v, w, x\}, ba) = \emptyset.$$

Several works have been done on labeled graph C*-algebras (for example, see [1], [2], [3] or [4]) with the restriction that the graph has no sinks. In this paper, we will present results on these algebras when the graph may have sinks. We will only consider finite graphs that have no loops, because infinite graphs or graphs that have loops or cycles produce infinite dimensional algebras which are beyond the intended scope of this paper. This paper's main aim is to provide algebra formulation, provide examples, and formulate a process describing the algebra given the graph and a labeling. We consider these results an introductory steppingstone towards characterizing algebras generated by different graphs and/or different labelling. Moreover, due to the level of this work, we have avoided discussion of the norms and completeness, as such, we are considering *-algebras as opposed to C*-algebras.

The paper is organized as follows. In section 2, we recall basic definitions and examples, and develop some terminologies for labeled graphs. In section 3, we briefly describe labeled graph *-algebras. In section 4, we formulate a means by which we compute labeled graph *-algebras for finite graphs with no cycles.

2. Preliminaries

We will start this section by recalling the definition of a ring. A ring R is a set with two binary operations, addition denoted *a+b*, and multiplication denoted *ab* such that ∀*a, b, c* ϵ R

1. *a+b = b+a*
2. *(a+b)+c = a+(b+c)*
3. There is an element, typically denoted **0** ϵ R such that ∀a ϵ R, a+**0** = a
4. For each a ϵ R there is an element, typically denoted *-a* in R , with *a+(-a)* = **0**
5. *a(bc)=(ab)c*
6. *a(b+c) = ab+ac, (b+c)a = ba+ca*

In this paper, we will mainly discuss algebras, which are rings that also interact with a field of scalars. For much of our discussion, this field will be the set of complex numbers, ℂ. For all a, b ϵ R and α, β ϵ ℂ we have

1. α a ϵ R
2. α (a + b) = α a + α b
3. (α + β) a = α a + β b
4. **0** a = **0** = α **0**

The algebras discussed in this paper are called *-algebra. A *-algebra ***A*** is an algebra along with an operation * (called adjoint) with the following ∀ A, B ϵ ***A*** and λ ϵ **C**

1. A* ϵ ***A***
2. (A+B)* = A*+B*
3. (λ A)* = $\bar{\lambda}$ A*
4. (A*)* = A
5. (AB)* = B* A*

Example 2.1 We provide a few examples:

1. The set of complex numbers ℂ is a *-algebra, where the adjoint of a complex number is simply the complex conjugate.

2. The set of 2 X 2 matrices of complex numbers, $M_2(\mathbb{C})$ is a *-algebra where the adjoint of a matrix A is the conjugate transpose of A. In fact, $M_n(\mathbb{C})$, the set of n X n matrices of complex numbers is a *-algebra.

3. The set of complex valued continues functions on [0, 1] is a *-algebra, where addition and multiplication are the usual addition and multiplication of functions and the adjoint is the complex conjugate.

We call P ϵ ***A*** a ***projection*** if P^2 = P=P*. In this paper, projections will generally be denoted by P. Note that any identity matrix is a projection, as is any (0,1)-matrix with 1s only on the diagonal. We call Sϵ ***A*** a ***partial isometry*** if S*S is a projection. We note that if S*S is a projection then SS* is also a projection.

Example 2.2

1. Let $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $S^*S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ which is a projection, so S is a partial isometry.

2. Let $V = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix}$. Then

$$V^*V = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2}i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ -\frac{1}{2}i & \frac{1}{2} \end{bmatrix}$$

is a projection.

Notice that $VV^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}i & -\frac{1}{2}i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is also a projection.

Remark 2.3
We will use the following standard facts on partial isometries:

If S is a partial isometry, so is S*. Also, S is a partial isometry if and only if S = SS*S, these are also equivalent to S* = S*SS*.

