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| RESEARCH ARTICLE

δ-Small Submodule and Prime Modules

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| ABSTRACT
In this paper, we introduced and studied δ-small submodule over prime module. Two concepts are very important namely strongly prime submodule and completely prime submodule. Multiple results led to obtaining a δ-small submodule of a singular, divisible and Bezout module with R is local. Important terms that appeared in this article, together with some terms, produced the submodule that we were interested in.

| KEYWORDS

Completely prime submodule, fractional submodule, δ-small submodule and strongly prime submodule

| ARTICLE INFORMATION

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1. Introduction
All rings in this paper are commutative with 1 and all modules with unitary. An R-module $\mathcal{M}$ is called multiplication if every submodule A of $\mathcal{M}$, there exists an ideal J such that $A = J\mathcal{M}$ [Singh, 2001]. The prime ideal was extended to module by several researchers. Any proper submodule A of $\mathcal{M}$ is called prime submodule of $\mathcal{M}$ if for each ideals J of R and $A_1 \leq \mathcal{M}$ such that $JA_1 \subseteq A$, so $A_1 \subseteq A$ or $J\mathcal{M} \subseteq A$. A definition of prime module [Ssevviiri, 2011]. Any submodule A of $\mathcal{M}$ is called a completely prime submodule if for every $r \in R$, $m \in \mathcal{M}$ such that $rm \in A$, so $m \in A$ module in [Ssevviiri, 2013]. A module $\mathcal{M}$ is called pseudo valuation module if $A \leq \mathcal{M}$ is a strongly prime submodule of $\mathcal{M}$. Note that the strongly prime submodule (s-pr-submodule) in [Moghaderi, 2011]. Fractional ideal and fractional submodule with more details in [Abed, 2019]. Any module $\mathcal{M}$ is called singular if $Z(\mathcal{M}) = \mathcal{M}$ and non-singular $Z(\mathcal{M}) = 0$ where $Z(\mathcal{M}) = \{x \in \mathcal{M}: ann_T(x) \ \leq_{ess} T\}$ [Kasch, 1982]. Any submodule A of $\mathcal{M}$ is called small ($A \ll \mathcal{M}$) if there exists another submodule B in $\mathcal{M}$ such that $A + B \neq \mathcal{M}$ [Leonard, 1966]. Also, A is called δ-small if there exists a non-zero submodule B of $\mathcal{M}$ such that $A + B \neq \mathcal{M}$ with $\mathcal{M}/B$ is a singular module ($A \ll_\delta \mathcal{M}$) [Wang, 2007]. Torsion module in and simple module in [Kasch, 1982]. A module $\mathcal{M}$ is called indecomposable if $\mathcal{M} = \{0\} + \mathcal{M}$ [Janusz, 1968]. The module $\mathcal{M}$ is called uniform if every submodule A of $\mathcal{M}$ is essential in $\mathcal{M}$[Dauns, 1980].

2- S-pr-submodule:
Definition 2.1: [Ssevviiri, 2011] Any submodule A of an R-$module$ $\mathcal{M}$ is called prime if:

i) $A \neq \mathcal{M}$.
ii) $r \in R$, $m \in \mathcal{M}$, $rm \in A \Rightarrow m \in A$ or $r \in (A{:}\mathcal{M})$ such that $(A{:}\mathcal{M}) = \{rm \subseteq A; r \in R\}$.

Remark 2.2: If A is a prime submodule of $\mathcal{M}$, so $(A{:}\mathcal{M})$ is a prime ideal of R.

Definition 2.3: [Kasch, 1982] Let $\mathcal{M}$ be an R-module over integral domain R with quotient field. Then $\mathcal{M}$ is said to be torision free module if $T(\mathcal{M}) = 0$ where $T(\mathcal{M})$ refere to any torsion elements in $\mathcal{M}$.

