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| RESEARCH ARTICLE

The Generalized Lucas Primes in the Landau's and Shanks' Conjectures

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| ABSTRACT
Landau's conjecture and Shanks' conjecture state that there are infinitely many prime numbers of the forms x^2+1 and x^4+1 for some nonzero integer x, respectively. In this paper, we present a technique for studying whether or not there are infinitely many prime numbers of the form x^2+1 or x^4+1 derived from some Lucas sequences of the first kind $\{U_n(P,Q)\}\,(or\ simply, \{U_n\})$ or the second kind $\{V_n(P,Q)\}\,(or\ simply, \{V_n\})$, where $P \geq 1$ and $Q \in \{-1,1\}$. Furthermore, as applications, we represent the procedure of this technique in the case of $x \in Z$, $x \in \{U_n\}\ or\ x \in \{V_n\}$ with $x \geq 1$ and $1 \leq P \leq 20$.

| KEYWORDS

Lucas sequences, Diophantine equation, Landau's conjecture, Shanks' conjecture, prime numbers.

| ARTICLE INFORMATION

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1. Introduction
In 1912, Edmund Landau [1913] (see also [Landau, 1985]) listed four unsolvable problems related to prime numbers during his talk at the International Congress of Mathematicians at Cambridge University. Indeed, these problems are still unattackable conjectures in number theory, and they are as follows:

> **Goldbach's conjecture**: If *x* is an even integer greater than 2, can every *x* be written as $p+q$ such that *p* and *q* are prime numbers?
> **Twin prime conjecture:** If *q* is a prime number, are there infinitely many *q*'s such that $q+2$ is a prime number?
> **Legendre's conjecture:** If *a* and *b* are consecutive perfect squares, is there always a prime number *q* between *a* and *b*?
> **Landau's conjecture (Near-square primes):** If p is a prime number, are there infinitely many p's such that $p-1$ is a prefect square?

In fact, the latter conjecture can be rephrased as "are there infinitely many primes p such that $p=x^2+1$ for some integer x? However, many authors have attempted to prove these conjectures, but no complete proofs have been produced for all of them" until 2020 when Vega [n.d] proposed a proof for Landau's conjecture to be true. Such attempts or studies related to Landau's conjecture can be found in, e.g. [Shanks, 1959] or [Shanks, 1960]. Similar conjectures regarding infinite numbers of special types of primes were also proposed, such as $p=x^4+1$ that was conjectured in 1961 by Shanks [22]. For simplicity or later use, we call it the Shanks' conjecture. In fact, many authors have tried to find the number of such primes under certain ranges of intervals. For example, in 1967, Lal [1967] reported 172 primes in the case of $1 \leq x \leq 4004$. Later in 1973, Bohman [1973] extended this range to report all the possible primes with $4002 \leq x \leq 10000$. Here, the number of actual primes is 790. Other popular open problems in number theory are related to the prime numbers in some types of linear recurrence sequences. For instance,

Fibonacci and Lucas primes conjecture: Are there infinitely many primes in the Fibonacci sequence $\{F_n\}$ or Lucas sequence $\{L_n\}$?

These sequences are given by the recurrence relations

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \; for \; n \geq 2, \tag{1}$$

and

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \; for \; n \geq 2. \tag{2}$$

However, this conjecture has remained open, and authors have determined the Fibonacci and Lucas primes under certain values of n. In fact, up to now, many Fibonacci and Lucas primes that are formed with thousands of digits have been obtained. For more details, see, for instance, [Dubner 1999], and the references are given there. Indeed, this latter conjecture is also extended by Lawrence and Michal [25] in the case of more general sequences that are respectively called the Lucas sequences of the first kind $\{U_n(P,Q)\}$ or the second kind $\{V_n(P,Q)\}$ which are defined by the relations:

$$U_0(P,Q) = 0, U_1(P,Q) = 1, \; U_n(P,Q) = PU_{n-1}(P,Q) - QU_{n-2}(P,Q), \tag{3}$$