Lemma 2.4
Given a set A ∈ B, if $\alpha = a_1a_2 \ldots a_n$ is in L*(E), then $r(A,\alpha) = r(r(\ldots r(r(A,a_1),a_2),\ldots ,a_{n-1}),a_n)$.

Proof
For simplicity we will prove that r(A, ab) = r(r(A, a), b). The complete proof follows inductively.
We first wish to show r(r(A,a)) ⊆ r(A,ab). Let x ∈ r(r(A,a),b). Then ∃ y ∈ r(A,a) and $\lambda_2 \in E^1$ with $L(\lambda_2)=b$ with $s(\lambda_2) = y$, $r(\lambda_2) = x$. Then ∃ z ∈ A and $\lambda_1 \in E^1$ with $L(\lambda_1)=a$, $s(\lambda_1) = z$, $r(\lambda_1) = y$. Then because $r(\lambda_1) = s(\lambda_2)$ we define $\lambda := \lambda_1 \lambda_2$. Also, L(λ) = ab. Then x ∈ r(A,ab). By the arbitrary choice of x, r(r(A,a),b) ⊆ r(A,ab).

We now wish to show r(A,ab) ⊆ r(r(A,a),b). Let x ∈ r(A,ab). Then ∃ $\lambda \in E^2$ with L(λ) = ab and z ∈ A with s(λ) = z and r(λ) = x.
By the nature of $\lambda \in E^2$, ∃ $\lambda_1,\lambda_2 \in E^1$ with $r(\lambda_1)=s(\lambda_2)$ and $\lambda = \lambda_1 \lambda_2$. Also, $L(\lambda_1) = a$, $L(\lambda_2)=b$. This means $s(\lambda_2) \in r(A,a)$. Because $r(\lambda_2) = x$, x ∈ r(r(A,a),b). Then by the arbitrary choice of x ∈ r(A,ab), r(A,ab) ⊆ r(r(A,a),b).

By the two way inclusion of the sets, we have shown r(A,ab) = r(r(A,a),b).

QED

We say B is closed under relative ranges if r(A,α) ∈ B for any A∈B and any α ∈ L*(E).

A set B is said to be ***accommodating*** for a labeled graph (E, L) if

1. r(α)∈ B for each α ∈ L*(E)
2. B is closed under relative ranges
3. B is closed under finite intersections and unions.

A labeled space is a triple (E, L, B) where B is accommodating for a labeled graph (E, L). We will assume that B ≠q {Φ}.
An accommodating labeled space (E, L, B) is called ***weakly left-resolving*** if for any A, B ∈ B and any α ∈ L*(E)
r(A ∩ B, α) = r(A,α) ∩ r(B,α).

Given sets A, B ∈ B and α ∈ L*(E), in general r(A ∩ B,α) ⊆ r(A,α) ∩ r(B,α).

Proof
Suppose v ∈ r(A ∩ B, α). Then ∃ x ∈ A ∩ B such that x ∈ s(α) and v ∈ r(α). Then v ∈ r(A,α) and v ∈ r(B,α). By our arbitrary choice of v, r(A ∩ B, α) ⊆ r(A,α)∩ r(B,α).

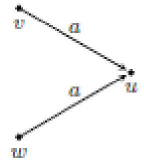
QED

Theorem 2.5
If a labeled graph (E, L) is left-resolving then the labeled triple (E, L, B) is weakly left-resolving.

Proof
We will proceed by proving the contrapositive of the theorem. Suppose (E, L, B) is not weakly left-resolving. Then because r(A ∩ B, α) ≠ r(A, α)∩ r(B, α) and by Lemma 2.2, r(A ∩ B, α) ⊂ (r(A, α) ∩ r(B, α)).
Then ∃v ∈ r(A, α) ∩ r(B, α) \ r(A ∩ B, α). This implies ∃ x ∈ A \ B, y ∈ B \ A such that x, y ∈ s(α). Then there exist distinct $\lambda_1, \lambda_2 \in E^*$ with $s(\lambda_1) = x$, $r(\lambda_1) = v$, $s(\lambda_2) = y$, $r(\lambda_2) = v$ and $L(\lambda_1)=L(\lambda_2) = \alpha$. Note $\lambda_1, \lambda_2 \in r^{-1}(v)$ and $\lambda_1 \neq \lambda_2$.
This shows $L : r^{-1} \to A$ is not injective. Then (E, L) is not left-resolving.

