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| RESEARCH ARTICLE

On Absolute Valued Algebras Containing a Central Algebraic Element

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| ABSTRACT

Let A be an absolute valued algebra containing a nonzero central algebraic element. Then A is a pre-Hilbert algebra and is finite dimensional in the following cases:
1) A satisfies $(x, x, x) = 0$.
2) A satisfies $(x^2, x^2, x^2) = 0$.
3) A satisfies $(x, x^2, x) = 0$.
In these cases A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$. It may be conjectured that every absolute valued algebra containing a nonzero central element is pre-Hilbert algebra.

| KEYWORDS

Absolute valued algebra, pre-Hilbert algebra, algebraic element, central element.

| ARTICLE INFORMATION

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1. Introduction

Let A be a non-necessarily associative real algebra which is normed as real vector space. We say that real algebra is a pre-Hilbert algebra, if it's norm $\|.\|$ come from an inner product $(./.)$, and it's said to be absolute valued algebras, if it's norm satisfies the equality $\|ab\| = \|a\|\|b\|$, for all $a, b \in A$. Note that, the norm of any absolute valued algebras containing a nonzero central idempotent (or finite dimensional) comes from an inner product [El − Mallah, 1987] and [El − Mallah, 1980]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathrm{H}$ and O, and that every finite dimensional absolute valued algebra has dimension 1, 2, 4 or 8 [Albert, 1947]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by $\mathbb{R}, \mathbb{C}, \mathrm{H}$ and O [Urbanik, 1960]. It is easily seen that the one-dimensional absolute valued algebras are classified by $\mathbb{R}$, and it is well-known that the two-dimensional absolute valued algebras are classified by $\mathbb{C}, \mathbb{C}^*$, $^*\mathbb{C}$ or $\overset{*}{\mathbb{C}}$ (the real algebras obtained by endowing the space $\mathbb{C}$ with the product $x * y = \bar{x}y$, $x * y = x\bar{y}$, and $x * y = \bar{x}\,\bar{y}$ respectively) [Rodriguez, 1984]. As main results El-Mallah [El-Mallah, 1990] proved that any absolute valued algebra with a nonzero central idempotent and satisfying $(x, x, x) = 0$ (where $(x, y, z) = (xy)z - x(yz)$), is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$. The same conclusion is true for any absolute valued algebras with a nonzero central idempotent and $(x^2, x^2, x^2) = 0$ or $(x, x^2, x) = 0$ [4] and [6]. It is natural to study those absolute valued algebras by replacing the original assumption central idempotent with a weaker one central algebraic element, we prove that, if A is an absolute valued real algebra containing a central algebraic element a and satisfying one of the following identities $(x, x, x) = 0$, $(x^2, x^2, x^2) = 0$ or $(x, x^2, x) = 0$. Then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$, this result is an important generalization of a results given in [El-Mallah, 1990], [El-Mallah, 2004] and [Rochdi, 2009].

In section 2 we introduce the basic tools for the study of absolute valued algebras containing a central algebraic element. We also give some properties related to central algebraic elements satisfying some restrictions on commutativity (proposition 2.9, lemma 2.9 and corollaries 2.10, 2.11), and some conditions imply that such an algebra A is an inner product space. Moreover, section 3 is

devoted to classifying all absolute valued algebras with a central algebraic element and satisfying one of the following identities $(x,x,x) = 0$, $(x^2,x^2,x^2) = 0$ or $(x,x^2,x) = 0$. The paper ends with the following main results:
Theorem. Let A be an absolute valued algebra containing a nonzero central algebraic element. Then the following assertions are equivalent:
1) A satisfies $(x,x,x) = 0$,
2) A satisfies $(x^2,x^2,x^2) = 0$,
3) A satisfies $(x, x^2, x) = 0$,
4) A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

2. Notation and preliminaries results

In this paper all the algebras are considered over the real numbers field $\mathbb{R}$.