$$V_0(P,Q) = \; 2, V_1(P,Q) = P, \; V_n(P,Q) = PV_{n-1}(P,Q) - QV_{n-2}(P,Q), \tag{4}$$

for $n \geq 2$ where the parameters P and Q are nonzero relatively prime integers. In fact, it is also known that the Lucas sequences of the first and second kind are connected in the identity

$$V_n^2(P,Q) = DU_n^2(P,Q) + 4Q^n, \tag{5}$$

where $D = P^2 - 4Q$. For simplicity, these sequences are also called Lucas sequences, and their numbers are known as the generalized Lucas numbers. Concerning these sequences, we have their characteristics; polynomial is defined by

$$X^2 - PX + Q = 0,$$

where

$$\alpha \; = \frac{P + \sqrt{D}}{2} \; and \; \beta = \frac{P - \sqrt{D}}{2}$$

are the roots of the latter polynomial. Hence, these sequences can also be defined by

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n(P,Q) = \alpha^n + \beta^n \; for \; n \geq 0.$$

Thus, if α/β is not a root of unity, then these sequences are said to be nondegenerate and degenerate otherwise. As a result, they degenerate only with $(P,Q) \in \{(\pm 1, 1), (\pm 2, 1)\}$, for more details, see, e.g. [13] or [19]. Moreover, such results concerning primes in the Lucas sequences, we recommend, e.g. [Leyendekkers, 2015], [Trojovsk´y, 2019 or [Sun, 2010]. In fact, we call this extension by the generalized Lucas primes conjecture.

In this paper, we answer the following question that is derived from combining all of the above mentioned conjectures: Question: Are there also infinitely many generalized Lucas primes of the form $p = x^2 + 1$ or $x^4 + 1$?

In other words, we investigate the nonnegative integer solutions (x, n) of the Diophantine equation

$$p = x^2 + 1 \tag{6}$$

or

$$p = x^4 + 1, \tag{7}$$

where the prime number $p = U_n = U_n(P,Q) \; or \; V_n = V_n(P,Q)$. Clearly, here we have that $x \geq 1$ and $n \geq 2$ or $n \geq 0$ with $p = U_n$ or $p = V_n$, respectively. Furthermore, for the simplicity of presenting our results, we assume these sequences are nondegenerate with $P \geq 1$ and $Q = \pm 1$. We indeed answer this question negatively by showing this equation has only finitely many solutions. As applications, we solve these equations completely where $P \leq 20$ under the following cases: $x \in Z^+, x \in \{U_n\} \; or \; x \in \{V_n\}$.

In the literature, Diophantine equations connected to linear recurrence sequences have been studied by many authors. For instance, by using the congruence argument techniques with some identities related to Lucas sequences, Keskin and Yosma [2011] studied the solutions (x,n) of some equations of the form

$$V_n(P,-1) = kx^2,$$

Where P is odd, and the integer k divides P. They also determined the solutions (x,n) of the equations

$$V_n(P,-1) = 3V_m(P,-1)x^2 \ \ and \ \ V_n(P,-1) = 6V_m(P,-1)x^2,$$

Where P is also an odd integer. Furthermore, if P is even, they also determine the solutions of the equations

$$V_n(P,-1) = 3x^2 \ \ and \ \ V_n(P,-1) = 3V_m(P,-1)x^2.$$

In the case of the Lucas sequences of the first kind, under different assumptions on the values of P, Karaatli and Keskin [10] studied the solutions of the equations

$$U_n(P,-1) = 5x^2 \ \ and \ \ U_n(P,-1) = 5U_m(P,-1)x^2.$$

Another interesting result related to such equations was introduced by Alekseyev and Tengely [2014], in which they described an argument for finding the generalized Lucas numbers, under some conditions, of the form $am^2 + b$ with some fixed integers $a \neq 0$ and b. Their argument is based on reducing such equations to a finite number of Thue equations that have a finite number of solutions. However, this result can answer the question about the generalized Lucas primes in Landau's conjecture negatively; in this paper, we use a straight forward approach, described in Section 2, that proves the finiteness result of equation (6) or equation (7) and solves it completely with the help of the Magma software [5]. Other results related to the solutions of such equations connected to some sequences can be found in, e.g. [Ait-Amrane, 2017], [S¸iar] 2016, [Duman, 2018] and [Kiss, 1993].

