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| RESEARCH ARTICLE

A Hybrid Analytical Approximate Technique for Solving Two-dimensional Incompressible Flow in Lid-driven Square Cavity Problem

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| ABSTRACT
This paper suggests a new technique for finding the analytical approximate solutions to two-dimensional kinetically reduced local Navier-Stokes equations. This new scheme depends combines the q-Homotopy analysis method (q-HAM) , Laplace transform, and Padé approximant method. The power of the new methodology is confirmed by applying it to the flow problem of the lid-driven square cavity. The numerical results obtained by using the proposed method showed that the new technique has good convergence, high accuracy, and efficiency compared with the earlier studies. Moreover, the graphs and tables demonstrate the new approach's validity.

| KEYWORDS

Navier-Stokes equations; q-Homotopy Analysis method; Padé approximation; Convergence Analysis;

| ARTICLE INFORMATION

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1. Introduction

Fluid flow is one of the most significant engineering phenomena that has drawn significant interest in both theoretical and applied scientific research. Several of these studies concentrate on mathematical models that simulate these phenomena. The researchers spent a lot of time and effort in order to obtain analytical and numerical solutions to the Navier-Stokes equations, which represent the basic model for describing fluid motion. This study investigates the two-dimensional Navier-Stokes (NS) equations for studying viscous incompressible fluid flows. The equations for non-dimensional momentum and continuity have the following form:

$$\left.\begin{aligned} u_t &= -(uu_x + vu_y + p_x) + \frac{1}{Re}(u_{xx} + u_{yy}), \\ v_t &= -(uv_x + vv_y + p_y) + \frac{1}{Re}(v_{xx} + v_{yy}), \\ &u_x + v_y = 0 \end{aligned}\right\} \quad (1)$$

Where t is the physical time, $u(x, y, t)$ and $v(x, y, t)$ are the fluid velocity components, $p(x, y, t)$ is the pressure, and Re is the Reynolds number. Since there is no direct formula for determining pressure, numerous researchers have proposed alternative equations for the thermodynamic description of incompressible fluid flows. One of these alternate equations is the kinetically reduced local Navier-Stokes (KRLNS) equation. [1-7] are generated by replacing the pressure by

$$p = g + \frac{u^2+v^2}{2} \tag{2}$$

and the continuity equation by

$$g_t = -\frac{1}{(Ma)^2}\left(u_x + v_y\right) + \frac{1}{Re}\left(g_{xx} + g_{yy}\right), \tag{3}$$

Where $g(x, y, t)$ is the grand potential, and Ma is the Mach number. $t_{KRLNS}(\tau) = Ma \times t_{NS}$ relates the time scale in NS equations to that in KRLNS equations. As a result, the KRLNS equation system has the following structure

$$\left.\begin{aligned} u_t &= -\left(2uu_x + vv_x + vu_y + g_x\right) + \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \\ v_t &= -\left(uv_x + uu_y + 2vv_y + g_y\right) + \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \\ g_t &= -\frac{1}{(Ma)^2}\left(u_x + v_y\right) + \frac{1}{Re}\left(g_{xx} + g_{yy}\right) \end{aligned}\right\} \tag{4}$$

Many papers have used numerical methods to model the incompressible flow problems of the KRLNS equations proposed in [8], to simulate the low Mach number flow, and were modeled numerically using the 3D Taylor-Green vortex flow spectral element technique. Borok et al. [3] simplified and solved the KRLNS system two-dimensional for Taylor-Green vortex flow and a two-dimensional cap-driven cavity at the steady state and compared it with a Chorin artificial compression approach. Hashimoto et al.[4] solved the KRLNS equations for simulating two - dimensional shear layers and homogeneous perturbations decomposition using the fourth-order Runge-Kutta (RK~4) method and center difference scheme, they compared it with the method of artificial compression and Boltzmann's lattice method. Hashimoto et al. [6], applied the Higher-order approach of (KRLNS) equations for two-dimensional simulations of Womersley problem and doubly periodic shear layers, and this approach is accurate and efficient. The lid-driven cavity problem concerns the flow in a box cavity with no wall slippage and one or more moving walls that move continuously. It has been widely utilized as a benchmark case for the research of computing techniques for solving Navier-Stokes equations because of the geometry's and boundary conditions' simplicity. Through the use of various numerical techniques in cavities that are rectangular or square, numerous research papers have offered answers to this problem. Marchi et al.[9] the finite volume method is used in conjunction with repeated Richardson extrapolations and numerical approximations of second-order accuracy to solve the problem of flow inside a square cavity with constant velocity. Sahin and Owens[10], described how to solve the steady and unsteady two-dimensional lid-driven cavity problem at high Reynolds numbers using the implicit cell-vertex finite volume method. Khanafer et al.[11], a lid-driven cavity with unsteady laminar mixed convection heat transport is studied using a finite element method based on the Galerkin method of weighted residuals. Mramor et al.[12], local radial basis function collocation method (LRBFCM) is tested on a lid-driven cavity benchmark case. Ambethkar and Kushawaha [13], numerical simulations of two-dimensional fluid flow and heat transfer in a four-sided lid-driven rectangular domain were carried out, by utilizing the quadratic upstream interpolation for the convective kinematics (QUICK) scheme of the finite volume method. Albensoeder and Kuhlmann[14], the lid-driven cavity at high Reynolds numbers was solved using the vorticity-stream formulation of the Navier-Stokes equation with the strong-stability-preserving Runge-Kutta four and five-order scheme in a very fine grid mesh. Poochinapan [15-16] , the nonlinear convective terms of the two-dimensional incompressible Navier-Stokes equations time-dependent biharmonic equation in the current function form are numerically solved using the finite difference method with internal iterations. Hubert [17], explored the states of incompressible flow in a four-sided lid-driven square cavity by using the hyperbolic method. Ambethkara and Kumarb[18], used the flow-vortex $(\psi - \xi)$ method to solve the problem of steady two-dimensional incompressible viscous flow in a driven square cavity with moving top and bottom walls. Gürbüz and Tezer [19], applied the Stokes approximation method to the Stokes equations for two-dimensional magneto hydrodynamics to study the effect of the magnetic field on flow in a cap-driven cavity.Ali et al.[20-21], the Forchheimer model's porous medium with changeable viscosity was used to theoretically examine the continuous dependency of double-diffusive convection . Many previous researchers on this problem concentrated on numerical solutions. To obtain approximative analytical solutions, first, the q-Homotopy analysis is employed in this context, the reason is the lack of prior use of this method to address this problem, and if it exists, it differs with boundary conditions, as in [22]. Second, the integral transform is the Laplace transform (LT), suggested by Simon in 1942. It was initially used to solve partial differential equation [23-25]. Third, a method of approximation known as Padé approximation was introduced by Henri Padé in his doctoral thesis in 1892. It has garnered a lot of attention because it has been utilized by several authors to address a variety of themes [26-30]. The intent of combining analytical methods with a suitable transform reduces the time consumption in investigating solutions to nonlinear problems describing fluid flow. q-HALPM is an amalgamation of q-HAM with Laplace transform and Padé approximate. Padé approximation order was chosen at random. By expanding the solution field at this point, the Padé approximation improves the accuracy and convergence of the truncated series solution. The primary objective of this study is to propose a new analytical method for solving KRLNS equations. The q-HALP processes are most efficient and accurate in solving problems of unsteady incompressible viscous flow at low Mach numbers and for various Reynolds numbers, according to the calculations presented in the tables and figures. In addition, by solving this problem, we have demonstrated the usefulness of q-HALPM as a tool for solving nonlinear fluid flow problems.

2- Preliminaries

we mention some necessary definitions that will help us achieve the aim of this work.

Deffinition.2.1.

For t ≥ 0, let $m(t)$ be given, the Laplace transform of $m(t)$, which is denoted by $l\{m(t)\}$or $M(s)$ is defined by $M(s) = \int_0^\infty e^{-st} m(t)dt$.

Deffinition.2.2.

Suppose that, we are given a power series $w(y) = \sum_{i=0}^{\infty} d_i y^i$,then the Padé approximant of a function $w(y)$ is a rational fraction, so that

$$\sum_{i=0}^{\infty} d_i y^i = \frac{\sum_{i=0}^{p} d_i y^i}{1+\sum_{i=1}^{r} d_i y^i}$$ (For details of Padé approximate method see Ref [27])

2.1- Build the q-HALP Algorithm:

we provide the basic ideas of q-HALP depend on the algorithms of q-HAM, Laplace transform and Padé approximate. To clarify the fundamental ideas of these techniques, let us look at the generic nonlinear differential equation:

$$L\big(z(x,y,t)\big) + N[z(x,y,t)] - W(x,y,t) = 0 \tag{5}$$

where, L is a linear operator , N is a nonlinear operator, (x,y,t) are an independent variables, $W(x,y,t)$ is known function and $z(x,y,t)$ is unknown function, according to the zero-order deformation equation :

$$(1-\delta q)L[V(x,y,t;q) - z_0(x,y,t;q)] - hqH(x,y,t)N[V(x,y,t;q) - W(x,y,t)] = 0 \tag{6}$$

where $q \in [0, \frac{1}{\delta}]$ is an embedding parameter, $\delta \geq 1, h \neq 0$ and an auxiliary operator, $H(x,y,t)$ is a non-zero an auxiliary function, $z_0\ (x,y,t;q)$ is the initial guess of $z(x,y,t)$ and $V(x,y,t;q)$ are the unknown functions. It is evident that when $q = 0 \; and \; q = \frac{1}{\delta}$ equation (6) becomes :

$V(x,y,t,0) = z_0\,(x,y,t), \quad V\left(x,y,t,\frac{1}{\delta}\right) = z(x,y,t),$

respectively. The solution $V(x,y,t;q)$ changes from the initial guess $z_0\,(x,y,t)$ to the solution $z(x,y,t)$.

For $q \in [0,\frac{1}{\delta}]$. From Tayler's expansion of $V(x,y,t;q)$, we have

$$V(x,y,t;q) = z_0(x,y,t) + \sum_{i=1}^{\infty} z_i(x,y,t)q^i \tag{7}$$

If $q = \frac{1}{\delta}$, then equation (7) becomes

$$V\left(x,y,t;\frac{1}{\delta}\right) = z_0\,(x,y,t) + \sum_{i=1}^{\infty} z_i(x,y,t)(\frac{1}{\delta})^i \tag{8}$$

We define the following vector $\vec{z}_i(x,y,t) = \{z_1(x,y,t), z_2(x,y,t), z_3(x,y,t), \dots, z_n(x,y,t)\}$.