Remark 2.4: From [Al-Bahrani, 2017]; $\mathcal{M}_T = \{\frac{x}{t}: \ x \in \mathcal{M}, t \in T\}$ where $T = R - \{0\}$. Therefore, suppose R has no zero divisors with a quotient field F and $0 = T(\mathcal{M})$;

$$\forall\, A \leq \mathcal{M} \wedge y = \frac{r}{k} \in F, b = \frac{x}{t} \in \mathcal{M}_T$$

Implies $yb \in A$ if $\exists a \in A \ni rx = sta$.

Now all the tools became available to present new definition named strongly prime submodule (s-pr-submodule).

Definition 2.5: [Moghaderi, 2011] For all non-zero elements $a, b \in R$ such that $ab \neq 0$ with qutiont field F and $T(\mathcal{M}) = 0$, we say A is a s-pr-submodule of $\mathcal{M}$, if $y \in F$, $x \in \mathcal{M}, yx \in A$ implies that $x \in A$ or $y \in (A:\mathcal{M})$.

From [Abed, 2019], fractional ideal (FI) means an ideal I of a finitely generated Z-module Z with Ii is the principal ideal over the ring Ri; where i represent every maximal ideal in Z.

And from [Al-Bahrani, 2017]; any submodule A of $\mathcal{M}$ over the integral domain R is called a fractional submodule if rA is a subset of $\mathcal{M}$. On the other hand; if A is a s-pr-submodule, so $\mathcal{M}$ is a prime module. In the next proposition; we utilize two concepts are fractional ideal and fractional submodule A in order to get A is a δ-small in $\mathcal{M}$. But before that, we need to state the following lemma:

Lemma 2.6: Let $\mathcal{M}$ be an R-module and let A be a proper submodule of $\mathcal{M}$. If J is a fractional ideal of R and B is a fractional submodule (FS) of $\mathcal{M}$ such that $JB \subseteq A$; implies $B \subseteq A$ or $J \subseteq (A:\mathcal{M})$, then A is a s-pr-submodule [Al-Bahrani, 2017] and hence $\mathcal{M}$ is a prime module.

Proposition 2.7: Let $\mathcal{M}$ be an $R - module$. If the following statements are hold:

i) $Z(\mathcal{M}) = \mathcal{M}$.
ii) All conditions in lemma 2.6 are hold;

Then $A \ll_\delta \mathcal{M}$.

Proof: Assume that condition (ii) is hold. We need to explain how A is s-pr-submodule. Suppose that J is a fractional ideal of R and B is a fractional submodule of $\mathcal{M}$. It is clear A is a prime submodule of $\mathcal{M}$. For $b \in F$ and $a \in \mathcal{M}_T$, $ba \in A$. If we put $Rb = J$, so a fractional ideal of R and $Ra = B$ is a fractional submodule (FS) of $\mathcal{M}$. Then JB subset of A and hence $Ra = B$ subset of A or $Rb = J$ subset of $(A:\mathcal{M})$. Hence $a \in A$ or $b \in (A:\mathcal{M})$. So A is a s-pr-submodule of $\mathcal{M}$ (by definition of strongly prime submodule). Therefore $\mathcal{M}$ is a prime module. But $Z(\mathcal{M}) = \mathcal{M}$. Thus A is a δ-small of $\mathcal{M}$[Al-Bahrani, 2017].

Remark 2.8: The converse of lemma 2.6 is true in general because if A is a s-pr-submodule of $\mathcal{M}$, $a \in B - A$ and $b \in J$, then $ba \in JB$. Also, since JB subset of A, a is an element of $\mathcal{M}_T - A$ and $b \in F$, then $b \in (A:\mathcal{M})$. Hence J subset of $(A:\mathcal{M})$.

Proposition 2.9: Let $\mathcal{M}$ be a singular R-module and let A be a submodule of $\mathcal{M}$. If A is a comparable to each (FS) of $\mathcal{M}$, then $A \ll_\delta \mathcal{M}$.