QED

The converse of Theorem 2.5 is not true. The labeled graph below is not left resolving. However, if
B = {{v,w}, {u}, Φ} then (E, L, B) is weakly left-resolving.



Notice also that some labeled graphs cannot be made weakly left-resolving labeled spaces; the labeled graph in Example 1.1 is an example of such.

We say (E, L, B) is ***non-degenerate*** if B is closed under relative complements. A ***normal*** labeled space is accommodating and non-degenerate.

3. Labeled Graph *-algebras

Let (E, L, B) be a weakly left-resolving labeled space. A representation of (E, L, B) in a *-algebra consists of projections $\{p_A : A \in B\}$, and partial isometries $\{s_a : a \in A\}$, satisfying:

1. If $A,\ B \in \mathcal{B}$, then $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$.
2. For any $a,\ b \in \mathcal{A}$, if $a \neq b$ then $s_a^* s_b = 0$.
3. For any $a \in \mathcal{A}$, $s_a^* s_a = p_{r(a)}$.
4. For any $a \in \mathcal{A}$ and $A \in \mathcal{B}$, $s_a^* p_A = p_{r(A,a)} s_a^*$.
5. For $A \in \mathcal{B}$ with $\mathcal{L}(AE^1)$ finite and A does not contain a sink we have

$$p_A = \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*.$$

The *labeled graph *-algebra* is the *-algebra generated by a (non-zero) representation of (E, L, B). Thus, the labeled graph *-algebra is the sums of products of the elements in the set $\{P_A : A \in B\} \cup \{s_a : a \in A\} \cup \{s_a^* : a \in A\}$ following the rules/axioms listed above. We use A(E, L, B) for the labeled graph *-algebra.

Remark 3.1 The following will be useful when constructing the *-algebra of a given labeled space.

1. Since s_a is a partial iscometry, $s_a = s_a s_a^* s_a = s_a p_{r(a)}$.
2. If a, b ϵ A then $s_a s_b = s_a p_{r(a)} s_b = s_a s_b p_{r(r(a), b)}$. And this is either 0, if ab is not in L*(E) or $s_a s_b p_{r(ab)}$. Thus, if ab is a legal word (i.e., is in L*(E)) then $s_a s_b = s_a s_b p_{r(ab)}$, otherwise $s_a s_b = 0$.
3. For a word $\mu = a_1 \ldots a_n$ we write s_μ to mean $s_{a1} \ldots s_{an}$. Therefore from (2) we get $s_\mu = s_\mu p_{r(\mu)}$.
4. Taking the adjoint: $(s_\mu)^* = (s_\mu p_{r(\mu)})^* = p_{r(\mu)}^* s_\mu^* = p_{r(\mu)} s_\mu^*$. That is, $s_\mu^* = p_{r(\mu)} s_\mu^*$.
5. Given A ϵ B and μ ϵ L*(E), since Poops $s_\mu = s_\mu p_{r(A,\mu)}$, taking the adjoint gives us: $s_\mu^* P_A = p_{r(A,\mu)} s_\mu^*$.

Lemma 3.2

Let $\mu = a_1 \ldots a_n$ be a word in L*(E), then $s_\mu^* s_\mu = p_{r(\mu)}$.

Proof

For simplicity, we will demonstrate this using a word of length two μ = ab. The complete proof can be done by induction.