Differentiating equation (6) i term with respect to the embedding parameter q and then dividing them by $i!$ after that set $q = 0$, obtain the i^{th} order deformation equation:

$$L[z_i(x,y,t) - k_i z_{i-1}(x,y,t)] = hH(x,y,t)K_i[\overrightarrow{z_i}(x,y,t)], \tag{9}$$

where $K_{\,i}[\vec{z}_{i-1}(x,y,t)] = \frac{1}{(i-1)!}\frac{d^{i-1}N[V(x,y,t;q)]}{dq^{i-1}}\Big|_{q=0}$ and $k_i = \begin{cases} 0, i \le 1 \\ \delta\,, i > 1 \end{cases}$

To calculate the functions $z_i(x,y,t)$, the inverse linear operator is applied to equation (9), then

$$z_i(x,y,t) = k_i z_{i-1}(x,y,t) + hH(x,y,t)L^{-1}(K_i[\overrightarrow{z_i}(x,y,t)]) \tag{10}$$

Hence

$$\begin{cases} q^1: z_1 = hH(x,y,t)L^{-1}(K_1[\overrightarrow{z_0}(x,y,t)]), \quad K_1[\overrightarrow{z_0}(x,y,t)] = \frac{1}{(0)!}\frac{d^{1-1}N[V(x,y,t;q)]}{dq^{1-1}}\Big|_{q=0} \\ q^2: z_2 = \delta z_1(x,y,t) + hH(x,y,t)L^{-1}(K_2[\overrightarrow{z_1}(x,y,t)]), \; K_2[\overrightarrow{z_1}(x,y,t)] = \frac{1}{(1)!}\frac{d^{1}N[V(x,y,t;q)]}{dq^{1}}\Big|_{q=0} \\ q^3: z_3 = \delta z_2(x,y,t) + hH(x,y,t)L^{-1}(K_3[\overrightarrow{z_2}(x,y,t)]) \quad , K_3[\overrightarrow{z_2}(x,y,t)] = \frac{1}{(2)!}\frac{d^{2}N[V(x,y,t;q)]}{dq^{2}}\Big|_{q=0}, \\ \vdots \end{cases}$$

Then the series solution of q-HAM is $z(x,y,t) = \sum_{i=0}^{n} z_i(x,y,t)(\frac{1}{\delta})^i, \; n = 1,2,3 \dots$ (11)

Now, by taking the Laplace transform for both sides of equation (11), we have

$$l(z(x,y,t)) = l(\sum_{i=0}^{n} z_i(x,y,t)\left(\frac{1}{\delta}\right)^i, t, s) \;, \tag{12}$$

$$Z(x,y,s) = l\left(\sum_{i=0}^{n} z_i(x,y,t)\left(\frac{1}{\delta}\right)^i, t, s\right)$$

After that, applying Padé approximate method on equation (12) , then

$$P_r^p[Z(x,y,s)] = \text{Padé}\left(Z\left(x,y,\frac{1}{s}\right),[p,r]\right), 0 \le p, r \le p+r+1 \tag{13}$$

Finally, to obtain the approximate solution take invers Laplace transform of equation (13) ,we have

$$z(x,y,t) = l^{-1}(Pade\left(Z\left(x,y,\frac{1}{s}\right)\right), s, t) .$$

3- Application q-HALPM on NSEs.

We solve the mentioned system in an equation (5) by using q-HALPM. To obtain the approximate analytical solutions of the unsteady lid-driven cavity flow problem.

Now, before we start applying the q-HALPM on KRLNS equation taking the initial conditions of u, v, P as [31-33];

$$\begin{cases} u(x,y) = 8d(x)\ b'(y), \\ v(x,y) = -8d'(x)\, b(y) \\ P(x,y,0) = \frac{8}{Re}\left(F(x)b'''(x) + d'(x)\, b'(y)\right) + 64F_1(x)(b(y)b''(y) - \left(b'(y)\right)^2)) \end{cases} \tag{14}$$

where

$$\begin{cases} d(x) = x^4 - 2x^3 + x^2,\ b(y) = y^4 - y^2, \\ F(x) = \int d(x)dx, \quad F_1(x) = \int r(x)\, d'(x)dx \end{cases} \tag{15}$$

such that the stream function ψ and vorticity ω are defined as

$$\begin{cases} \psi = 8d(x)c(y), such\ that\ \ \psi_y = u, and\ \psi_x = -v \\ \omega = v_x - u_y = -8(d''(x)\ b(y) + d(x)\, b''(y)) \end{cases} \tag{16}$$

Then, the basic steps of the new technique are illustrated as follows:

Firstly, applying equation (7) on system equations in(4), we get

$$\left.\begin{aligned} (1-\delta q)L[u(x,y,t;q) - u_0(x,y,t;q)] - hqN\left[-\left(2uu_x + vv_x + vu_y + g_x\right) + \frac{1}{Re}\left(u_{xx} + u_{yy}\right)\right] = 0, \\ (1-\delta q)L[v(x,y,t;q) - v_0(x,y,t;q)] - hqN\left[-\left(uv_x + uu_y + 2vv_y + g_y\right) + \frac{1}{Re}\left(v_{xx} + v_{yy}\right)\right] = 0, \\ (1-\delta q)L[g(x,y,t;q) - g_0(x,y,t;q)] - hqN\left[-\frac{1}{(Ma)^2}\left(u_x + v_y\right) + \frac{1}{Re}\left(g_{xx} + g_{yy}\right)\right] = 0 \end{aligned}\right\}, \tag{17}$$

such that $L = \frac{d}{dt}$

Suppose the series solution of $u(x,y,t), v(x,y,t), g(x,y,t)$ in equations (17) by the following as :

$$\left.\begin{aligned} u(x,y,t) = \textstyle\sum_{i=1}^{\infty} u_i(x,y,t)q^i\ , \\ v(x,y,t) = \textstyle\sum_{i=1}^{\infty} v_i(x,y,t)q^i\ , \\ g(x,y,t) = \textstyle\sum_{i=1}^{\infty} g_i(x,y,t)q^i \end{aligned}\right\} \tag{18}$$

After substituting $u(x,y,t), v(x,y,t), g(x,y,t)$in equation (17) becomes at the following

$$
\begin{cases}
(1-\delta q)L\left[\sum_{i=1}^{\infty}u_i(x,y,t)q^i-u_0(x,y,t)\right]=hqN\left[-\left(\begin{array}{c}\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_x+\\ (\sum_{i=1}^{\infty}v_i(x,y,t)q^i)(\sum_{i=1}^{\infty}v_i(x,y,t)q^i)_x\\ +2\left(\sum_{i=1}^{\infty}v_i(x,y,t)q^i\right)\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_y\\ +\left(\sum_{i=1}^{\infty}g_i(x,y,t)q^i\right)_x\end{array}\right)+\frac{1}{Re}\left(\begin{array}{c}\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_{xx}+\\ \left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_{yy}\end{array}\right)\right]\\
(1-\delta q)L[\sum_{i=1}^{\infty}v_i(x,y,t)q^i-v_0\,(x,y,t)]=hqN\left[-\left(\begin{array}{c}\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)\left(\sum_{i=1}^{\infty}v_i(x,y,t)q^i\right)_x+\\ \left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_y+\\ 2\left(\sum_{i=1}^{\infty}v_i(x,y,t)q^i\right)\left(\sum_{i=1}^{\infty}v_i(x,y,t)q^i\right)_y\\ +\left(\sum_{i=1}^{\infty}g_i(x,y,t)q^i\right)_y\end{array}\right)+\frac{1}{Re}\left((\sum_{i=1}^{\infty}v_i(x,y,t)q^i)_{xx}+\left(\sum_{i=1}^{\infty}v_i(x,y,t)q^i\right)_{yy}\right)\right]\\
(1-\delta q)L\left[\sum_{i=1}^{\infty}g_i(x,y,t)q^i-g_0\,(x,y,t;q)\right]=hqN\left[-\frac{1}{(Ma)^2}\left(\begin{array}{c}\left(\sum_{i=1}^{\infty}u_i(x,y,t)q^i\right)_x\\ +(\sum_{i=1}^{\infty}v_i(x,y,t)q^i)_y\end{array}\right)+\frac{1}{Re}\left((\sum_{i=1}^{\infty}g_i(x,y,t)q^i)_{xx}+(\sum_{i=1}^{\infty}g_i(x,y,t)q^i)_{yy}\right)\right]
\end{cases}
\tag{19}
$$

we obtained the iterative solutions from q-HAM, such that:

$$u_0(x,y,t)=32x^2y\left(y^2-\frac{1}{2}\right)(x-1)^2$$

$$u_1(x,y,t)=256hx^3y^2(8y^6-6y^4-y^2+3)(3x-2)(x-1)t$$

$$
\begin{aligned}
&u_2(x,y,t)=-36864h(y(x-1)x^2\Big(\Big(\Big(y^4-\Big(\tfrac{1}{3}\Big)y^2+\tfrac{1}{6}\Big)h\Big(y^2-\tfrac{1}{6}\Big)tx^9-\Big(5\Big(y^4-\Big(\tfrac{1}{3}\Big)y^2+\\
&\tfrac{1}{6}\Big)\Big)h\Big(y^2-\tfrac{1}{6}\Big)tx^8-\Big(\tfrac{47}{3}\Big)h\Big(y^8-\Big(\tfrac{224}{141}\Big)y^6+\Big(\tfrac{49}{94}\Big)y^4-\Big(\tfrac{11}{94}\Big)y^2+\tfrac{1}{141}\Big)tx^7+\Big(52\Big(y^8-\Big(\tfrac{139}{117}\Big)y^6+\\
&\Big(\tfrac{1}{3}\Big)y^4-\Big(\tfrac{4}{117}\Big)y^2-\tfrac{1}{234}\Big)\Big)htx^6+\Big(\tfrac{28}{3}\Big)h\Big(y^{10}-\Big(\tfrac{83}{8}\Big)y^8+\Big(\tfrac{863}{84}\Big)y^6-\Big(\tfrac{433}{112}\Big)y^4+\Big(\tfrac{227}{336}\Big)y^2+\\
&\tfrac{13}{336}\Big)tx^5-\Big(\tfrac{52}{3}\Big)h\Big(y^{10}-\Big(\tfrac{1933}{312}\Big)y^8+\Big(\tfrac{293}{52}\Big)y^6-\Big(\tfrac{607}{208}\Big)y^4+\Big(\tfrac{147}{208}\Big)y^2+\tfrac{5}{624}\Big)tx^4+\Big(\tfrac{38}{3}\Big)y^2h\Big(y^8-\Big(\tfrac{1211}{228}\Big)y^6+\\
&\Big(\tfrac{92}{19}\Big)y^4-\Big(\tfrac{421}{152}\Big)y^2+\tfrac{317}{456}\Big)tx^3-\Big(\tfrac{64}{9}\Big)y\Big(hty^9-\Big(\tfrac{477}{128}\Big)hty^7+\Big(\tfrac{3}{128}\Big)\tfrac{\delta_1}{Ma}y^6+\\
&\Big(\tfrac{53}{16}\Big)hty^5-\Big(\tfrac{9}{512}\Big)\tfrac{\delta_1}{Ma}y^4-\Big(\tfrac{383}{256}\Big)hty^3-\Big(\tfrac{3}{1024}\Big)\tfrac{\delta_1}{Ma}y^2+\Big(\tfrac{73}{256}\Big)hty+\Big(\tfrac{9}{1024}\Big)\tfrac{\delta_1}{Ma})\Big)x^2+\Big(\tfrac{25}{9}\Big)y\Big(hty^9-2.5hty^7+\\
&0.04\tfrac{\delta_1}{Ma}y^6+2hty^5-0.03\tfrac{\delta_1}{Ma}y^4-0.5hty^3-\Big(\tfrac{1}{200}\Big)\tfrac{\delta_1}{Ma}y^2+\Big(\tfrac{3}{200}\Big)\tfrac{\delta_1}{Ma}\Big)x-\Big(\tfrac{1}{3}\Big)y^4h(y-1)^2\Big(y^2-\tfrac{1}{2}\Big)(y+\\
&1)^2t\Big)Re^2-0.04ht(\Big(y^2-\tfrac{1}{6}\Big)x^9+(-\Big(\tfrac{9}{2}\Big)y^2+3/4)x^8+(-(\tfrac{34}{3})y^2+59y^4+\tfrac{1}{9})x^7+(-\tfrac{35}{9}+\Big(\tfrac{182}{3}\Big)y^2-\Big(\tfrac{413}{2}\Big)y^4)x^6+\\
&\Big(279y^4-\Big(\tfrac{172}{3}\Big)y^2-\Big(\tfrac{350}{3}\Big)y^6+\tfrac{125}{36}\Big)x^5+(-\Big(\tfrac{475}{4}\Big)y^4-\Big(\tfrac{175}{12}\Big)y^2+\Big(\tfrac{1225}{9}\Big)y^6+\tfrac{25}{36})x^4+(55y^6-(\tfrac{100}{3})y^4-\Big(\tfrac{170}{3}\Big)y^8+\\
&15y^2-\tfrac{35}{36})x^3+(\Big(\tfrac{155}{3}\Big)y^8-\Big(\tfrac{415}{6}\Big)y^6+\Big(\tfrac{235}{12}\Big)y^4+\Big(\tfrac{115}{12}\Big)y^2)x^2-\Big(\tfrac{20}{3}\Big)y^2(y^6-(5/24)y^4-(31/48)y^2+25/\\
&48)x-\Big(\tfrac{5}{6}\Big)y^4(y-1)^2(y+1)^2)Re+(\tfrac{1}{16})yh(x^2+\Big(\tfrac{1}{6}\Big)y^2-x+\tfrac{1}{12})t)t/Re^2
\end{aligned}
$$

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.

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and

$$v_0(x,y,t) = -32x\left(x-\frac{1}{2}\right)(x-1)(y+1)y^2(y-1),$$

$$v_1(x,y,t) = -768h\Bigg(\left(\left(y^4-\left(\tfrac{1}{3}\right)y^2+\tfrac{1}{6}\right)Reyx^8-\left(4\left(y^4-\left(\tfrac{1}{3}\right)y^2+\tfrac{1}{6}\right)\right)Reyx^7+(-12Rey^7+14Rey^5-$$
$$2Rey^3)x^6+\left(-\tfrac{1}{20}+\left(\tfrac{76}{3}\right)y^7Re-22y^5Re+2y^3Re+\left(\tfrac{4}{3}\right)yRe\right)x^5+\left(\tfrac{1}{8}-\left(\tfrac{46}{3}\right)y^7Re+14y^5Re-2y^3Re-\right.$$
$$\left.\left(\tfrac{5}{6}\right)yRe\right)x^4+\left(-y^2+\left(\tfrac{8}{3}\right)y^7Re-4y^5Re+\left(\tfrac{4}{3}\right)y^3Re+\tfrac{1}{12}\right)x^3+\left(\left(\tfrac{3}{2}\right)y^2-\left(\tfrac{2}{3}\right)y^7Re+y^5Re-\left(\tfrac{1}{3}\right)y^3Re-\right.$$
$$\left.\tfrac{1}{4}\right)x^2+\left(\tfrac{1}{12}-\left(\tfrac{1}{4}\right)y^4-\left(\tfrac{1}{4}\right)y^2\right)x+\left(\tfrac{1}{8}\right)y^4-\left(\tfrac{1}{8}\right)y^2\Bigg)\frac{t}{Re}$$

$$v_2(x,y,t) = \frac{22880ht}{Re^3}\Bigg(\Bigg(\Bigg(yh(y^2-1)\left(y^4-\left(\tfrac{1}{5}\right)y^2+\tfrac{1}{30}\right)t(x^8-4.5x^7)-\left(\tfrac{52}{5}\right)y\left(y^8-\left(\tfrac{133}{52}\right)y^6+\left(\tfrac{85}{52}\right)y^4-\right.$$
$$\left.\left(\tfrac{193}{624}\right)y^2+\tfrac{25}{624}\right)htx^6+\left(ht\left(\left(\tfrac{128}{5}\right)y^9-\left(\tfrac{829}{15}\right)y^7+\left(\tfrac{175}{6}\right)y^5-\left(\tfrac{59}{10}\right)y^3+\left(\tfrac{11}{15}\right)y\right)-\frac{\left(\frac{1}{160}y^4-\frac{1}{480}y^2+\frac{1}{960}\right)\delta_1}{Ma}\right)x^5+$$
$$\left(\left(-\left(\tfrac{12}{5}\right)y^{11}-\left(\tfrac{271}{15}\right)y^9+\left(\tfrac{469}{10}\right)y^7-\left(\tfrac{811}{30}\right)y^5+\left(\tfrac{79}{12}\right)y^3-\left(\tfrac{13}{20}\right)y\right)ht+\frac{\left(\frac{3}{160}y^4-\frac{1}{160}y^2+\frac{1}{320}\right)\delta_1}{Ma}\right)x^4+\left(ht\left(4y^{11}+\right.\right.$$
$$\left.\left.\left(\tfrac{44}{15}\right)y^9-\left(\tfrac{589}{30}\right)y^7+\left(\tfrac{227}{15}\right)y^5-\left(\tfrac{259}{60}\right)y^3+\left(\tfrac{13}{60}\right)y\right)+\frac{\left(\frac{3}{40}y^6-\frac{11}{160}y^4+\frac{1}{160}y^2+\frac{1}{320}\right)\delta_1}{Ma}\right)x^3+\left(ht\left(-2y^{11}+0.03y^9+\right.\right.$$
$$\left.\left.\left(\tfrac{353}{60}\right)y^7-\left(\tfrac{163}{30}\right)y^5+\left(\tfrac{91}{60}\right)y^3\right)-\left(\tfrac{1}{12}y^6-\tfrac{11}{160}y^4+\tfrac{1}{160}y^2-\tfrac{1}{192}\right)\tfrac{\delta_1}{Ma}\right)x^2+y^2(y^2-1)\Bigg(\left(\tfrac{4}{15}\right)\Bigg(ht(y^7+1.5y^5-$$
$$2.5y^3+0.81y)+\left(\tfrac{3}{64}y^2-\tfrac{3}{128}\right)\tfrac{\delta_1}{Ma}\Bigg)x-\left(\tfrac{1}{240}\right)\left(y^2-\tfrac{1}{2}\right)\tfrac{\delta_1}{Ma}\Bigg)\Bigg)(x-1)x^2Re^2+\Bigg(\tfrac{9}{25}y\left(y^2-\tfrac{11}{72}\right)ht\Bigg(x^8-$$
$$\left(\tfrac{36}{25}\right)yx^7\Bigg)-\left(\tfrac{17}{8}\right)yh(y^4-1.265y^2+0.091)tx^6+\left(ht\left(\left(\tfrac{143}{40}\right)y^5-\left(\tfrac{91}{40}\right)y^3-\left(\tfrac{13}{80}\right)y\right)+\left(\tfrac{1}{3200}\right)\tfrac{\delta_1}{Ma}\right)x^5+$$
$$\left(ht\left(\left(\tfrac{33}{80}\right)y^5+\left(\tfrac{139}{240}\right)y^3-2y^7+\left(\tfrac{13}{96}\right)y\right)-\tfrac{1}{1280}\tfrac{\delta_1}{Ma}\right)x^4+\Bigg((3.6y^7-2.3y^5+0.03y^3+0.1y)ht+$$
$$\frac{\left(\frac{1}{160}y^2-\frac{1}{1920}\right)\delta_1}{Ma}\Bigg)x^3+\Bigg(\frac{\left(\left(\frac{1}{640}\right)-\frac{3}{320}y^2\right)\delta_1}{Ma}-\left(0.8y^7-\left(\tfrac{41}{80}\right)y^5-\left(\tfrac{1}{16}\right)y^3+\left(\tfrac{9}{160}\right)y\right)\Bigg)x^2+\left(\tfrac{1}{640}\tfrac{\delta_1}{Ma}\right)\Bigg(y^4+y^2-$$
$$\tfrac{1}{3}\Bigg)x+\left(\tfrac{1}{240}\right)y^2(y^2-1)\left(hty^3-\left(\tfrac{1}{2}\right)hty-\left(\tfrac{3}{16}\right)\tfrac{\delta_1}{Ma}\right)\Bigg)Re-\tfrac{1}{64}\left(x^2+\left(\tfrac{12}{5}\right)y^2-x-\tfrac{2}{5}\right)\left(x-\tfrac{1}{2}\right)ht\Bigg)$$