Definition 2.1 Let B be an arbitrary algebra.
i) B is called a division algebra if the operators $\mathrm{L_x}$ and $\mathrm{R_x}$ of left and right multiplication by x are bijective for all $\mathrm{x} \in \mathrm{B} \setminus \{0\}$.
ii) We say that B is algebraic, if for every x in B, the subalgebra $\mathrm{B}(x)$ of B generated by x is finite dimensional.
iii) We mean by a nonzero central element in B, a nonzero element which commute with all elements of the algebra B.
iv) B is called a normed algebra (respectively, absolute valued algebra) if it is endowed with a space norm: $\|\,.\,\|$ such that $\|xy\| \leq \|x\|\|y\|$ (respectively, $\|xy\| = \|x\|\|y\|$, for all $\mathrm{x}, \mathrm{y} \in \mathrm{B}$).
v) B is called a pre-Hilbert algebra if it is endowed with a space norm that comes from an inner product (./.) such that

$$(./.) : \mathrm{B} \times \mathrm{B} \longrightarrow \mathbb{R}$$
$$(\mathrm{x},\mathrm{y}) \mapsto \frac{1}{4}(\|\mathrm{x}+\mathrm{y}\|^2 - \|\mathrm{x}-\mathrm{y}\|^2)$$

The most natural examples of absolute valued algebras are $\mathbb{R}, \mathbb{C}, \mathrm{H}$ (the algebra of Hamilton quaternion) and O (the algebra of Cayley numbers) with norms equal to their usual absolute values [El-Mallah, 1987] and [Urbanik, 1960]. The algebra $\overset{*}{\mathbb{C}}$ (repectively, $\overset{*}{\mathrm{H}}$ and $\overset{*}{\mathrm{O}}$) obtained by replacing the product of $\mathbb{C}$ (respectively, H and O) with the one defined by $\mathrm{x} * \mathrm{y} = \bar{\mathrm{x}}\,\bar{\mathrm{y}}$, where $\mathrm{x} \rightarrow \bar{\mathrm{x}}$ is the standard conjugation of $\mathbb{C}$ (respectively, H and O).

We need the following relevant results:

Theorem 2.3 [El-Mallah, 1980] The norm of any absolute valued algebra containing a nonzero central idempotent comes from an inner product.

Theorem 2.4 [El-Mallah, 1980] Any absolute valued algebra containing a central idempotent and satisfying $(x,x,x) = 0$ for all $x \in A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Theorem 2.5 [4] Any absolute valued algebra containing a central idempotent and satisfying $(x^2,x^2,x^2) = 0$ for all $x \in A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Theorem 2.6 [Rochdi, 2009] Any absolute valued algebra containing a central idempotent and satisfying $(x,x^2,x) = 0$ for all $x \in A$ is isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Lemma 2.7 [Urbanik, 1960] If all the elements of a subset B of any absolute valued algebra A commute with each other, then the linear hull spanned by B is pre-Hilbert space.

We give some conditions that imply that A is an inner product space.

Proposition 2.8 Let A be an absolute valued algebra containing a central element a and let x be a element in A. If x is orthogonal to a in the inner product space $[a,x]$, then the following are equivalent:
1) $x^2a^2 = a^2x^2$,
2) $x^2 = -||x||^2a^2$,
3) A is an inner product space.
Proof. 1) ⇒ 2) Assuming that $||x|| = 1$, we have $||x^2 - a^2|| = ||x - a||||x + a|| = 2$. According to Lemma 2.7, we get $x^2 = -a^2$.
2) ⇒ 1) is clear
2) ⇒ 3) Let $u = \alpha a + \beta x$ and $v = \gamma a + \delta y$ be norm-one elements in A where $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ and $x, y \in \{a\}^{\perp}$ such that $\left||x|\right| = ||y|| = 1$. According to Schoenberg's Theorem [14], it is sufficient to show that the inequality $||u + v||^2 + ||u - v||^2 \geq 4$ holds.
Using Lemma 2.7 and since $x^2 = y^2 = -a^2$, then