2. Main Approach

Our main approach is simply based on combining equation (6) or equation (7) with identity (5) to obtain elliptic curves of the form

$$y^2 = ax^4 + bx^2 + c, \tag{8}$$

where $a,b,c \in Z$ and $\Delta = 16ac(b^2 - 4ac)^2 \neq 0$ defines the discriminant of the curve. Hence, the integral points of such curves can be obtained by either using the Magma software [Bosma, 1963] with the algorithm **SIntegralLjunggrenPoints(),** which is implemented based on Tzanakis' results in [Tzanakis, 1996] or by following an argument described by Tengely and Alekseyev in [Alekseyev, 2014]. Indeed, by the result of Baker [Baker, 1969] (or its refinement in [Hajdu, 1998]) in which upper bounds for the solutions of elliptic curves of the form

$$y^2 = b_0x^m + b_1x^{m-1} + \ldots + b_m,$$

where $m \geq 3$ and $b_0 \neq 0, b_1, \ldots, b_m \in Z$ are given, we conclude the finiteness for the number of solutions of equation (6) or (7).

Remark 1. In general, our described approach can be applied to show the finiteness result for the solutions of the equations

$$U_n(P,\pm1) = x^2 + 1 \ \ and \ \ V_n(P,\pm1) = x^2 + 1$$

or

$$U_n(P,\pm1) = x^4 + 1 \ \ and \ \ V_n(P,\pm1) = x^4 + 1,$$

but in this paper, we only focus on Landau's and Shanks' problems assuming that the general terms of the nondegenerate Lucas sequences $U_n(P,\pm1)$ and $V_n(P,\pm1)$ are prime numbers with $P \geq 1$.

3. Main Results

Theorem 1. Suppose that $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ are nondegenerate Lucas sequences with $P \geq 1$ and $Q \in \{-1,1\}$. If p is a prime number such that $p = U_n(P,Q)$ or $V_n(P,Q)$, then equation (6) (or equation (7)) has finitely many solutions (x,n) that can be determined effectively.

Proof. Note that in the following, we only focus on proving the finiteness result in the case of equation (6) since this result can be applied similarly for equation (7) as it can be written in the form (6), namely $p = x^4 + 1 = (x^2)^2 + 1$. Moreover, as mentioned earlier that we deal here with $x \geq 1$ and $n \geq 2$ or $n \geq 0$ in the case of $p = U_n(P,Q)$ or $p = V_n(P,Q)$, respectively. We split the proof into two cases regarding the Lucas sequences of the first kind or the second kind:

- **Case 1:** If $p = U_n(P,Q)$. Here, equation (6) gives

$$U_n(P,Q) = x^2 + 1.$$

Combining the latter equation with identity (5) leads to the equation

$$y^2 = Dx^4 + 2Dx^2 + (D + 4Q^n), \qquad (9)$$

with $y = V_n$ and $D = P^2 - 4Q$ such that $P \geq 1$ and $Q = \pm 1$. Indeed, we claim that the latter equation presents an elliptic curve. Therefore, in order to prove this claim, we have to show this equation has nonzero discriminants. As mentioned earlier that the elliptic curve of the form (8) has the discriminant

$$\Delta = 16ac(b^2 - 4ac)^2.$$

Hence, the discriminant of equation (9) is presented by

$$\Delta_U = 4096D^3Q^{2n}(D + 4Q^n).$$

If $Q = 1$, then $D = P^2 - 4 > 0$ as $(P,Q) \notin \{(\pm 1, 1), (\pm 2, 1)\}$ since we assumed that the Lucas sequences are nondegenerate. Furthermore, we have that $D + 4Q^n = P^2 - 4 + 4 = P^2 > 0$ as $P \geq 1$. As a result, we obtain that $\Delta_U > 0$. On the other hand, if $Q = -1$, then we get that

$$\Delta_U = 4096(P^2 + 4)^3(P^2 + 4 \pm 4),$$

which is clearly greater than zero as $P \geq 1$. Hence, equation (9) presents an elliptic curve in the case of $p = U_n(P,Q)$, where $n \geq 2$.