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and

$$g_1(x,y,t) = \frac{1}{Re^2}\Big(19200h\Big(-\left(\frac{4}{5}\right)Re\left(x^4-\left(\frac{2}{3}\right)x^3-\left(\frac{1}{2}\right)x^2+\left(\frac{3}{10}\right)x-\frac{1}{60}\right)y^8+\Big(\frac{28}{75}\Big(x^6-11x^5+20x^4-\left(\frac{62}{7}\right)x^3-\left(\frac{31}{14}\right)x^2+\left(\frac{9}{7}\right)x-\frac{1}{14}\Big)\Big)Rey^6+Re\left(x^8-4x^7+\left(\frac{98}{25}\right)x^6+\left(\frac{86}{25}\right)x^5-\left(\frac{37}{5}\right)x^4+\left(\frac{44}{15}\right)x^3+\left(\frac{9}{25}\right)x^2-\left(\frac{6}{25}\right)x+\frac{1}{75}\right)y^4+\left(-\left(\frac{1}{25}\right)x+\frac{1}{50}\right)y^3-\left(\frac{9}{25}\right)x^2Re\left(x^6-4x^5+\left(\frac{10}{3}\right)x^4+\left(\frac{34}{9}\right)x^3-\left(\frac{49}{9}\right)x^2+\left(\frac{4}{3}\right)x+\frac{1}{9}\right)y^2-\left(\frac{2}{25}\left(x-\frac{1}{2}\right)\right)\left(x^2-x-\frac{1}{4}\right)y+\left(\frac{1}{50}\right)x^4Re(x^2-2x-1)(x-1)^2\Big)t\Big),$$

$$g_2(x,y,t) = -1920h\Bigg(hx^2\Bigg(\left(\left(y^4-\left(\frac{1}{5}\right)y^2+\frac{1}{30}\right)x^6+\left(-4y^4+\left(\frac{4}{5}\right)y^2-\frac{2}{15}\right)x^5+\left(-\left(\frac{84}{5}\right)y^6+14y^4-\left(\frac{6}{5}\right)y^2\right)x^4+\left(\left(\frac{532}{15}\right)y^6-22y^4+\left(\frac{6}{5}\right)y^2+\frac{4}{15}\right)x^3+\left(-\frac{1}{6}-8y^8-\left(\frac{232}{15}\right)y^6+15y^4-\left(\frac{21}{5}\right)y^2\right)x^2+\left(\frac{32}{3}\right)y^2\left(y^6-\left(\frac{2}{5}\right)y^4-\left(\frac{1}{2}\right)y^2+\frac{9}{20}\right)x+\left(\frac{22}{15}\right)y^6-\left(\frac{16}{5}\right)y^8+\left(\frac{7}{5}\right)y^4-\left(\frac{7}{5}\right)y^2\right)tRe^3+\Bigg(-(10\delta_1)\left(y^4-\left(\frac{9}{25}\right)y^2+\frac{1}{50}\right)x^8+(40\delta_1)\left(y^4-\left(\frac{9}{25}\right)y^2+\frac{1}{50}\right)x^7-\left(\frac{56}{15}\delta_1\right)\left(y^6+\left(\frac{21}{2}\right)y^4-\left(\frac{45}{14}\right)y^2+\frac{3}{14}\right)x^6+\left(\frac{616}{15}\right)y^2\left(y^4-\left(\frac{129}{154}\right)y^2+\frac{51}{154}\right)\delta_1x^5+8\delta_1\left(y^8-\left(\frac{28}{3}\right)y^6+\left(\frac{37}{4}\right)y^4-\left(\frac{49}{20}\right)y^2+\frac{1}{40}\right)x^4-\left(\frac{2}{5}\right)y\left(\left(\frac{40}{3}\right)\delta_1y^7-\left(\frac{248}{3}\right)\delta_1y^5+\left(\frac{220}{3}\right)\delta_1y^3-12\delta_1y+ht\right)x^3+\left(\frac{3}{5}\right)y\left(-\left(\frac{20}{3}\right)\delta_1y^7+\left(\frac{124}{9}\right)\delta_1y^5-6\delta_1y^3+\left(\frac{2}{3}\right)\delta_1y+ht\right)x^2-\left(\frac{1}{5}\right)y\left(-12\delta_1y^7+24\delta_1y^5-12\delta_1y^3+hty^2+\left(\frac{1}{2}\right)ht\right)x+\left(\frac{1}{10}\right)y\left(-\left(\frac{4}{3}\right)\delta_1y^7+\left(\frac{8}{3}\right)\delta_1y^5-\left(\frac{4}{3}\right)\delta_1y^3+hty^2-\left(\frac{1}{2}\right)ht\right)\Bigg)Re^2+60Ma\Bigg(htx^6-\left(\frac{364}{15}\right)h\left(y^4-\left(\frac{159}{455}\right)y^2+\frac{19}{910}\right)tx^5-\left(\frac{14}{5}\left(y^6-\left(\frac{61}{6}\right)y^4+\left(\frac{26}{7}\right)y^2-\frac{79}{420}\right)\right)htx^4+\left(-\left(\frac{196}{45}\right)hty^6-\left(\frac{38}{15}\right)hty^4+\left(\frac{2}{3}\right)hty^2+\left(\left(\frac{1}{75}\right)\frac{\delta_1}{Ma}\right)y-\left(\frac{2}{25}\right)ht\right)x^3+\left(-\left(\frac{4}{5}\right)hty^8+\left(\frac{28}{3}\right)hty^6-\left(\frac{142}{15}\right)hty^4+\left(\frac{58}{25}\right)hty^2+\left(-\left(\frac{1}{50}\right)\frac{\delta_1}{Ma}\right)y-\left(\frac{2}{75}\right)ht\right)x^2+\left(\frac{4}{15}\right)y\left(hty^7-\left(\frac{52}{5}\right)hty^5+10hty^3+\left(\left(\frac{1}{40}\right)\frac{\delta_1}{Ma}\right)y^2-\left(\frac{9}{5}\right)hty+\left(\frac{1}{80}\right)\frac{\delta_1}{Ma}\right)x+\left(\frac{1}{15}\right)y\left(hty^7-\left(\frac{17}{15}\right)hty^5-\left(\frac{1}{10}\right)hty^3+\left(-\left(\frac{1}{20}\right)\frac{\delta_1}{Ma}\right)y^2+\left(\frac{1}{10}\right)hty+\left(\frac{1}{40}\right)\frac{\delta_1}{Ma}\right)\Bigg)Re-\left(\frac{18}{5}\right)yh\left(x-\frac{1}{2}\right)Mat\Bigg)\frac{t}{Re^3Ma}\ ,\delta_1 = Ma(h+\delta)$$

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Substituting $u_i(x,y,t)$ *and* $v_i(x,y,t)$ in equation (19) respectively to find the solution $u(x,y,t)$ *and* $v(x,y,t)$. Solve of the equation(4) by q-HAM when $\delta = 7, h > -1, Re = 5$ *and* $Ma = 0.1$.

$$u(x,y,t) = 32y(x-1)^2x^2\left(y^2-\frac{1}{2}\right)-\left(\frac{64}{175}\right)x^3y^2(8y^6-6y^4-y^2+3)(3x-2)(x-1)t-\left(\frac{2304}{765625}\right)t\Bigg((25(x-1))\left(t\left(y^4-\left(\frac{1}{3}\right)y^2+\frac{1}{6}\right)\left(y^2-\frac{1}{6}\right)x^9-5t\left(y^4-\left(\frac{1}{3}\right)y^2+\frac{1}{6}\right)\left(y^2-\frac{1}{6}\right)x^8-\left(\frac{47}{3}\left(y^8-\right.\right.\right.$$

$$\Big(\frac{224}{141}\Big) y^6 + \Big(\frac{49}{94}\Big) y^4 - \Big(\frac{11}{94}\Big) y^2 + \frac{1}{141}\Big)\Big) t x^7 + 52t \Big(y^8 - \Big(\frac{139}{117}\Big) y^6 + \Big(\frac{1}{3}\Big) y^4 - \Big(\frac{4}{117}\Big) y^2 - \frac{1}{234}\Big) x^6 + \Big(\frac{28}{3}\Big(y^{10} -$$

$$\Big(\frac{83}{8}\Big) y^8 + \Big(\frac{863}{84}\Big) y^6 - \Big(\frac{433}{112}\Big) y^4 + \Big(\frac{227}{336}\Big) y^2 + \frac{13}{336}\Big)\Big) t x^5 - \Big(\frac{52}{3}\Big(y^{10} - \Big(\frac{1933}{312}\Big) y^8 + \Big(\frac{293}{52}\Big) y^6 - \Big(\frac{607}{208}\Big) y^4 +$$

$$\Big(\frac{147}{208}\Big) y^2 + \frac{5}{624}\Big)\Big) t x^4 + \Big(\frac{38}{3}\Big) t y^2 \Big(y^8 - \Big(\frac{1211}{228}\Big) y^6 + \Big(\frac{92}{19}\Big) y^4 - \Big(\frac{421}{152}\Big) y^2 + \frac{317}{456}\Big) x^3 + \Big(\Big(\frac{53}{2}\Big) t y^8 - \Big(\frac{212}{9}\Big) t y^6 +$$

$$\Big(\frac{383}{36}\Big) t y^4 - \Big(\frac{73}{36}\Big) t y^2 - \Big(\frac{64}{9}\Big) t y^{10} + \Big(\frac{233}{2}\Big) y^7 - \Big(\frac{699}{8}\Big) y^5 + \Big(\frac{699}{16}\Big) y - \Big(\frac{233}{16}\Big) y^3\Big) x^2 + \Big(\frac{25}{9}\Big) y \Big(t y^9 - \Big(\frac{5}{2}\Big) t y^7 +$$

$$2t y^5 - \Big(\frac{699}{25}\Big) y^6 - \Big(\frac{1}{2}\Big) y^3 t + \Big(\frac{2097}{100}\Big) y^4 + \Big(\frac{699}{200}\Big) y^2 - \frac{2097}{200}\Big) x - \Big(\frac{1}{3}\Big) (y-1)^2 t \Big(y^2 - \frac{1}{2}\Big) (y+1)^2 y^4\Big) y x^2 -$$

$$\Bigg(\frac{1}{4}\Bigg(\Big(y^2 - \frac{1}{6}\Big) x^9 - \Big(\Big(\frac{9}{2}\Big) y^2 - \frac{3}{4}\Big) x^8 + \Big(59 y^4 - \Big(\frac{34}{3}\Big) y^2 + \frac{1}{9}\Big) x^7 + \Big(-\frac{35}{9} - \Big(\frac{413}{2}\Big) y^4 + \Big(\frac{182}{3}\Big) y^2\Big) x^6 +$$

$$\Big(-\Big(\frac{350}{3}\Big) y^6 + \frac{125}{36} + 279 y^4 - \Big(\frac{172}{3}\Big) y^2\Big) x^5 + \Big(\frac{25}{36} + \Big(\frac{1225}{9}\Big) y^6 - \Big(\frac{475}{4}\Big) y^4 - \Big(\frac{175}{12}\Big) y^2\Big) x^4 + \Big(15 y^2 - \Big(\frac{170}{3}\Big) y^8 -$$

$$\frac{35}{36} + 55 y^6 - \Big(\frac{100}{3}\Big) y^4\Big) x^3 + \Big(\Big(\frac{115}{12}\Big) y^2 + \Big(\frac{155}{3}\Big) y^8 - \Big(\frac{415}{6}\Big) y^6 + \Big(\frac{235}{12}\Big) y^4\Big) x^2 - \Big(\frac{20}{3}\Big(y^6 - \Big(\frac{5}{24}\Big) y^4 - \Big(\frac{31}{48}\Big) y^2 +$$