Proof: Suppose that an element b in F; $b = \frac{r}{z_1}$ and $a \in \mathcal{M}_T$; $a = \frac{K}{Z_2}$. So

$$ba \in A, a \notin A \wedge b \notin (A:\mathcal{M})$$

We know $a \notin A$. But A comparable to each (FS) of $\mathcal{M}$. Hence

$$A \subseteq Ra \wedge ab \in Ra. \text{ So } \in R\ .$$

Also, $b \notin (A:\mathcal{M})$, Therefore:

$$A \subseteq b\mathcal{M} \text{ and } ab \in b\mathcal{M}. \text{ So } a \in \mathcal{M}$$

$\forall b \in R$ and $a \in \mathcal{M}$, then $ba \in A \ni A$ satisfies all conditions of prime submodule. Then

$$a \in A \text{ or } b \in (A:\mathcal{M}).$$

But this contradiction. Hence A is a s-pr-submodule, so $\mathcal{M}$ is a prime module with singular property implies by proposition 2.7; $A \ll_\delta \mathcal{M}$.

Remark 2.10: A s-pr-submodule not imply A is comparable to each (FS) of $\mathcal{M}$.

Note that, the best example to satisfies Remark 2.10 is the following:

Example 2.11: Suppose that $R = \mathbb{R}$ where $\mathbb{R}$ is Euclidian space So $\mathcal{M} = \mathbb{R}\oplus\mathbb{R}$ and $A = \mathbb{R}\oplus(0)$ is s-pr-submodule, but when $a = (0,1)$, then $Ra \nsubseteq A$ and $A \nsubseteq Ra$.

Corollary 2.12: If for every b is an element of F, $b^{-1}A$ subset of A or $b \in (A{:}\,\mathcal{M})$ where A is a prime submodule of singular module $\mathcal{M}$, then A is a δ-small of $\mathcal{M}$.

Proof: Assume that $b \in F$ and $a \in \mathcal{M}_T$. So $ba \in A$. When $b^{-1}A$ subset of A, this means

$$a = b^{-1}(ba) \in A$$

Otherwise $b \in (A{:}\,\mathcal{M})$. Hence A is a s-pr-submodule of $\mathcal{M}$. Then $\mathcal{M}$ is a prime module with singularty, So $A \ll_\delta \mathcal{M}$.

Note that there is a relationship between δ-small submodule and another concept namely pseudo valuation module. Therefore, we need to study and present pseudo valuation module with some examples.

Definition 2.13: [Moghaderi, 2011] Let $\mathcal{M}$ be an R-module and let A be a prime submodule of $\mathcal{M}$. We say $\mathcal{M}$ is a pseudo valuation module if A is a s-pr-submodule.

Remarks and Examples 2.14:

1- A module of rational numbers Q over the ring Z is pseudo valuation and hence any prime submodule A of $Q = \mathcal{M}$ is s-pr-submodule. So $\mathcal{M}$ is a prime module.

2- The module Z as a Z-module is not pseudo valuation module.

Proposition 2.15: Let $\mathcal{M}$ be a singular divisible R-module. Then any submodule A of $\mathcal{M}$ is δ-small.

Proof: Assume that A is a prime submodule in $\mathcal{M}$ and let $b = \frac{c}{d} \in K$ where K is a field with $a \in A$. Put $b = 0$ $(\frac{c}{d} = 0)$. So b belongs to the ideal $(A{:}\,\mathcal{M})$. Suppose that $b \neq 0$ $\left(\frac{c}{d} \neq 0\right)$. So $x\mathcal{M} = \mathcal{M}$. Hence

$$\exists h \in \mathcal{M} \ni a = ch.$$

But $a \in A$ with A is prime submodule, then $h \in A$ or $c \in (A{:}\,\mathcal{M})$. If $c \in (A{:}\,\mathcal{M})$ this implies that $\mathcal{M} = c\mathcal{M} \subseteq A$, but this contradiction. Therefore $h \in A$ with

$$b^{-1}a = \frac{d}{c}a$$
$$= \frac{d}{c}ah$$
$$= dh \in A$$

Then $b^{-1}A \subseteq A$ and A is s-pr-submodule and hence $\mathcal{M}$ is a prime module. But $Z(\mathcal{M}) = \mathcal{M}$, thus $A \ll_\delta \mathcal{M}$.