- **Case 2:** If $p = V_n(P,Q)$. Similarly, equation (6) becomes

$$V_n(P,Q) = x^2 + 1,$$

which leads to the equation

$$y^2 = Dx^4 + 2Dx^2 + (D - 4DQ^n), \qquad (10)$$

where $y = DU_n, P \geq 1$ and $Q = \pm 1$. Indeed, the discriminant of equation (10) is defined by

$$\Delta_V = 4096D^6Q^{2n}(1 - 4Q^n).$$

Since the Lucas sequences of the second kind are assumed to be nondegenerate, then $D \neq 0$. Therefore, $\Delta_V \neq 0$ with $Q = -1$ or 1 for all $n \geq 0$. Again, we obtain that equation (10) represents an elliptic curve equation.

Finally, as mentioned in Section 2 by the result of Alan Baker [3] and its best improvement by Hajdu and Herendi [9], we conclude that the elliptic curves (9) and (10) have a finite number of solutions. This completes the proof of Theorem 1.

4. Applications

Theorem 2. Let the sequence $\{U_n(P,Q)\}$ be nondegenerate with $1 \leq P \leq 20$ and $Q \in \{-1, 1\}$. If the prime number $p = U_n(P,Q)$, then the complete set of the solutions (P,Q,x,n) with $x \geq 1$ and $n \geq 2$ of equation (6) is given by

$$(P,Q,x,n) \in \{(5,1,2,2), (17,1,4,2), (1,-1,1,3), (1,-1,2,5), (2,-1,$$

$2,3), (2,-1,1,2), (4,-1,4,3), (5,-1,2,2), (6,-1,6,3), (10,-1,10,3),$

$(14,-1,14,3), (16,-1,16,3), (17,-1,4,2), (20,-1,20,3)\}.$

Corollary 1. Suppose that $\{U_n\}$ is nondegenerate. If $p = U_n(P_1, Q_1)$ and $x = U_k(P_2, Q_2)$ such that $n \geq 2, k \geq 1, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then the solutions of equation (6) are completely given by

$$((P_1, Q_1), (P_2, Q_2), n, k) \in \{((5,1), (1,-1), 2, 3), ((5,1), (2,-1), 2,$$

$$2), ((17,1), (4,1), 2, 2), ((17,1), (4,-1), 2, 2), ((1,-1), (A,B), 3, 1),$$

$$((1,-1), (1,-1), 5, 3), ((1,-1), (2,-1), 5, 2), ((2,-1), (A,B), 2, 1),$$

$$((4,-1), (4,1), 3, 2), ((4,-1), (4,-1), 3, 2), ((5,-1), (1,-1), 2, 3),$$

$$((5,-1), (2,-1), 2, 2), ((6,-1), (6,1), 3, 2), ((6,-1), (6,-1), 3, 2),$$

$$((10,-1), (10,1), 3, 2), ((10,-1), (10,-1), 3, 2), ((14,-1), (14,1),$$

$$3, 2), ((14,-1), (14,-1), 3, 2), ((16,-1), (16,1), 3, 2), ((16,-1),$$

$$-1), 3, 2), ((17,-1), (4,1), 2, 2), ((17,-1), (4,-1), 2, 2), ((20,-1),$$

$$(20.1), 3, 2), ((20,-1), (20,-1), 3, 2)\}.$$

Corollary 2. Assume that $\{U_n\}$ and $\{V_n\}$ are nondegenerate. If $p = U_n(P_1, Q_1)$ and $x = V_k(P_2, Q_2)$ such that $n \geq 2, k \geq 0, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then the set of solutions to equation (6) is as follows