$$\frac{25}{48}\Big)\Big) y^2 x - \Big(\frac{5}{6}\Big) y^4 (y-1)^2 \Big(y + \Big(\frac{1}{16}\Big(x^2 + \Big(\frac{1}{6}\Big) y^2 - x + \frac{1}{12}\Big)\Big) t y\Big)^2\Bigg)\Bigg) t\Bigg) \qquad (20)$$

and

$$\mathrm{v(x,y,t)} = -\Big(32\Big(x-\frac{1}{2}\Big)\Big) x(x-1)(y-1)y^2(y+1) + \Big(\frac{192}{875}\Big) t \Big(\Big(5\Big(y^4 - \Big(\frac{1}{3}\Big) y^2 + \frac{1}{6}\Big)\Big) y x^8 - \Big(20\Big(y^4 - \Big(\frac{1}{3}\Big) y^2 + \frac{1}{6}\Big)\Big) y x^7 +$$

$$(-60y^7 + 70y^5 - 10y^3) x^6 + \Big(\Big(\frac{380}{3}\Big) y^7 - 110 y^5 + 10 y^3 + \Big(\frac{20}{3}\Big) y - \frac{1}{20}\Big) x^5 + \Big(-\Big(\frac{230}{3}\Big) y^7 + 70 y^5 - 10 y^3 - \Big(\frac{25}{6}\Big) y +$$

$$\frac{1}{8}\Big) x^4 + \Big(\Big(\frac{40}{3}\Big) y^7 - 20 y^5 + \Big(\frac{20}{3}\Big) y^3 - y^2 + \frac{1}{12}\Big) x^3 + \Big(-\Big(\frac{10}{3}\Big) y^7 + 5 y^5 - \Big(\frac{5}{3}\Big) y^3 + \Big(\frac{3}{2}\Big) y^2 - \frac{1}{4}\Big) x^2 + \Big(-\Big(\frac{1}{4}\Big) y^4 - \Big(\frac{1}{4}\Big) y^2 +$$

$$\frac{1}{12}\Big) x - \Big(\frac{1}{8}\Big) y^2 + \Big(\frac{1}{8}\Big) y^4\Big) + \Big(\frac{1536}{153125}\Big) t \Bigg(\Bigg(25\Big(\Big(y^4 - \Big(\frac{1}{5}\Big) y^2 + \frac{1}{30}\Big)(y-1)t(y+1) y x^8 - \Big(\frac{9}{2}\Big(y^4 - \Big(\frac{1}{5}\Big) y^2 + \frac{1}{30}\Big)\Big)(y-1)t(y+$$

$$1) y x^7 - \Big(\frac{52}{5}\Big) t y \Big(y^8 - \Big(\frac{133}{52}\Big) y^6 + \Big(\frac{85}{52}\Big) y^4 - \Big(\frac{193}{624}\Big) y^2 + \frac{25}{624}\Big) x^6 + \Big(\Big(\frac{128}{5}\Big) t y^9 - \Big(\frac{829}{15}\Big) t y^7 + \Big(\frac{175}{6}\Big) t y^5 - \Big(\frac{59}{10}\Big) y^3 t +$$

$$\Big(\frac{699}{160}\Big) y^4 - \Big(\frac{233}{160}\Big) y^2 + \Big(\frac{11}{15}\Big) t y + \frac{233}{320}\Big) x^5 + \Big(-\Big(\frac{12}{5}\Big) t y^{11} - \Big(\frac{271}{15}\Big) t y^9 + \Big(\frac{469}{10}\Big) t y^7 - \Big(\frac{811}{30}\Big) t y^5 + \Big(\frac{79}{12}\Big) y^3 t - \Big(\frac{2097}{160}\Big) y^4 +$$

$$\Big(\frac{699}{160}\Big) y^2 - \Big(\frac{13}{20}\Big) t y - \frac{699}{320}\Big) x^4 + \Big(4 t y^{11} + \Big(\frac{44}{15}\Big) t y^9 - \Big(\frac{589}{30}\Big) t y^7 + \Big(\frac{227}{15}\Big) t - \Big(\frac{2097}{40}\Big) y^6 + \Big(\frac{7689}{160}\Big) y^4 - \Big(\frac{699}{160}\Big) y^2 - \Big(\frac{259}{60}\Big) y^3 t +$$

$$\Big(\frac{13}{60}\Big) t y - \frac{699}{320}\Big) x^3 + \Big(-2 t y^{11} + \Big(\frac{1}{30}\Big) t y^9 + \Big(\frac{353}{60}\Big) t y^7 - t y^5 + \Big(\frac{233}{4}\Big) y^6 - \Big(\frac{7689}{160}\Big) y^4 + \Big(\frac{699}{160}\Big) y^2 + \Big(\frac{91}{60}\Big) y^3 t + \frac{233}{64}\Big) x^2 +$$

$$\Big(\frac{4}{15}(y-1)\Big)(y+1)\Big(t y^7 + \Big(\frac{3}{2}\Big) t y^5 - \Big(\frac{5}{2}\Big) y^3 t + \Big(\frac{7}{8}\Big) t y - \Big(\frac{2097}{64}\Big) y^2 + \frac{2097}{128}\Big) y^2 x + \Big(\frac{233}{80}\Big) y^6 +$$

$$\Big(\frac{233}{160}\Big) y^2 - \Big(\frac{699}{160}\Big) y^4\Big)\Big)(x-1) y x^2 + \Big(\frac{9}{5}\Big(y^2 - \frac{11}{72}\Big)\Big) t y x^8 - \Big(\frac{36}{5}\Big(y^2 - \frac{11}{72}\Big)\Big) t y x^7 - \Big(\frac{85}{8}\Big) t y \Big(y^4 - \Big(\frac{1613}{1275}\Big) y^2 + \frac{233}{2550}\Big) x^6 +$$

$$\Big(5\Big(\Big(\frac{143}{40}\Big) t y^5 - \Big(\frac{91}{40}\Big) y^3 t - \Big(\frac{13}{80}\Big) t y - \frac{699}{3200}\Big)\Big) x^5 + \Big(5\Big(-2 t y^7 + \Big(\frac{33}{80}\Big) t y^5 + \Big(\frac{139}{240}\Big) y^3 t + \Big(\frac{13}{96}\Big) t y + \frac{699}{1280}\Big)\Big) x^4 +$$

$$\Big(5\Big(\Big(\frac{8}{3}\Big) t y^7 - \Big(\frac{23}{10}\Big) t y^5 - \Big(\frac{699}{160}\Big) y^2 + \Big(\frac{1}{30}\Big) y^3 t + \Big(\frac{1}{10}\Big) t y + \frac{233}{640}\Big)\Big) x^3 + \Big(5\Big(-\Big(\frac{9}{160}\Big) t y - \Big(\frac{4}{5}\Big) t y^7 + \Big(\frac{41}{80}\Big) t y^5 + \Big(\frac{2097}{320}\Big) y^2 +$$

$\left(\frac{1}{16}\right)y^3 t-\frac{699}{640}\right)\Big)x^2+\left(5\left(-\left(\frac{699}{640}\right)y^4-\left(\frac{699}{640}\right)y^2+\frac{233}{640}\right)\right)x+\left(\frac{1}{48}\left(y^3 t-\left(\frac{1}{2}\right)ty+\frac{2097}{16}\right)\right)(y-1)(y+$
$1)y^2-\left(\frac{1}{64}\left(x-\frac{1}{2}\right)\right)t\left(x^2+\left(\frac{12}{5}\right)y^2-x-\frac{2}{5}\right)\Bigg)$

by applying the Laplace transform to $u(x,y,t)$, $v(x,y,t)$ in equation (20), we have

$$U(x,y,s)=\frac{8}{765625s^3}(100[(-1344x^6+384x^5-432x^4-2848x^3-1424x^2+448x-48)x^2y^{11}+(2256x^8-9744x^7+2143x^6-29408x^5-25152x^4-13504x^3+4816x^2-1120x+120)x^2y^9+(-16788sx^3+27980sx^2-11192sx+47.7)x^2y^8+(-144x^{10}+864x^9-2704x^7+27872x^5+896x^3-4192x^2-96)x^2y^7+(12591sx^5+196x^4+79.2x^3-99.6x^2+2x+2.4)xy^6+(72x^{10}+432x^9+1536x^8-3672x^7+692x^6-12480x^5+12336x^4-6584x^3+1732x^2-224x)x^2y^5+(401.76x^5+2098.5sx^5+171x^4-48x^3+28.2x^2+6.2x-1.2)y^4+(-23x^{12}+192x^{11}-242x^{10}+520x^9-1164x^8+2672x^7-3032x^6+1560x^5-30625s^2x^4+6125s^2x^3-292x^4-0.06)y^3+(1.44x^9-6.48x^8-16.23x^7+87.36x^6-6295.5sx^5 10492.5sx^4-82.56x^5+21.6x^3-4197sx^3+13.8x^2-5x)y^2+(-24x^{11}+36x^{10}+16x^9-84x^8+72x^7-20x^6-15312.5s^2x^4+30625s^2x^3-15312.5s^2x^2-0.36x^2+0.36x-0.03)y]-24x^9+108x^8+16x^7-560x^6+100x^4-140x^3-120y^8$$
(21)

$$V(x,y,s)=\frac{2}{765625s^3}(100[(-4608x^7+12288x^6-11520x^5+4352x^4-512x^3)y^{12}+(-19968x^9+69120x^8-83840x^7+40320x^6-5568x^5+1920x^4-256x^3)y^{10}+(1920x^{11}-10560x^{10}+59712x^9-157184x^8+19616x^7-127744x^6+48992x^5-13344x^4+2058x^3)y^8+(-2304x^{11}+12672x^{10}-43008x^9+88640x^8-107904x^7+80960x^6-39488x^5+12160x^4-33576x^3)y^6+(-100728sx^6+212648sx^5-128708sx^4-768x^4+22384sx^3-5596sx^2+1024x^3-307.2x^2+1.6)y^7+(8394sx^8-33576sx^7+1175160sx^6-816x^6-184668sx^5+1372.8x^5+117516sx^4+158.4x^4-33576sx^3-88320x^3-2.4+19680x^2+8394sx^2)y^5+(44800x^{11}-2464x^{10}+8192x^9-17504x^8+23968x^7-20928x^6+11200x^5-3360x^4-122500s^2x^3+183750s^2x^2-61250s^2x+448x^3-419.7sx+209.85s)y^4+(-2798sx^8+13824x^8+11192sx^7-55296x^7-16788sx^6+103232x^6+16788sx^5-873.6x^5-16788sx^4+22240x^4+11192sx^3-2798sx^2)y^3+(-64x^{11}+352x^{10}-1088x^9+2208x^8-2656x^7+1664x^6-416x^5+122500s^2x^3-167880sx^3+1280x^3+251820sx^2+61250s^2x-20985s-288x)y^2+1399sx^8y-2112x^8y-5596sx^7y+8448x^7y+11192sx^5y-7456x^6y-183750s^2x^2-8394sx^5-699500sx^4y-6240x^5y+20985sx^4+5200x^4y+2400x^2y^3+13990sx^3-41970sxy^2+3840x^3y-41970sx^2-120x^3-2160x^2y+80y^3+13990sx+180x^2+144y^2-12y-24)$$