$$||u + v||^2 + ||u - v||^2 = ||(\alpha + \gamma)a + (\beta x + \delta y)||^2 + ||(\alpha - \gamma)a + (\beta x - \delta y)||^2$$

$$= (\alpha + \gamma)^2 + ||\beta x + \delta y||^2 + (\alpha - \gamma)^2 + ||\beta x - \delta y||^2$$
$$= (\alpha + \gamma)^2 + (\alpha - \gamma)^2 + ||(\beta x + \delta y)^2|| + ||(\beta x - \delta y)^2||$$
$$\geq (\alpha + \gamma)^2 + (\alpha - \gamma)^2 + ||(\beta x + \delta y)^2 + (\beta x - \delta y)^2||$$
$$= 2\alpha^2 + 2\gamma^2 + || - 2(\beta^2 + \delta^2)a^2||$$
$$= 2\alpha^2 + 2\gamma^2 + 2\beta^2 + 2\delta^2 = 4$$

This implies that A is an inner product space.
3) ⇒ 2) Assuming that $||x|| = 1$, we have $||(a + x)^2||^2 = 4$ and since $(a^2/ax) = (x^2/ax) = (a/x) = 0$, then $(a^2/x^2) = -1$. Moreover

$$||a^2 + x^2||^2 = ||a^2||^2 + 2(a^2/x^2) + ||x^2||^2 = 1 - 2 + 1 = 0 \quad (1)$$

So (1) gives $x^2 = -a^2$, and then $x^2 = -||x||^2 a^2$ for all $x \in \{a\}^{\perp}$.

Lemma 2.9 Let A be an absolute valued algebra containing a central element a. If there exists $b \in A$ such $||b|| = 1$ and $a = a^2 b$, then A is an inner product space.

Proof. Let x be a norm one in A and suppose that $(a/x) = 0$ in the inner product space $[a, x]$, then we have

$$2 = ||x - a||||x + a|| = ||x^2 - a^2|| = ||x^2 b - a^2 b|| = ||x^2 b - a||.$$

As $(x^2 b)a = a(x^2 b)$, then by Lemma 2.7 we get $x^2 b = -a = -a^2 b$. Hence the result is concluded by a simplification by b and using proposition 2.8.

Corollary 2.10 Let A be an absolute valued algebra containing a central algebraic element a, then A is an inner product space.
Proof. As $A(a)$ is finite dimensional, then $A(a)$ is a division algebra. Therefore the operator L_{a^2} of left multiplication by a^2 on $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}.(a)$ is bijective, and there exists $b \in A(a)$ such $||b|| = 1$ and $a = L_{a^2}(b) = a^2 b$. Then the result is consequence of the lemma 2.8.

Corollary 2.11 Let A be an absolute valued algebra whose norm comes from an inner product and contains a nonzero central element a. Then $xz + zx = 2(z|a)ax - 2(x|z)a^2$ for all $x, z \in A$ with $(a|x) = 0$.
Proof. Let $y = z - (z|a)a$, we have $x, y \in \{a\}^{\perp}$. Then $(x + y)^2 = -||x + y||^2 a^2$ (proposition 2.8), hence $xy + yx = -2(x|y)a^2$.
Therefore that $xz + zx = 2(z|a)ax - 2(x|z)a^2$.