Corollary 2.16: Every injective R-module $\mathcal{M}$ with $Z(\mathcal{M}) = \mathcal{M}$ is a divisible module and hence any submodule A of $\mathcal{M}$ is δ-small.

Proof: By proposition 2.15.

Corollary 2.17: Every singular uniform module $\mathcal{M}$ over serial Noetherian ring R with J is prime ideal of R has $A \ll_\delta \mathcal{M}$.

Proof: Assume $\mathcal{M}$ is a uniform module. Suppose that $x, y \in \mathcal{M}$ with $0 \to \ker(F) \to R\oplus R \to aR + bR \to 0$ is exact sequence. We have R is Noetherian module ($aR + bR$ is uniserial), because every uniform module is indecomposable. Hence

$$aR \subseteq bR \text{ or } bR \subseteq aR$$

Then $\mathcal{M}$ is uniserial R-module. But $\mathcal{M} = Z(\mathcal{M})$, so $\mathcal{M}$ is pseudo valuation module ($A \leq \mathcal{M}$ is s-pr-submodule). Then $\mathcal{M}$ is prime module. Now $\mathcal{M}$ is prime module with $Z(\mathcal{M}) = \mathcal{M}$, implies that $A \ll_\delta \mathcal{M}$.

Corollary 2.18: Every submodule A of Bezout module $\mathcal{M}$ over local ring R is δ-small in $\mathcal{M}$.

Proof: Assume that $x, y \in \mathcal{M}$. We must prove that $x \in y\mathcal{M}$ or $y \in x\mathcal{M}$. There exists $a, b, c, d \in Rx(1 - ac) = ybc$ and $y(1 - bd)$. But R has a unique maximal ideal (local ring). Then $1 - ac \in U(R)$ or $ac \in U(R)$ where $U(R)$ is a unit element. If $x = ybc(1 - ac)^{-1}yR$ and $\frac{a,cR}{J(R)}$ with R local ring, then $a, c \in U(R)$. Put $d \in U(R)$. So $ad \in U(R)$ and $x = y(1 - bd)(ad)^{-1} \in yR, d \in \frac{R}{U(R)} = J(R), 1 - bd \in U(R)$ and $y = y, d(1 - bd)^{-1} \in yR$. Hence $\mathcal{M}$ is serial module. So $\mathcal{M}$ is pseudo valuation module ($A \leq \mathcal{M}$is s-pr-submodule). Then $\mathcal{M}$ is prime module with $Z(\mathcal{M}) = \mathcal{M}$, implies that $A \ll_\delta \mathcal{M}$.

Corollary 2.19: Every distributed artinain module $\mathcal{M}$ over the ring R with $Z(\mathcal{M}) = \mathcal{M}$ has submodule A is δ-small of $\mathcal{M}$.

Proof: Clear ; Every distributive artinian module is Bezout module with $Z(\mathcal{M}) = \mathcal{M}$ implies that $\mathcal{M}$ has A submodule is δ-small.

3- C-prime module :

In this section, we discuss completely prime modules and their interrelations with small property of some submodules.

Definition 3.1: [Ssevviiri, 2011] An R-module $\mathcal{M}$ is called prime if $rRm = 0$, so $rm = 0$ or $a = 0$ $\forall r \in R, m \in \mathcal{M}$.

Definition 3.2: [Ssevviiri, 2013] An R-module $\mathcal{M}$ is called completely prime module (c-pr-module) if $rm = 0$, so $r \in ann_R(\mathcal{M})$ or $m = 0$ $\forall r \in R, m \in \mathcal{M}$.

Remark 3.3: If the ring R is commutative, so the two definitions 3.1 and 3.2 are same.