The Padé approximate for $U(x,y,s),V(x,y,s)$ in equation (21), then

$$U_{[5,0]}(x,y,s)=\frac{(32x^4y^3-64x^3y^3-16x^4y+32x^2y^3+32x^3y-16x^2y)}{s}+\left(\left(\frac{1}{30625}\right)(-134304y^2+44768y^4+268608y^6-268608y^8)x^3-\left(\frac{1}{6125}\right)(-53721y^2+22384y^4+134304y^6-179072y^8)x^4-\left(\frac{1}{30625}\right)(201456y^2-53721y^4-402912y^6-537216y^8)x^5\right)/s^2+\big([(-1.4043x^8+4.0204x^7-4.5139x^6+0.2976x^5-1.4879x^4+0.05022x^2+0.4681x^3)y^{11}+(2.3573x^{10}-10.1814x^9+22.3943x^8-30.7283x^7+26.2813x^6-14.1103x^5+0.7189x^4-1.1702x^3+0.1253)y^9+(128x^{12}-768x^{11}+1152x^{10}+512x^9+2304x^7)y+(-4608x^{12}+27648x^{11}-137728x^{10}-726528x^8+891904x^7-732672x^6+391168x^5-134144x^4)y^7+(2304x^{12}-13824x^{11}-49152x^{10}-117504x^9+246144x^8-79872x^7+394752x^6-210688x^5+55424x^4)y^5]+(0.54x^9+0.2006x^{11}-0.4430x^{10}-0.0334x^{12}-1.2162x^8+2.7919x^7-$$

$3.168x^6+1.300x^5-0.3051x^4-0.00001)y^3+13.0403x^9y^7-0.0208x^6y-3.13e-5y+(0.0015x^9-0.0067x^8-0.0170x^7+0.0912x^6-0.0862x^5-0.0219x^4+0.0225x^3)y^2+0.0777x^2y^8-0.0100y^8x-0.1040x^2y^6+0.0020y^6x-0.0877yx^8-0.1755y^6x^5-0.0852y^8x^3+0.2048y^6x^4+0.0827y^6x^3-0.0012y^8+0.0025y^6-0.0003x^2y+0.0003xy+0.9362x^3y^7-0.8474x^2y^7-0.2340x^3y^5+0.2118x^2y^5++0.0144y^2x^2-0.0052y^2x-0.0003x^3+0.0010x^4+0.0052x^5-0.0058x^6+0.0001x^7+0.0011x^8-0.0003x^9\big)/s^3$ (22)

and

$$V_{[5,0]}(x,y,s)=\frac{(-32x^3+48x^2-16x)y^4+(32x^3-48x^2+16x)y^2}{s}$$
$$+(0.0365x+(0.3654y+2.1927y^5-0.7309y^3)x^8-26.3126y^7x^6-0.0219x^5+(-8.7708y^5+2.9236y^3-1.4618y)x^7+55.5488x^5y^7+30.6980x^6y^5-33.626x^4y^7-48.2398x^5y^5+5.8472x^3y^7-4.3854x^6y^3+30.6980x^4y^5-1.4618x^2y^7+4.3854x^5y^3-8.7708x^3y^5+2.1927x^2y^5+2.9236x^5y-0.4385x^3y^2-0.1096xy^4+0.0548y^4+0.6578y^2x^2-0.1096y^2x+0.054x^4+0.0365x^3-4.3854x^4y^3+2.9236x^3y^3-1.8272x^4y-0.7309x^2y^3-0.1096x^2-0.0548y^2)/s^2$$
$$+(-0.0194x^6y-03.13e-5x-0.2842x^9y^2+0.5767x^8y^2+6.2610x^7y^4-0.6938x^7y^2-5.4669x^6y^4+0.4346x^6y^2-0.0055yx^8+0.0361y^3x^8+12.7979y^8x^5-3.4857y^8x^4-10.3152y^6x^5+0.5349y^8x^3+3.1764y^6x^4+2.9257y^4x^5-0.4513y^6x^3-0.8777y^4x^4-0.1086y^2x^5-0.0006y^5+0.0004y^7-0.2131x^6y^5-0.1996x^4y^7-0.1444x^7y^3+0.3586x^5y^5+0.2674x^3y^7+0.2696x^6y^3+0.0413x^4y^5-0.0802x^2y^7+0.0220x^7y-0.2282x^5y^3-0.2307x^3y^5+0.0163x^2y^5-0.0163x^5y+0.1170x^3y^4-0.0007y^2x+0.0002y^3-0.0003x^3+0.0580x^4y^3+0.0033x^3y^3+0.0135x^4y+0.0062x^2y^3+0.0100x^3y+0.0004x^2+0.003y^2-0.0056x^2y+0.0501x^4y^{10}-0.0167x^{11}y^2+2.1399x^9y^4-28.1871x^7y^6-0.0668x^3y^{10}+0.0919x^{10}y^2-41.0603x^8y^8+10.5325x^6y^{10}+1.1368x^4y^{12}+0.1170x^{11}y^4-11.2347x^9y^6+51.2417x^7y^8-1.4544x^5y^{10}-0.1337x^3y^{12}-0.6436x^{10}y^4+23.1549x^8y^6-33.3698x^6y^8-5.2161x^9y^{10}-1.2037x^7y^{12}-2.7585x^{10}y^8+18.0558x^8y^{10}+3.2099x^6y^{12}-0.6018x^{11}y^6+15.5982x^9y^8-21.9010x^7y^{10}-3.0093x^5y^{12}+3.3102x^{10}y^6+0.5015x^{11}y^8-4.5724x^8y^4+21.1487y^6x^6-6.26e-5)/s^3$$

Applying invers laplace transform of $U_{[5,0]}(x,y,s)\ and\ V_{[5,0]}(x,y,s)$ in equation (22) to obtain on the approximate solution of the velocity $u(x,y,t) and\ v(x,y,t)$.

$u(x,y,t)=32x^4y^3-64x^3y^3-16x^4y+32x^2y^3+32x^3y-16x^2y+\left(\frac{22384}{30625}(-24x^5y^8+40x^4y^8+18x^5y^6-16x^3y^8-30x^4y^6+3x^5y^4+12x^3y^6-5x^4y^4-9x^5y^2+2x^3y^4+15x^4y^2-6x^3y^2)\right)t+\left(\frac{4}{765625}(-14400x^{12}y^7+225600x^{10}y^9-134400x^8y^{11}+86400x^{11}y^7-974400x^9y^9+384000x^7y^{11}+7200x^{12}y^5-430400x^{10}y^7+2143200x^8y^9-432000x^6y^{11}-43200x^{11}y^5+1248000x^9y^7-2940800x^7y^9+284800x^5y^{11}-3200x^{12}y^3+153600x^{10}y^5-2270400x^8y^7+2515200x^6y^9-142400x^4y^{11}+19200x^{11}y^3-367200x^9y^5+2787200x^7y^7-1350400x^5y^9+44800x^3y^{11}+400x^{12}y-42400x^{10}y^3+769200x^8y^5-2289600x^6y^7+481600x^4y^9-4800x^2y^{11}-2400x^{11}y+52000x^9y^3-1248000x^7y^5+1222400x^5y^7-112000x^3y^9+3600x^{10}y+144x^9y^2-116400x^8y^3+8496x^7y^4+1233600x^6y^5-16800x^5y^6-419200x^4y^7-8160x^3y^8+12000x^2y^9+1600x^9y-648x^8y^2+267200x^7y^3-29736x^6y^4-658400x^5y^5+19600x^4y^6+89600x^3y^7+7440x^2y^8-24x^9-8400x^8y-1632x^7y^2-303200x^6y^3+40176x^5y^4+173200x^4y^5+7920x^3y^6-9600x^2y^7-960xy^8+108x^8+7200x^7y+8736x^6y^2+156000x^5y^3-17100x^4y^4-22400x^3y^5-9960x^2y^6-120y^8+16x^7-$

$2000x^6y - 8256x^5y^2 - 29200x^4y^3 - 4800x^3y^4 + 2400x^2y^5 + 200xy^6 - 560x^6 - 2100x^4y^2 + 2820x^2y^4 + 240y^6 + 500x^5 + 2160x^3y^2 + 620xy^4 + 100x^4 + 1380x^2y^2 - 120y^4 - 140x^3 - 36x^2y - 500xy^2 - 6y^3 + 36xy - 3y)\Big)t^2$

and

$$
\begin{aligned}
v(x,y,t) = {} & -32x^3y^4 + 48x^2y^4 + 32x^3y^2 - 16xy^4 - 48y^2x^2 + 16y^2x \\
& + \frac{2798}{765625}(600x^8y^5 - 7200x^6y^7 - 2400x^7y^5 + 15200x^5y^7 - 200x^8y^3 + 8400x^6y^5 \\
& - 9200x^4y^7 + 800x^7y^3 - 13200x^5y^5 + 1600x^3y^7 + 100x^8y - 1200x^6y^3 + 8400x^4y^5 \\
& - 400x^2y^7 - 400x^7y + 1200x^5y^3 - 2400x^3y^5 - 1200x^4y^3 + 600x^2y^5 + 800x^5y \\
& + 800x^3y^3 - 6x^5 - 500x^4y - 120x^3y^2 - 200x^2y^3 - 30xy^4 + 15x^4 + 180x^2y^2 + 15y^4 \\
& + 10x^3 - 30xy^2 - 30x^2 - 15y^2 + 10x)t \\
& + \Bigg(\frac{4}{765625}(48000x^{11}y^8 - 499200x^9y^{10} - 115200x^7y^{12} - 264000x^{10}y^8 + 1728000x^8y^{10} \\
& + 307200x^6y^{12} - 57600x^{11}y^6 + 1492800x^9y^8 - 2096000x^7y^{10} - 288000x^5y^{12} \\
& + 316800x^{10}y^6 - 3929600x^8y^8 + 1008000x^6y^{10} + 108800x^4y^{12} + 11200x^{11}y^4 \\
& - 1075200x^9y^6 + 4904000x^7y^8 - 139200x^5y^{10} - 12800x^3y^{12} - 61600x^{10}y^4 \\
& + 2216000x^8y^6 - 3193600x^6y^8 + 4800x^4y^{10} - 1600x^{11}y^2 + 204800x^9y^4 - 2697600x^7y^6 \\
& + 1224800x^5y^8 - 6400x^3y^{10} + 8800x^{10}y^2 - 437600x^8y^4 + 2024000x^6y^6 - 333600x^4y^8 \\
& - 27200x^9y^2 + 3456x^8y^3 + 599200x^7y^4 - 20400x^6y^5 - 987200x^5y^6 - 19200x^4y^7 \\
& + 51200x^3y^8 + 55200x^8y^2 - 13824x^7y^3 - 523200x^6y^4 + 34320x^5y^5 + 304000x^4y^6 \\
& + 25600x^3y^7 - 528x^8y - 66400x^7y^2 + 25808x^6y^3 + 280000x^5y^4 + 3960x^4y^5 \\
& - 43200x^3y^6 - 7680x^2y^7 + 2112x^7y + 41600x^6y^2 - 21840x^5y^3 - 84000x^4y^4 \\
& - 22080x^3y^5 - 1864x^6y - 10400x^5y^2 + 5560x^4y^3 + 11200x^3y^4 + 4920x^2y^5 + 40y^7 \\
& - 1560x^5y + 320x^3y^3 + 1300x^4y + 600x^2y^3 - 60y^5 + 960x^3y - 30x^3 - 540x^2y - 72xy^2 \\
& + 20y^3 + 45x^2 + 36y^2 - 3x - 6)\Bigg)t^2
\end{aligned}
$$

The graphs and tables shows the finding that were achieved after the problem was solved.