3. Main Results

3.1. Absolute Valued Algebras Satisfying $(x, x, x) = 0$

In this section we prove that if A is an absolute valued algebra containing a central element a and satisfying $(x, x, x) = 0$. Then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Lemma 3.1.1 Let A be an absolute valued algebra containing a central element a and satisfying $(x, x, x) = 0$ for all $x \in A$, then $xa^2 = a^2 x$ and A is an inner product space.
Proof. Let x be a element in A, we have $(x + a, x + a, x + a) = 0$. Then

$$0 = (x + a, x + a, x + a) = (x, x, a) + (x, a, x) + (x, a, a) + (a, x, x) + (a, a, x)$$

Replacing x by $-x$, we get $(x, x, a) + (x, a, x) - (x, a, a) + (a, x, x) - (a, a, x) = 0$
Adding these two equalities, we have $0 = (x, a, a) + (a, a, x) = xa^2 - a^2 x$
Then $xa^2 = a^2 x$ for all $x \in A$, by proposition 2.8 we conclude that A is an inner product space.

Theorem 3.1.2 Let A be an absolute valued algebra containing a central element a and satisfying $(x, x, x) = 0$ for all $x \in A$, then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.
Proof. By lemma 3.1.1, A is an inner product space. We assume that a and a^2 are linearly independent, and let $x \in \{a^2, a\}^{\perp}$ such that $||x|| = 1$, we have $xa^2 = a^2 x$ (lemma 3.1.1) and $ax = xa$, then $-(a^2)^2 = x^2 = -a^2$ (proposition 2.8). This means that $a^2 = \pm a$, which is absurd. Therefore a is a central idempotent, hence A is finite dimensional and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$ (theorem 2.4).

3.2 Absolute Valued Algebras Satisfying $(x^2, x^2, x^2) = (x, x^2, x) = 0$

In this section we prove that if A is an absolute valued algebra containing a central element a and satisfying $(x^2, x^2, x^2) = 0$ or $(x, x^2, x) = 0$. Then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Theorem 3.2.1 Let A be an absolute valued algebra containing a central algebraic element a and satisfying $(x^2, x^2, x^2) = 0$ for all $x \in A$, then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.
Proof. According to corollary 2.10, A is an inner product space. We assume that a and a^2 are linearly independent, and let $x \in$

$\{a^2,a\}^{\perp}$ such that $||x|| = 1$. Since $ax = xa$, then $x^2 = -a^2$ (proposition 2.8), and $((a + x)^2, (a + x)^2, (a + x)^2) = 0$. Then $(ax, ax, ax) = 0$, that is $(ax)^2(ax) = (ax)(ax)^2$ (2)
Moreover $A(a, a^2)$ is a finite dimensional division sub-algebra of A, then there exists a nonzero element $b \in A(a, a^2)$ such that $ab = ba = a$. Since $(ax/a) = (ax/ab) = (x/b) = 0$,
then $(ax)^2 = -a^2$. So (2) gives $a^2(ax) = (ax)a^2$, also $(ax/a^2) = (x/a) = 0$
this implies, by proposition 2.8, that $(ax)^2 = -(a^2)^2$. Hence $(a^2)^2 = a^2$, therefore $a^2 = \pm a$ $(aa^2 = a^2a)$, which is absurd. So a is a central idempotent, and the result is consequence of the theorem 2.5.

Theorem 3.2.2 Let A be an absolute valued algebra containing a central algebraic element a and satisfying $(x, x^2, x) = 0$ for all $x \in A$, then A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.
Proof. According to corollary 2.10, A is an inner product space. Let $x \in \{a^2, a\}^{\perp}$ such that $||x|| = 1$. Since $ax = xa$, then $x^2 = -a^2$ (proposition 2.8), and $(a + x, (a + x)^2, a + x) = 0$. Then $(a + x, ax, a + x) = 0$, replacing x by $-x$, we get $(a - x, ax, a - x) = 0$. Adding these two equalities, we obtain $(x, ax, x) = 0$.