Proposition 3.4: Let $\mathcal{M}$ be a c-pr-module. Then $\mathcal{M}$ is a pr-module.

Proof: Assume that $\mathcal{M}$ is a c-pr-module. Suppose that r belongs to R and m belongs to $\mathcal{M}$ such that $rRm = 0$. So $rm = 0$. Hence $rm = 0$. Thus $\mathcal{M}$ is a pr-module. Now it is very important to study the strong and clear relationship between c-pr-module and c-pr-submodule. So that we can use new concepts such as torision-free-module in order to obtain prime module and thus δ-small of any submodule of $\mathcal{M}$. Therefore, we need to present c-pr-module by another method. See the following definition.

Definition 3.5: Any R-module $\mathcal{M}$ is called c-pr-module if $0 \neq A \leq \mathcal{M}$ is a c-pr-submodule where c-pr-submodule means:

Any submodule A of $\mathcal{M}$ with $Rm \subseteq A$ is c-pr-submodule if :

$\forall r \in R, m \in \mathcal{M} \ni rm \in A$, then $m \in A$ or $r\mathcal{M} \subseteq A$.

Remark 3.6: If $A \leq \mathcal{M}$ is c-pr-submodule, so $\frac{\mathcal{M}}{A}$ is c-pr-module.

Lemma 3.7: Let $\mathcal{M}$ be an R-module. If $\mathcal{M}$ is torsion free module, then it is a c-pr-module.

Proof: Assume that $rm = 0$ $\forall r \in R, m \in \mathcal{M}$. If $m = 0$; there are nothing. Suppose $m \neq 0$. So from definition of torsion-free-module, $r = 0$ and $rm = 0$. Thus $\mathcal{M}$ is c-pr-module ($\mathcal{M}$ is pr-module).

Lemma 3.8: Let $\mathcal{M}$ be an R-module. If $\mathcal{M}$ is simple module with reduced property, then it is c-pr-module.

Proof: Assume that $rm = 0$. If $m = 0$ there are nothing. If $m \neq 0$, so $0 = rm \cap < m >= r\mathcal{M} \cap \mathcal{M} = r\mathcal{M}$. Thus $\mathcal{M}$ is c-pr-module.

Remark 3.9: Let R be a ring with 1 and let I be an ideal of R. If I is c-pr-ideal of R, then I is c-pr-submodule [Ssevviiri, 2013].

Proposition 3.10: Let $\mathcal{M}$ be a singular R-module and let A be submodule of $\mathcal{M}$. If the ideal $(A{:}\,\mathcal{M}) = (A{:}\,m), m \in \mathcal{M} - A$, then A is a c-pr-submodule of M and hence $\mathcal{M}$ is c-pr-module ($A \ll_\delta \mathcal{M}$).

Proof: Assume that $rm \in A, r \in R, m \in \mathcal{M}$. So $r \in (A{:}\,m)$. If $m \in A$ there are nothing. Let $m \notin A$. So $r \in (A{:}\,\mathcal{M}) \ni r\mathcal{M} \subseteq A$. Hence A is c-pr-submodule. Thus $\mathcal{M}$ is c-pr-module and then it is prime module. Therefore $A \ll_\delta \mathcal{M}$.

Corollary 3.11: Let $\mathcal{M}$ be a singular $R - module$ and let $A \leq \mathcal{M}$. If for all $r \in R$, $m \in \mathcal{M}$ with $(rm) \subseteq A$, then $(\mathcal{M}) \subseteq A$ or $< r\mathcal{M} > \subseteq A$ and so A is a δ-small in $\mathcal{M}$.