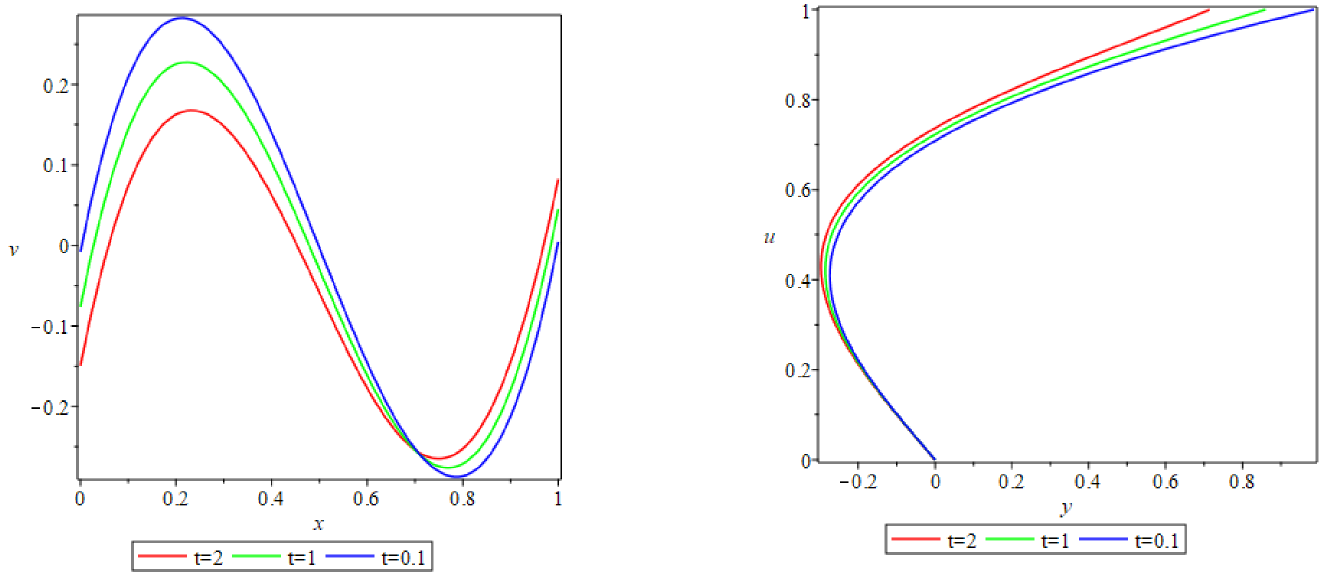

Fig 1: The approximate solutions by using q- HALPM of $u\ (y, 0.5, , t)$ and $v\ (x\ , 0.5, t)$ with $Re\ = 10\ and\ Ma = 0.01$

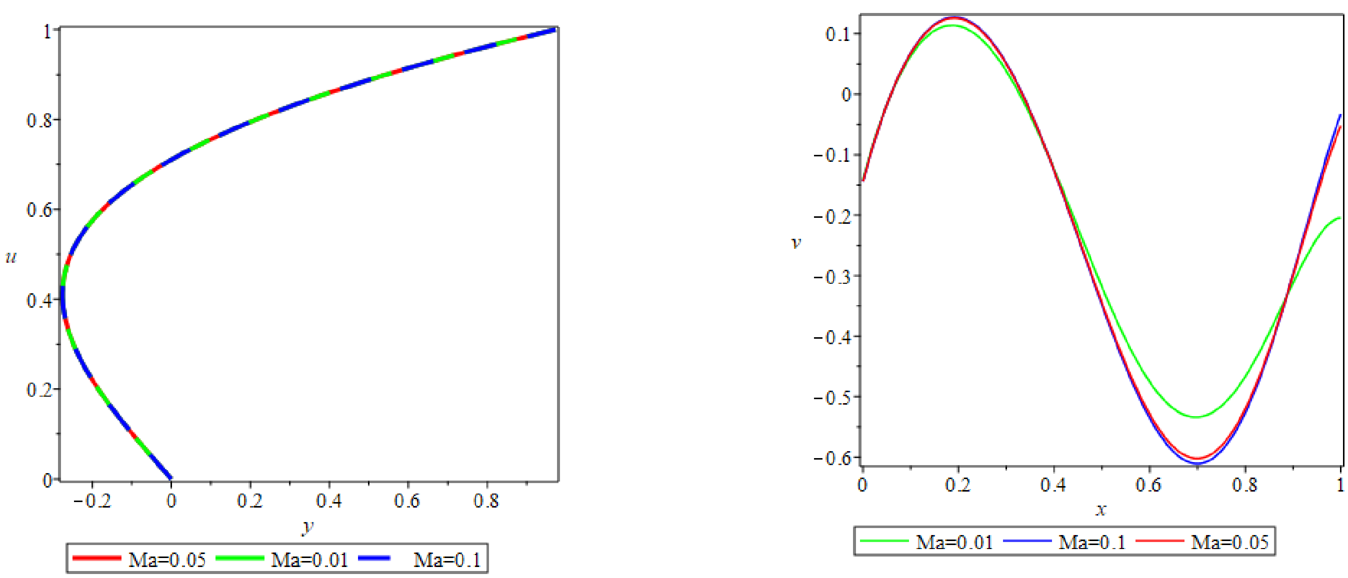

Fig 2:Aproximate solutions by using q- HALPM of $u\ (y, 0.5, ,2)$ and $v\ (x\ ,0.5,2)$ with $Re\ = 1$

Table 1. Comparison of the value of u and v between q-HALPM$[1/5]$, KRDTM , and KPIA at $t = 0.1$,

	q-HALPM		**KRDTM[1]**		**KPIA[1]**	
$u\ (0.5, y, 0$	$R = 10$	$R = 100$	$R = 10$	$R = 100$	$R = 10$	$R = 100$
0.0625	−0.062011719	−0.06201171	−0.06201162	−0.062011717	−0.06201162	−0.062011717
0.125	−0.121093641	−0.12109340	−0.12109341	−0.121093738	−0.12109341	−0.121093738
0.1875	−0.174316400	−0.17431649	−0.17431585	−0.174316394	−0.17431586	−0.174316394
0.25	−0.218750011	−0.21874998	−0.21874926	−0.251464874	−0.21874926	−0.218750003
0.3125	−0.251464850	−0.25146493	−0.25146396	−0.251464874	−0.25146396	0.2514648747
0.375	−0.269531394	−0.26953158	−0.26953025	−0.269531313	−0.26953025	−0.269531313
0.4375	−0.270019567	−0.27001919	−0.26953025	−0.270019617	−0.27001842	−0.270019617
0.5	−0.249999949	−0.24999949	−0.24999879	−0.250000079	−0.24999879	−0.250000079
0.5625	−0.206544493	−0.20654463	−0.20654166	−0.206543002	−0.20654165	−0.206543002
0.62	−0.136703246	−0.13671460	−0.13671735	−0.136718718	−0.13671735	−0.136718718
0.6875	−0.035307932	−0.03640696	−0.03759625	−0.037597612	−0.03759625	−0.037597613
0.75	0.0941705549	0.09395389	0.0937511524	0.0937498465	0.0937511505	0.0937498462
0.8125	0.2600880272	0.259873830	0.2602543372	0.2602531196	0.2602543328	0.2602531191
0.875	0.4649360750	0.464891237	0.464842753	0.4648416627	0.4648427455	0.4648416618
0.9375	0.7101083799	0.710294142	0.7104460577	0.7104451334	0.7104460475	0.7104451322
$v(x, 0.5, 0.1)$						
0.0625	0.1487649349	0.1532047528	0.152035224	0.1535990206	0.1520352173	0.1535990205
0.125	0.2410686981	0.2452113944	0.244369877	0.2458276287	0.2443698698	0.2458276285
0.1875	0.2804785725	0.2842108297	0.284023378	0.2853353452	0.2840233713	0.2853353451
0.25	0.2756046397	0.2788447849	0.279792490	0.2809293365	0.2797924846	0.2809293364
0.3125	0.2350926028	0.2377756304	0.240457582	0.2413982264	0.2404575786	0.2413982263
0.375	0.167710824	0.169801597	0.174789051	0.1755190079	0.1747890492	0.1755190079
0.4375	0.0827724218	0.083957753	0.091554664	0.0920648498	0.0915546634	0.0920648498
0.5	−0.00000	−0.00000000	−0.00047400	−0.00018802	−0.00047399	−0.000188024
0.5625	−0.082501182	−0.083197523	−0.092517322	−0.09245569	−0.09251732	−0.092455695
0.625	−0.183003438	−0.185728467	−0.175786459	−0.17594492	−0.17578645	−0.175944925
0.6875	−0.241638609	−0.255605798	−0.241485322	−0.24185484	−0.24148532	−0.241854848

0.75	−0.286746272	−0.233517415	−0.2808158	−0.28138169	−0.28081584	−0.281381696
0.8125	−0.298133138	−0.254397093	−0.2849851	−0.28572579	−0.28498514	−0.285725791
0.875	−0.25068970	−0.249412001	−0.2452139	−0.24609948	−0.24521391	−0.24609948
0.9375	−0.113146846	−0.154960887	−0.1527436	−0.15373414	−0.15274366	−0.153734143

Table.2. Comparison between the approximate solutions of q-HALPM[1/5], Ref[9], KPA, and KRDTM at $t = 0.1$, and $Re = 1$.