$s_\mu^* s_\mu = (s_a s_b)^*(s_a s_b) = s_b^* s_a^* s_a s_b = s_b^* p_{r(a)} s_b = s_b^* s_b p_{r(r(a), b)} = p_{r(b)} p_{r(ab)} = p_{r(b)\cap r(ab)} = p_{r(ab)} = p_{r(\mu)}$.

QED

In the next lemma, we will show that $s_\mu^* s_\nu = 0$ unless one of the words μ, ν extends the other.

Lemma 3.3 Let μ, ν ϵ L*(E). Then

$$s_\mu^* s_\nu = \begin{cases} p_{r(\mu)} & \text{if } \nu = \mu \\ s_\gamma p_{r(\nu)} & \text{if } \nu = \mu\gamma \\ p_{r(\mu)} s_\gamma^* & \text{if } \mu = \nu\gamma \\ 0 & \text{otherwise.} \end{cases}$$

Proof

If $\mu = \nu$ then by Lemma 3.2 we get $s_\mu^* s_\nu = s_\mu^* s_\mu = p_{r(\mu)}$.

If $\nu = \mu\gamma$, then $s_\mu^* s_\nu = s_\mu^* s_{\mu\gamma} = s_\mu^* s_\mu s_\gamma = p_{r(\mu)} s_\gamma = s_\gamma p_{r(r(\mu),\gamma)} = s_\gamma p_{r(\mu\gamma)} = s_\gamma p_{r(\nu)}$.

If $\mu = \nu\gamma$, then $s_\mu^* s_\nu = s_{\nu\gamma}^* s_\nu = (s_\nu s_\gamma)^* s_\nu = s_\gamma^* s_\nu^* s_\nu = s_\gamma^* p_{r(\nu)} = p_{r(\nu\gamma)} s_\gamma^* = p_{r(\mu)} s_\gamma^*$.

Lastly, suppose $\mu = a_1 a_2 \ldots a_m,\ \nu = b_1 b_2 \ldots b_n$, and neither μ nor ν extends the other. This means $a_k \neq b_k$ for some $k \leq min\{m,\ n\}$. We will assume that k is the smallest such index, that is, $a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_{k-1} = b_{k-1}$, but $a_k \neq b_k$. Then

$$\begin{aligned}
s_\mu^* s_\nu &= (s_{a_1 a_2 \ldots a_{k-1} a_k \ldots a_m})^* s_{b_1 b_2 \ldots b_{k-1} b_k \ldots b_n} \\
&= (s_{a_k \ldots a_m})^* (s_{a_1 a_2 \ldots a_{k-1}})^* s_{b_1 b_2 \ldots b_{k-1}} s_{b_k \ldots b_n} \\
&= (s_{a_k \ldots a_m})^* p_{r(b_1 b_2 \ldots b_{k-1})} s_{b_k \ldots b_n},\ \text{since } a_i = b_i,\ \text{for } i < k \\
&= (s_{a_k \ldots a_m})^* s_{b_k \ldots b_n} p_{r(b_1 b_2 \ldots a_n)} \\
&= s_{a_m}^* \ldots s_{a_k}^* s_{b_k} \ldots s_{b_n} p_{r(b_1 b_2 \ldots a_n)} \\
&= 0 \text{ because } a_k \neq b_k.
\end{aligned}$$

QED

Recall that the *-algebra of (E, L, B) is generated by the set of isometries $\{s_a : a \in A\}$ and the set of projections $\{P_A : A \in B\}$. This is the linear span of the products of these elements and their adjoints. However, from Remark 3.1 and the above two lemmas we see that the products of these elements and/or their adjoints is always one of the form P_A, $s_\mu P_A$, $P_A s_\nu^*$ or $s_\mu P_A s_\nu^*$. Therefore *A*(E, L, B) is the linear span of elements of this form. That is *A*(E, L, B) is the span of the union of the following four sets:

$$\begin{aligned}
S_1 &= \{p_A : A \in \mathcal{B}\} \\
S_2 &= \{s_\mu p_A : \mu \in \mathcal{L}^*(E),\ A \in \mathcal{B}\} \\
S_3 &= \{p_A s_\nu^* : \nu \in \mathcal{L}^*(E),\ A \in \mathcal{B}\} \\
S_4 &= \{s_\mu p_A s_\nu^* : \mu, \nu \in \mathcal{L}^*(E),\ A \in \mathcal{B}\}.
\end{aligned}$$