Proof: Assume that $r \in (A{:}\,\mathcal{M})$. Suppose that $m \in \mathcal{M} - A$. So $r\mathcal{M} \subseteq A$. Hence $rm \in A$ with $r \in (A{:}\,\mathcal{M})$. If $r \in (A{:}\,\mathcal{M})$ and $m \in \mathcal{M} - A$, then $rm \in A$ and $< rm > \subseteq A$. But $r\mathcal{M} \subseteq < r\mathcal{M} > \subseteq A$; because $< \mathcal{M} > \nsubseteq A$, $m \notin A$. Then $r \in (A{:}\,\mathcal{M})$. Hence, by proposition 3.10; A is a c-pr-submodule and then $\mathcal{M}$ is c-pr-module. Therefore $\mathcal{M}$ is a prime module with $Z(\mathcal{M}) = \mathcal{M}$ implies that $A \ll_\delta \mathcal{M}$.

Remark 3.12: By the same method of proof proposition 3.10, we can say; if $A_1 = (A{:}\,\mathcal{M})$ is a c-pr-ideal of R with $(A{:}\,\mathcal{M}) = (\bar{0}{:}\,\bar{m}) = A_1$ $\forall m \in \mathcal{M} - A$, so A is c-pr-submodule and hence $\mathcal{M}$ is c-pr-module ($\mathcal{M}$ is pr-module). Therefore, in the next result, we present the following:

Corollary 3.13: Let $\mathcal{M}$ be a singular module. If the set $\{(A{:}\,\mathcal{M}){:}\, m \in \mathcal{M} - A\}$ is a singular, the $A \ll_\delta \mathcal{M}, \forall A \leq \mathcal{M}$.

Proof: For all $m \in \mathcal{M} - A$; $A_1 = (A{:}\,\mathcal{M})$

$= \cap \{(A{:}\,\mathcal{M}){:}\, m \in \mathcal{M} - A\}$

$= (A{:}\,\mathcal{M})$ (by assumption)

But $(A:\mathcal{M}) = \{r \in R: rm \in A\}$

$= \{r \in R: r\ \bar{m} = \bar{0}\}$

$= (\bar{0}: \bar{m})$

Where $\bar{m} = m + A$. Suppose that $a_1a_2 \in (A:m), a_1, a_2 \in R, m \in \mathcal{M} - A$. So $a_1a_2 \in A$. If $a_2 \in (A:m)$, there is nothing. Assume that $a_2 \notin (A:m), a_2m \notin A$, so $a_1 \in (A:a_2m) = (A:m)$ and then $A_1 = (A:m) \forall m \in \mathcal{M} - A$ is c-pr-ideal. Thus from Remark 3.12; $\mathcal{M}$ is c-pr-submodule ($\mathcal{M}$ is c-pr-module) and hence is a prime module. But $\mathcal{M} = Z(\mathcal{M})$. Then $A \ll_\delta \mathcal{M}$.

Remark 3.14: Let $(A:F) = (A:\mathcal{M})$ where $F \subseteq \mathcal{M} - A$. If we take $F = \{m\}, m \in \mathcal{M} - A$ with $Z(\mathcal{M}) = \mathcal{M}$ then by pro.3.10; A is δ-small of $\mathcal{M}$.

Proposition 3.15: Let R be a ring and $\mathcal{M}$ be singular R-module such that $A_1 \neq R$. If A_1 is c-pr-ideal of R, then A is δ-small submodule of $\mathcal{M}$.

Proof: Suppose that A is a c-pr-ideal and suppose $\mathcal{M} = \frac{R}{A}$. Assume that $a \in A_1, b \in R$. Then $a(b + A_1) = ab + A_1 = A_1$; Hence

$$A \subseteq (0:\mathcal{M})_R.$$

If $c \in (0:\mathcal{M})_R$, then $c(r + A_1) = A_1, r \in R$. So $cR \subseteq A_1$. Since $A_1$is c-pr-ideal, then $c \in A_1$.Hence $(0:\mathcal{M})_R = A_1$.then $\mathcal{M}$ is c-pr-module if $c \in R$ and $m \in \mathcal{M} = \frac{R}{A_1}$such that $cm = \bar{0}$ then $m = m_1 + A_1$and $cm_1 \in A_1$. Since c-pr-submodule, $c \in A_1$or $m_1 \in A_1$and hence $cm = \bar{0}$ or $m = \bar{0}$.($\mathcal{M}$ is c-pr-module) with $Z(\mathcal{M}) = \mathcal{M}$, so $A \ll_\delta \mathcal{M}$.