	FVM[9]	**KRDTM [1]**	**KPIA[1]**	**q- HALP**
$\psi_{\min}$	−0.125	−0.1263030704	−0.1262878352	−0.1249479449
$x(\psi_{\min})$	0.5	0.5	0.5	0.5
$y(\psi_{\min})$	0.70703	0.70703125	0.703125	0.7142857141
$u_{\min}$	−0.2721659	−0.2720274424	−0.2720273443	−0.2711370241
$y(u_{\min})$	0.40869	0.41015625	0.40625	0.4285714285
$v_{\min}$	−0.2886756	−0.365196882	−0.3649970593	−0.2886297273
$x(v_{\min})$	0.78857	0.78515625	0.78125	0.7857142855
$v_{\max}$	0.2886756	0.3590485063	0.3588814265	0.2886296968
$x(v_{\max})$	0.21143	0.21484375	0.21875	0.2142857143
$u(0.5; 0.0625)$	−0.0620117187	−0.0619894102	−0.0619894551	−0.06201171355
$u(0.5; 0.125)$	−0.1210937499	−0.1210481233	−0.1210481633	−0.1210937383
$u(0.5; 0.1875)$	−0.1743164062	−0.1742486127	−0.1742486451	−0.1743164038
$u(0.5; 0.25)$	−0.2187499999	−0.2186617447	−0.2186617667	−0.2187500060
$u(0.5; 0.3125)$	−0.2514648436	−0.2513583768	−0.2513583858	−0.2514648229
$u(0.5; 0.375)$	0.2695312499	−0.2694093550	−0.2694093550	−0.2695312564
$u(0.5; 0.4375)$	−0.2700195312	−0.2698855135	−0.2698855135	−0.2700195660
$u(0.5; 0.5)$	−0.2500000000	−0.2498576781	−0.2498576781	−0.2499999628
$u(0.5; 0.5625)$	−0.2065429687	−0.2063966790	−0.2063966790	−0.2065430413
$u(0.5; 0.625)$	−0.1367187500	−0.1365733705	−0.1365733705	−0.1367051875
$u(0.5; 0.6875)$	−0.0375976562	−0.0374586584	−0.0374586584	−0.03609624939
$u(0.5; 0.75)$	0.0937499999	0.0938764774	0.0938764774	0.09412961961
$u(0.5; 0.8125)$	0.2602539062	0.2603609875	0.2603609875	0.2603581099
$u(0.5; 0.875)$	0.4648437499	0.4649238234	0.4649238234	0.4641393333
$u(0.5; 0.9375)$	0.7104492187	0.7104941432	0.7104941432	0.710398964
$v(x, 0.5, 0.1)$				
$v(0.0625; 0.5)$	0.15380859374	0.1365030994	0.1365023819	0.1365020622
$v(0.125; 0.5)$	0.24609374999	0.2298505049	0.2298497694	0.2298500264
$v(0.1875; 0.5)$	0.28564453123	0.2709291888	0.2709285329	0.2709289686
$v(0.25; 0.5)$	0.28124999999	0.2684293744	0.2684288605	0.2684292722
$v(0.3125; 0.5)$	0.2416992187	0.2310463824	0.2310460353	0.2310463296
$v(0.375; 0.5)$	0.17578125000	0.1674821344	0.1674819447	0.1674781124
$v(0.4375; 0.5)$	0.09228515625	0.0864478792	0.0864478112	0.08644782510
$v(0.5; 0.5)$	$2.3e-14$	−0.00333375	−0.003333749	−0.00033338999
$v(0.5625; 0.5)$	−0.09228515625	−0.0931287051	−0.0931286946	−0.09312953026
$v(0.625; 0.5)$	−0.17578125000	−0.1741943413	−0.1741943774	−0.1741938245
$v(0.6875; 0.5)$	−0.24169921874	−0.2377853663	−0.2377854934	−0.2377852184
$v(0.75; 0.5)$	−0.28124999998	−0.2751626592	−0.2751629021	−0.2751622548
$v(0.8125; 0.5)$	−0.28564453123	−0.2776041425	−0.2776044993	−0.2776041702
$v(0.875; 0.5)$	−0.24609374999	−0.2364163347	−0.2364167719	−0.2364156354

$v(0.9375; 0.5)$	-0.15380859374	-0.1429446548	-0.1429451057	

Table.3. Comparisons of the $L_\infty - errors$ between Ref [27], KRDTM, KPIA, and q-HALPM at $t = 0.1$ and $Ma = 0.001$.

Grid size	Ref. [27]		KRDTM[1]		KPIA[1]		q-HALPM		PYRDTM[7]	
	ψ	ω	ψ	ω	ψ	ω	ψ	ω	ψ	ω
					$Re = 10$					
21×21	3.23e−7	1.01e-5	2.01e-8	8.13e-7	5.84e-8	1.59e-6	1.56e-11	4.57e-10	2.00e-9	7.72e-9
41×41	2.35e−8	7.74e-7	3.41e-8	9.77e-7	7.23e-8	1.85e-6	1.60e-11	4.57e-10	7.23e-9	5.05e-8
81×81	1.56e−9	5.17e-8	4.46e-8	1.08e-7	9.29e-8	2.00e-6	1.59e-11	4.57e-10	2.47e-8	3.00e-7
					$Re = 100$					
21×21	8.09e−5	4.08e-3	4.25e-9	3.62e-7	4.25e-9	3.62e-7	2.71e-11	1.48e-10	4.69e-11	1.13e-8
41×41	7.12e−6	2.51e-4	5.11e-9	3.62e-7	5.11e-9	3.62e-7	2.71e-11	1.48e-10	3.86e-10	1.67e-8
81×81	4.93e−7	1.72e-5	5.11e-9	3.64e-7	5.66e-9	3.64e-7	2.71e-11	1.48e-10	1.13e-9	1.61e-8
					$Re = 1000$					
41×41	3.32e−4	1.45e-2	5.07e-9	3.65e-7	5.07e-9	3.64e-7	4.46e-10	1.37e-10	1.35e-9	3.60e-9
81×81	3.92e−5	1.66e-3	5.74e-9	3.64e-7	5.74e-9	3.64e-7	4.46e-10	1.37e-10	2.31e-10	2.53e-7
16×161	2.79e−6	1.46e-4	6.11e-9	3.64e-7	6.11e-9	3.64e-7	4.46e-10	1.37e-10	1.96e-9	1.57e-7

4- Results and discussion

In this section, we introduce the numerical computations of the velocity components u, v vorticity function ω, and stream function ψ that were produced by using a new analytical method (q-HALPM) , the results were shown graphically and tabular. The accuracy and efficiency of the current method was verified by comparing it with previous methods. All computations are performed using the Maple 2016 program, which uses different Reynolds numbers and Mach numbers in the domain$[0,1]^2$.

Figures 1and 2 demonstrate the approximate solution of the velocity u, v obtained by using the provided approach at $Re = 1$ and $Ma = 0.001$ and different values of t, Figure 2 at $t = 2$ and $Re = 10$ and various values of the Mach numbers ($Ma = 0.1,0.01,0.05$). It is clear from Figure 1 that the increase in values of t leads to an increase in the vertical velocity, but the horizontal velocity decreases, in Figure 2, the horizontal velocity decreases and the vertical velocity remains constant as Mach values increase. In table 1 we compare the results obtained from the suggested method with the methods KRDRM and KPIM of velocity u along the vertical line and v velocity along the horizontal line through the geometric center of the square cavity, the findings demonstrate that these values are identical with values stated in KRDRM and KPIM for various values of Renolds numbers at $t = 0.1$ and $Ma = 0.01$,was q-HALPM solutions from two iterations .Table 2 represented the results u and v that are obtained from q-HALPM with KRDRM, KPIM and finite volume method in Ref[9]. The $L_\infty - errors$ shows in Table 3 for the stream function ψ and the vorticity ω by using the current method and comparted with these provided methods in [1] and[9] and rational fourth-order compact finite difference method in [33] , for three different values of Reynolds numbers $Re = 10,100$ $and 1000$ at $Ma = 0.001$, We see that the number of grid points has no bear on the estimated errors, which are small for all Reynolds number values. The analytical approximate solution produced by the new approach is remarkably exact, as shown by the calculations included in the tables and figures. Additionally, we found that q-HALPM is a practical and efficient method for solving the nonlinear two-dimensional unsteady incompressible Navier-Stokes equations for low Mach numbers and for different Reynolds number values.

5-Convergence Analysis:

We will study the convergence analysis of approximate analytical solutions obtained by an application q-HALPM. The necessary convergence conditions for the series solution are outlined in the defintion and the following theorem:
Theorem.5.1: Assume that $\mathbb{N}$ is a nonlinear operator that satisfies the Lipschitz condition from a Hilbert space H into H. The series solution $\{s_i\}_{i=0}^{\infty}$ convergence when there is γ such that $0 < \gamma < 1, \gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $\|B_{i(k+1)}\| \leq \gamma_i \|B_{i(k)}\|$. (See for proof Ref. [1])

Definition.5.1 [1],[7]: If $i = 1,2,3$ and $k \in \mathbb{N} \cup \{0\}$,we define

$$\gamma_{ik} = \begin{cases} \frac{\|\mathrm{B}_{i(k+1)}\|}{\|\mathrm{B}_{ik}\|}, & \|\mathrm{B}_{ik}\| \neq 0 \\ 0 \quad , & \|\mathrm{B}_{ik}\| = 0 \end{cases}$$

Next, we declare that the series of approximations $\{s_i\}_{i=0}^{\infty}$ convergence to the exact solution u, when $\gamma_k = \gamma_{1k} + \gamma_{2k} + \gamma_{3k}$, and $0 < \gamma_k < 1$ for all $k \in \mathbb{N} \cup \{0\}$.

Table 4. Comparisons of Convergence solutions between q-HALPM, KPIM[1], KRDTM[1] and PYRDTM[7].

Re	Ma	t	Method	γ_0	γ_1
1	0.1	0.1	**KPIA**	0.9991283888	0.7958329986
			KRDTM	0.9991283888	0.7958329986
			PYRDTM	0.9966346559	0.7179153236
			q-HALPM	0.0304613068	0.0388670595
10	0.1	0.01	**KPIA**	0.0129736262	0.2936820858
			KRDTM	0.0129736262	0.2936820858
			PYRDTM	0.0122822273	0.0040514387
			q-HALPM	0.0030011290	0.0030035942

6- Conclusions

In the current study, q-HALPM is successfully applied to find the analytical solution of the two-dimensional incompressible Navier-Stokes equations using a square lid-driven cavity. Through comparing the numerical results of the vertical and horizontal velocities of the other methods together with the results of the current method, the q-HALPM validity is shown. The results reveal that q-HALPM has high accuracy and efficiency in finding the analytical approximate solutions. Even though the current method is intended for two-dimensional flows, a clear path has been laid out for its expansion to three-dimensional problems that will be taken into account in subsequent studies

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