4. Algebras

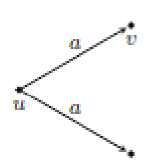
In this section we will consider some graphs and determine their labeled graph *-algebra. We will first demonstrate the full procedure on a simple graph and then discuss the results for the more involved graphs.

As the *-algebra of a labeled graph is generated by partial isometries assigned to labels and projections assigned to sets of vertices, we need a means of choosing which collection of sets of vertices we wish to consider. We will use the following convention:

1. $\mathcal{E} = \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\}$
2. $E^0_{\text{sink}} = \{v \in E^0 : s^{-1}(v) = \emptyset\}$
3. $\mathcal{E}^- = \cup E^0_{\text{sink}}$
4. $\mathcal{E}_0 =$ the smallest normal subset of $\mathcal{P}(E^0)$ that contains $\mathcal{E}^-$

Notation: We will write P_v, $P_{\{vw\}}$, etc. to mean $P_{\{\{v\}\}}$, $P_{\{\{v, w\}\}}$, etc. for ease of writing.

Example 4.1



We will consider the labeled graph given above for our first example. We note *E* = {{v, w}}, $E^- = E$, E_0 = {{v, w}, Φ}. We then choose B = {{v, w}, Φ}. Then, based on our definitions above, *A*(E, L, B) is generated by the linear span of $\{P_{\{vw\}}, S_a P_{\{vw\}}, P_{\{vw\}} S_a^*, S_a P_{\{vw\}} S_a^*\}$.

Further, $(S_a P_{\{vw\}})(S_a P_{\{vw\}})^*=S_a P_{\{vw\}}S_a^*$ and $(S_a P_{\{vw\}})^*(S_a P_{\{vw\}}) = P_{\{vw\}}S_a^*S_a P_{\{vw\}} = P_{\{vw\}}$. Then all units of the algebra can be expressed as products and adjoints of the partial isometry $S_a P_{\{vw\}}$. For this reason, we define a map Φ: *A*(E, L, B) → $M_2(\mathbb{C})$ we need only to define $\Phi(S_aP_{\{vw\}}) =\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$. We will define Φ to preserve multiplication, addition, and adjoint. Therefore $\Phi(P_{\{vw\}}S_a^*) = \Phi((S_aP_{\{vw\}})^*) =(\Phi(S_aP_{\{vw\}}))^* = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}^* =\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, $\Phi(P_{\{vw\}}) = \Phi(P_{\{vw\}}S_a^*S_aP_{\{vw\}}) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$, and $\Phi(S_aP_{\{vw\}}S_a^*) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$.

This gives us that

A(E, L, B) = $\text{span}\{P_{\{vw\}},S_aP_{\{vw\}},P_{\{vw\}}S_a\text{^}\{*\},S_aP_{\{vw\}}S_a\text{^}\{*\}\} \cong \text{span}\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}.\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\}$.

Therefore *A*(E, L, B) is isomorphic to $M_2(\mathbb{C})$.

Example 4.2

In the next example, we will consider the same labeled graph given above, and we will choose B = {{v}, {w}, {v, w}, Φ}, which also gives us a normal labeled space. *A*(E, L, B) is generated by the linear span of $\{P_v, P_w, P_{\{vw\}}, S_aP_v, S_aP_w, S_aP_{\{vw\}}, P_vS_a^*, P_wS_a^*, P_{\{vw\}}S_a^*, S_aP_vS_a^*, S_aP_wS_a^*, S_aP_{\{vw\}}S_a^*\}$. However, $P_{\{vw\}} = P_v + P_w$, $S_aP_{\{vw\}} = S_aP_v+S_aP_w$, similarly for the other terms. Therefore *A*(E, L, B) is the linear span of $\{P_v, P_w, S_aP_v, S_aP_w, P_vS_a^*, P_wS_a^*, S_aP_vS_a^*, S_aP_wS_a^*\}$.