Definition 3.16: [Singh, 2001] An ideal J of the ring R is called insertion of factor property (IFP) is $ab \in J, a, b \in R$, so $aRb \subseteq J$. Therefore a submodule A of $\mathcal{M}$ is called (IFP) if $am \in A, a \in R, m \in \mathcal{M}$, so $aRm \subseteq A$ and a module $\mathcal{M}$ has (IFP) if the zero submodule has (IFP).

Proposition 3.17: Let $\mathcal{M}$ be an $R - module$ and let A be a submodule of $\mathcal{M}$. If:

i) $Z(\mathcal{M}) = \mathcal{M}$.
ii) A has IFP and pr-submodule,

Then $A \ll_\delta \mathcal{M}$.

Proof: Suppose that $A \leq \mathcal{M}$ is c-pr-submodule. So A is prime and has IFP. Now we have $A \leq \mathcal{M}$ is a prime with IFP. Suppose that $rm \in A$. But A has IFP, $r < m > \subseteq A$. Since A is prime submodule, then $m \in A$ or $rm \subseteq A$. Hence A is c-$pr - submodule$. Then $\mathcal{M}$ is c-pr-module ($\mathcal{M}$ is pr-module). But $Z(\mathcal{M}) = \mathcal{M}$. Thus $A \leq \mathcal{M}$.

Lemma 3.18: Every maximal submodule with completely semi prime (c-$semi$-prime) is completely prime submodule (c-pr-submodule) [Dauns, 1980].

Definition 3.19: [Bland, 2011] Let $\mathcal{M}$ be an R-module and let $\emptyset \neq \mathcal{M}_1 \subseteq \mathcal{M} - \{0\}$ is called multiplicative system of $\mathcal{M}$ is $\forall r \in R, m \in \mathcal{M}, H \leq \mathcal{M} \ni (H + < m >) \cap \mathcal{M}_1 \neq \emptyset$ and $(H + < rm >) \cap \mathcal{M}_1 \neq \emptyset$, so $(H + < rm >) \cap \mathcal{M}_1 \neq \emptyset$.

Proposition 3.20: Let $\mathcal{M}$ be a singular R-module. If A is a submodule of $\mathcal{M}$ such that $\mathcal{M} - A$ is a multiplication system of $\mathcal{M}$, then $A \ll_\delta \mathcal{M}$.

Proof: Suppose that $r \in R, m \in \mathcal{M} \ni < rm > \subseteq A$. But $< m > \nsubseteq A$ with $< rm > \nsubseteq A$. Hence $< m > \cap K = \mathcal{M} - A \neq \emptyset$. Also, $< rm > \cap K \neq \emptyset$. But $\mathcal{M} - A$ is multiplication system, then $< rm > \cap K \neq \emptyset \ni < rm > \nsubseteq A$, contradiction. So A is c-pr-submodule. Then $\mathcal{M}$ is c-pr-module ($\mathcal{M}$ is prime module) with $Z(\mathcal{M}) = \mathcal{M}$ implies $A \ll_\delta \mathcal{M}$.

Corollary 3.21: Let $\mathcal{M}$ be a singular R-module and let $K \subseteq \mathcal{M}$ is a multiplicative system of $\mathcal{M}$ such that $A \leq \mathcal{M}$ is a maximal with respect $A \cap K = \emptyset$. Then $A \ll_\delta \mathcal{M}$.