Using corollary 2.11, we have
$$\begin{aligned} xx^2 + x^2x &= 2(x^2/a)ax - 2(x|x^2)a^2 \\ &= -2(a^2/a)ax - 2(x|x^2)a^2 \\ &= -2(a^2/a)ax \qquad ((x^2/x) = -(a^2/x) = 0) \end{aligned}$$
Also,
$$\begin{aligned} (xx^2)x + (x^2x)x &= -2(a^2/a)(ax)x \\ x(x^2x) + (x^2x)x &= -2(a^2/a)(ax)x \qquad ((x, x^2, x) = 0) \end{aligned}$$
According to corollary 2.11,
$$\begin{aligned} 2(x^2x/a)ax - 2(x^2x/x)a^2 &= -2(a^2/a)(ax)x \\ (x^2x/a)ax + (x^2x/x)x^2 &= -(a^2/a)(ax)x \end{aligned}$$
So
$$(x^2x/a)a + (x^2x/x)x = -(a^2/a)ax \qquad (3)$$
Thus
$$((x^2x/a)a + (x^2x/x)x/a) = -(a^2/a)(ax/a)$$

Since $A(a, a^2)$ is a finite dimensional division sub-algebra of A, then there exists a nonzero element $b \in A(a, a^2)$ such that $ab = ba = a$. So $(ax/a) = (ax/ab) = (x/b) = 0$, this implies that $(x^2x/a) = 0$. Therefore (3) gives $(x^2x/x)x = -(a^2/a)ax$, and we have the following two cases:

- If $(a^2/a) = 0$, then $(x^2x/x) = 0$, $xx^2 + x^2x = 0$ and $(a^2)^2 = -a^2 = e$. Moreover, we have $(e + x, (e + x)^2, e + x) = 0$ which means that
$$\begin{aligned} 0 &= (e + x, e, e + x) \\ &= (e, e, x) + (x, e, e) \qquad (x^2 = e) \\ &= ex - e(ex) + (xe)e - xe \\ &= 2ex + e(xe) + (xe)e \qquad (ex = -xe) \\ &= 2ex + 2(xe|a)ae - 2(xe|e)a^2 \qquad (corollary\ 2.11) \end{aligned}$$

Since $A(a, a^2)$ is a finite dimensional division sub-algebra of A, then there exists a nonzero element $c \in A(a, a^2)$ such that $ca^2 = a$, so $(xe/a) = (xe/ca^2) = -(xe/ce) = -(x/c) = 0$. Since $(xe/e) = (xe/e^2) = (x/e) = 0$, then $x = 0$. Which is absurd, in this case $A = A(a, a^2)$ is a commutative algebra and is isomorphic to $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$.

- If $(a^2/a) \neq 0$, then $ax = \pm x$. Thus $(x, x, x) = (x, ax, x) = 0$, hence the result is consequence of the theorem 3.1.2.

4. Conclusion
We have the following classical results:
Theorem. Let A be an absolute valued algebra containing a nonzero central algebraic element. Then the following assertions are equivalent:
1) A satisfies $(x, x, x) = 0$,
2) A satisfies $(x^2, x^2, x^2) = 0$,
3) A satisfies $(x, x^2, x) = 0$,
4) A is finite dimensional and isomorphic to $\mathbb{R}, \mathbb{C}, \mathrm{H}, \mathrm{O}, \overset{*}{\mathbb{C}}, \overset{*}{\mathrm{H}}$ or $\overset{*}{\mathrm{O}}$.

Based on the findings of this article, the following conclusions can be drawn:
1) In general, if A is a real absolute valued algebra containing a nonzero central algebraic element, then, A is a pre-Hilbert algebra. It may be conjectured that every absolute valued algebra containing a nonzero central element is pre-Hilbert algebra.
2) Note that, central idempotent is a central element. The reciprocal case does not hold in general, and the counter-example is given [Benslimane, 2011].
3) We give some conditions that imply that any absolute valued algebra A is an inner product space.

In future work, it is intended to classify all real absolute valued algebra containing a nonzero central algebraic element and satisfying $(x^2, x, x^2) = 0, (x^2, x, x) = 0, (x, x, x^2) = 0, (x^2, x^2, x) = 0$ *or* $(x, x^2, x^2) = 0$.

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