Further, after careful observation (and some computations) we see that the set $S_1 = \{P_v, S_aP_v, P_vS_a^*, S_aP_vS_a^*\}$ is closed under multiplication, so is $S_2 = \{P_w, S_aP_w, P_wS_a^*, S_aP_wS_a^*\}$. Moreover the product of any element from S_1 and any element from S_2 is zero, for instance $(S_aP_vS_a^*) (S_aP_w)= S_aP_vP_{\{vw\}}P_w = 0$, since $P_vP_{\{vw\}}P_w = (P_vP_{\{vw\}})P_w = P_vP_w = 0$.

Thus the building units of the algebra can be expressed as products and adjoints of the partial isometry S_aP_v and the partial isometry S_aP_w and the two subalgebras are “orthogonal". For this reason, we may define a map Φ: *A*(E, L, B) → $M_4(\mathbb{C})$ we need only to define

$$\phi(SaPv) = \begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix},\ \text{and}\ \phi(SaPw) = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}.$$

As before, we will want Φ to preserve multiplication, addition, and adjoint.

This gives us that *A*(E, L, B) = span(S_1 **U** S_2)

$$\cong span\left(\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}\bigcup\right.$$
$$\left.\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\right).$$

Therefore *A*(E, L, B) is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$

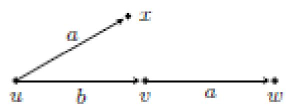
$$= \begin{bmatrix}a & b & 0 & 0\\c & d & 0 & 0\\0 & 0 & w & x\\0 & 0 & y & z\end{bmatrix} : a, b, c, d, w, x, y, z$$

For the remaining examples we will choose B to be E_0.

Example 4.3

Above is the next labeled graph we will consider. Note that E_0 = {{v}, {w}, {v,w}, Φ}. Choosing B as E_0, our representation of (E, L, B) gives projections {P_w,$S_aP_wS_a$*,$S_{aa}P_w$,S_{aa}*} and partial isometries {S_aP_w,$S_{aa}P_w$,$S_{aa}P_wS_a$*,P_wS_a*,P_wS_{aa}*,$S_{aa}P_wS_a$* }. It can be shown, as above, that the union of these sets is closed under multiplication, as such, the elements provide us with the matrix units we need. As we have three projections and six partial isometries, it can be shown above that *A*(E, L, B) is isomorphic to $M_3(\mathbb{C})$.

Example 4.4



Above is the next labeled graph we will consider. This gives us B = {{x}, {w}, {v}, {x,w}, {v,x,w}, Φ}. Then our representation of (E, L, B) has projections {P_x, $S_aP_xS_a$*, P_w, $S_aP_wS_a$*, $S_{ba}P_wS_{ba}$*}. It can be shown that ∀P ϵ {P_x, $S_aP_xS_a$*} and ∀Q ϵ {P_w,$S_aP_wS_a$*,$S_{ba}P_wS_{ba}$*}, PQ = QP = 0. Similarly, our representation of (E, L, B) gives two sets of partial isometries, {S_aP_x,P_xS_a*} and {S_aP_w, $S_{ba}P_w$, $S_{ba}P_wS_a$*, P_wS_a*, P_w, S_{ba}*,$S_aP_wS_{ba}$*}, in which the partial isometries are mutually orthogonal between sets. This goes to show that *A*(E, L, B) is isomorphic to $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