Proof: Assume that $r \in R, m \in \mathcal{M} \ni < rm > \subseteq A$. If $< m > \nsubseteq A$ and $< rm > \nsubseteq A$, so

$$(< m > + A) \cap K \neq \emptyset \text{ and } (< rm > + A) \cap K \neq \emptyset.$$

But K is a multiplicative system of $\mathcal{M}$. So $(< rm > + A) \cap K \neq \emptyset$. If $< rm > \subseteq A$ imply $A \cap K \neq \emptyset$, a contradiction. Therefore A is c-pr-submodule of $\mathcal{M}$. Hence $\mathcal{M}$ is c-pr-module. Thus $\mathcal{M}$ is a prime module. But $\mathcal{M}$ is singular module. Then $A \ll_\delta \mathcal{M}$.

Corollary 3.22: Let $\mathcal{M}$ be an R-module. If:

i) $Z(\mathcal{M}) = \mathcal{M}$.
ii) $A \leq \mathcal{M} \ni A$ is $c\text{-}pr$-ideal of R.
iii) $A \neq R$.

Then $A \ll_\delta \mathcal{M}$.

Proof: Suppose that A is $c\text{-}pr$-idael and let $\mathcal{M} = \frac{R}{A}$. Clear that $\mathcal{M}$ is an R-module. If $a \in A, r \in R$, so $a(r + A) = ar + A = A$. Then $A \subseteq (0{:}\mathcal{M})_R$. Since $a_1 \in (0{:}\mathcal{M})_R$, then $a_1(r_1 + A) = A, r_1 \in R$. Hence $a, R \subseteq A$. But A is $c\text{-}pr$-module and hence is prime module. But $Z(\mathcal{M}) = \mathcal{M}$. So $A \ll_\delta \mathcal{M}$.

Proposition 3.23: Let $\mathcal{M}$ be a singular module. If $\mathcal{M}$ is contained in every non-zero invariant submodule of $\mathcal{M}_1$where $\mathcal{M}_1$ is injective hull of $\mathcal{M}$. then any submodule A of $\mathcal{M}$ is δ-small.

Proof: Assume that $0 \neq A \leq \mathcal{M}$. Clear that $ann_R(\mathcal{M}) \subseteq ann_R(A)$. To prove that $ann_R(A) \subseteq ann_R(\mathcal{M})$. Suppose that there exists $m \in \mathcal{M}, rm \neq 0$. If $0 \neq A, \exists 0 \neq b \in A$. $\pi(Rb, \mathcal{M}_1) = \sum \emptyset(Rb), \emptyset \in Hom(Rb, \mathcal{M}_1)$. Since $Rb \subseteq \mathcal{M} \subseteq \mathcal{M}_1$, then $\pi(Rb, \mathcal{M}_1)$ is a non-zero submodule of $\mathcal{M}_1$. So $\pi(Rb, \mathcal{M}_1)$ is an invariant non-zero submodule of $\mathcal{M}_1$. Thus by assumption $\mathcal{M} \subseteq \pi(Rb, \mathcal{M}_1)$. Hence

$$\exists r_1, r_2, \dots, r_k \in R \ni Q_1, Q_2, \dots, Q_k \in Hom(Rb, \mathcal{M}_1)$$

$$\ni rm = \sum Q_i(r_i b).\text{Thus}$$

$$\begin{aligned} rm &= \sum r Q_i(r_i b) \\ &= \sum Q_i(r r_i\, b) \\ &= 0 \end{aligned}$$

This contradiction. Then $ann_R(A) \subseteq ann_R(\mathcal{M})$. Therefore $\mathcal{M}$ is a prime module. But $Z(\mathcal{M}) = \mathcal{M}$. Thus $A \ll_\delta \mathcal{M}$.

4. Conclusion

In this manuscript, we study δ-small submodules based on prime modules. We showed the relationship between some submodules, such as singular submodule with completely prime modules, to get a small concept. It is observed that if $\mathcal{M}$ is a singular R-module have a completely prime ideal then it is a δ-small submodule of $\mathcal{M}$. Moreover, we introduce that every submodule A of Bezout module $\mathcal{M}$ over local ring R is δ-small in $\mathcal{M}$.

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