In the example above, if we drop the non-degenerate property for B, then we will not have {x} as an element in B. In this case, it is not difficult to see that the projection P_{wx}-P_w functions the same as the projection as P_x would. We deem the non-degenerate property ineffective. Accordingly, we choose a modified version of E_0, giving B = {{x,w},{w},{v}, {v, x, w}, Φ}. Then our representation of (E, L, B) has projections

{P_{xw}-P_w,$S_a(P_{xw}$-$P_w)S_a$*,P_w,$S_aP_wS_a$*,$S_{ba}P_wS_{ba}$*}. It can be shown that ∀P ϵ {P_{xw}-P_w,$S_a(P_{xw}$-$P_w)S_a$*} and ∀Q ϵ {P_w,$S_aP_wS_a$*,$S_{ba}P_wS_{ba}$*}, PQ = QP = 0. Similar to the case in the example, our representation of (E, L, B) gives two sets of partial isometries, {$S_a(P_{xw}$-$P_w)$,$(P_{xw}$-$P_w)S_a$*} and {S_aP_w, $S_{ba}P_w$, $S_{ba}P_wS_a$*,P_wS_a*,P_w,S_{ba}*,$S_aP_wS_{ba}$*}, in which the partial isometries are mutually orthogonal between sets. This gives us *A*(E, L, B) as being isomorphic to $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Remark 4.5

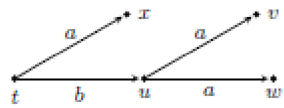
When carefully observing the examples above, and many more experiments, we noticed a method of determining the *-algebra representation of a labeled graph. For a label space (E, L, B), where B = E_0 as described at the beginning of this section:

- First, let V = {E^0_{sink}} **∩** r(μ) : μ ϵ L*(E)}, since we have a finite set, we can call it V = {V_1, V_2, ... V_k}.
- Next disjointize the members of set V, call the resulting set C = {C_1, C_2, ... , C_n}. The disjointization process is briefly described below.
- Then *A*(E, L, B) $\cong \oplus_{C_i \in C} M_{n_i+1}(\mathbb{C})$ where n_i = the number of words μ with $C_i \subseteq$ r(μ).

There are a few ways to distontize a group of sets. We will describe a hand on and visual process. To disjointize a finite group of sets such as V = {V_1, V_2, ... V_k}. First take the intersection of all, if it is non-empty, then that is the first set. $\cap_{i=1}^{n} V_i$. Next take the intersection of all but one, and then subtract the first set (set difference); do this for each set V_1, V_2, ... , V_k. Then the intersection of all but two, subtracts the previously found sets. Continue this process by taking intersections of less and a smaller number of sets and subtracting more and more.

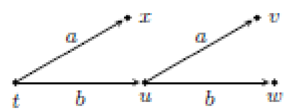
We now consider the following labeled graphs for our final examples. We have determined the *-algebra representation by traditional means as well to confirm our findings but here will be using the proposed methodology.

Example 4.6



We note that, in following the terminology above, V = {{x, v, w}, {v, w},Φ }. Then we disjointize V to get C = {{x},{v, w}}. Then, as one word, a, has {x} as a subset of its range, and two words, a, ba, have {v,w} as a subset of their ranges, it can be shown that A(E, L, B) ≅ $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Example 4.7



In this labeled graph, we note that V = {{x, v}, {w}, {v}}. Then we disjointize V to obtain C = {{x}, {v}, {w}}. Then, as one word, a, has {x} as a subset of its range, two words, a,ba, have {v} as a subset of their ranges, and two words, *b*,*bb*, have {w} as a subset of their ranges, it can be shown that A(E, L, B) ≅ $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$.

We should emphasize the fact that the algebra that is created by a labeled space is dependent on three things: the graph, the labeling and the choice of set B. As such, the method described in Remark 4.5 only applies to the set B = E_0, where E_0 is the smallest subset of P(E) that contains E^0_{sink} and makes (E, L, B) a normal labeled space. A similar, but more involving, process may be developed for any labeled space (E, L, B) so long as B contains E^0_{sink}.

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