$$((P_1, Q_1), (P_2, Q_2), n, k) \in \{\big((5,1), (A,B), 2, 0\big)\big((5,1), (2,-1), 2,1\big),$$

$$\big((17,1), (4,1), 1, 1\big), \big((17,1), (4,-1), 1, 1\big), \big((1,-1), (1,-1), 3, 1\big),$$

$$((1,-1), (1,-1), 5, 0), ((2,-1), (1,-1), 2, 1), ((2,-1), (A,B), 3, 0),$$

$$\big((2,-1), (2,-1), 3, 1\big), \big((4,-1), (4,1), 3, 1\big), \big((4,-1), (4,-1), 3, 1\big),$$

$$((5,-1), (A,B), 2, 0), ((5,-1), (2,1), 2, 1), ((6,-1), (6,1), 3, 1),$$

$$((6,-1), (6,-1), 3, 1), ((10,-1), (10,1), 3, 1), ((10,-1), (10,-1),$$

$$3, 1), ((14,-1), (14,1), 3, 1), ((14,-1), (14,-1), 3, 1), ((16,-1),$$

$$(16,1), 3, 1), ((16,-1), (16,-1), 3, 1), ((17,-1), (4,1), 2, 1), ((17,$$

$$-1), (1.-1), 2, 3), ((17,-1), (4,-1), 2, 1), ((20,-1), (20,1), 3, 1),$$

$$((20,-1), (20,-1), 3, 1)\},$$

for all $1 \leq A \leq 20$ and $B = \pm 1$.

Theorem 3. Let $\{V_n(P,Q)\}$ be a nondegenerate with $n \geq 0, 1 \leq P \leq 20$ and $Q \in \{-1, 1\}$. If p is a prime number such that $p = V_n(P,Q)$, then the complete list of solutions of equation (6) is as follows (assuming that $x \geq 1$)

$$(P, Q, x, n) \in \{(3,1,1,0), (4,1,1,0), (5,1,1,0), (5,1,2,1), (6,1,1,0),$$

$$(7,1,1,0), (8,1,1,0), (9,1,1,0), (10,1,1,0), (11,1,1,0), (12,1,1,0),$$

$$(13,1,1,0), (14,1,1,0), (15,1,1,0), (16,1,1,0), (17,1,1,0), (17,1,4,$$

$1), (18, 1, 1, 0), (19, 1, 1, 0), (20, 1, 1, 0), (1, -1, 1, 0), (2, -1, 1, \{0, 1\}),$

$(3, -1, 1, 0), (4, -1, 1, 0), (5, -1, 1, 0), (5, -1, 2, 1), (6, -1, 1, 0), (7, -1,$

$1, 0), (8, -1, 1, 0), (9, -1, 1, 0), (10, -1, 1, 0), (11, -1, 1, 0), (12, -1, 1,$

$0), (13, -1, 1, 0), (14, -1, 1, 0), (15, -1, 1, 0), (16, -1, 1, 0), (17, -1, 1,$

$0), (17, -1, 4, 1), (18, -1, 1, 0), (19, -1, 1, 0), (20, -1, 1, 0)\}.$

Corollary 3. Suppose that $\{U_n\}$ and $\{V_n\}$ are nondegenerate. If $p = V_n(P_1, Q_1)$ and $x = U_k(P_2, Q_2)$ such that $n \geq 0, k \geq 1, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then the complete solutions to equation (6) are given in the following table:

(P_1, Q_1)	(P_2, Q_2)	n	k		(P_1, Q_1)	(P_2, Q_2)	n	k
(3, 1)	(A, B)	0	1		(2, -1)	(1, -1)	0	2
(3, 1)	(1, -1)	0	2		(2, -1)	(A, B)	1	1
(4, 1)	(A, B)	0	1		(2, -1)	(1, -1)	1	2
(4, 1)	(1, -1)	0	2		(3,-1)	(A, B)	0	1
(5, 1)	(A, B)	0	1		(3,-1)	(1, -1)	0	2
(5, 1)	(1, -1)	0	2		(4, -1)	(A, B)	0	1
(5, 1)	(2, -1)	1	2		(4, -1)	(1, -1)	0	2
(6, 1)	(A, B)	0	1		(5, -1)	(A, B)	0	1
(6, 1)	(1, -1)	0	2		(5, -1)	(1, -1)	0	2
(7, 1)	(A, B)	0	1		(5, -1)	(2, -1)	1	2
(7, 1)	(1, -1)	0	2		(6, -1)	(A, B)	0	1
(8, 1)	(A, B)	0	1		(6, -1)	(1, -1)	0	2
(8, 1)	(1, -1)	0	2		(7, -1)	(A, B)	0	1
(9, 1)	(A, B)	0	1		(7, -1)	(1, -1)	0	2
(9, 1)	(1, -1)	0	2		(8, -1)	(A, B)	0	1
(10, 1)	(A, B)	0	1		(8, -1)	(1, -1)	0	2
(10, 1)	(1, -1)	0	2		(9, -1)	(A, B)	0	1
(11, 1)	(A, B)	0	1		(9, -1)	(1, -1)	0	2
(11, 1)	(1, -1)	0	2		(10, -1)	(A, B)	0	1
(12, 1)	(A, B)	0	1		(10, -1)	(1, -1)	0	2
(12, 1)	(1, -1)	0	2		(11, -1)	(A, B)	0	1
(13, 1)	(A, B)	0	1		(11, -1)	(1, -1)	0	2
(13, 1)	(1, -1)	0	2		(12, -1)	(A, B)	0	1
(14, 1)	(A, B)	0	1		(12, -1)	(1, -1)	0	2
(14, 1)	(1, -1)	0	2		(13, -1)	(A, B)	0	1
(15, 1)	(A, B)	0	1		(13, -1)	(1, -1)	0	2
(15, 1)	(1, -1)	0	2		(14, -1)	(A, B)	0	1
(16, 1)	(A, B)	0	1		(14, -1)	(1, -1)	0	2

(16, 1)	(1, -1)	0	2	(15, -1)	(A, B)	0	1
(17, 1)	(A, B)	0	1	(15, -1)	(1, -1)	0	2
(17, 1)	(1, -1)	0	2	(16, -1)	(A, B)	0	1
(17, 1)	(4, 1)	1	2	(16, -1)	(1, -1)	0	2
(17, 1)	(4, -1)	1	2	(17, -1)	(A, B)	0	1
(18, 1)	(A, B)	0	1	(17, -1)	(1, -1)	0	2
(18, 1)	(1, -1)	0	2	(17, -1)	(4, 1)	1	2
(19, 1)	(A, B)	0	1	(17, -1)	(4, -1)	1	2
(19, 1)	(1, -1)	0	2	(18, -1)	(A, B)	0	1
(20, 1)	(A, B)	0	1	(18, -1)	(1, -1)	0	2
(20, 1)	(1, -1)	0	2	(19, -1)	(A, B)	0	1
(1, -1)	(A, B)	0	1	(19, -1)	(1, -1)	0	2
(1, -1)	(1, -1)	0	2	(20, -1)	(A, B)	0	1
(2, -1)	(A, B)	0	1	(20, -1)	(1, -1)	0	2

where $1 \leq A \leq 20$ and $B = \pm 1$.

Corollary 4. Assume that $\{V_n\}$ is nondegenerate. If $p = V_n(P_1, Q_1)$ and $x = V_k(P_2, Q_2)$ such that $n, k \geq 0, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then the solutions to equation (6) are fully given by the following table:

(P_1, Q_1)	(P_2, Q_2)	n	k	(P_1, Q_1)	(P_2, Q_2)	n	k
(3, 1)	(1, -1)	0	1	(2, -1)	(1, -1)	{0, 1}	1
(4, 1)	(1, -1)	0	1	(3, -1)	(1, -1)	0	1
(5, 1)	(1, -1)	0	1	(4, -1)	(1, -1)	0	1
(5, 1)	(A, B)	1	0	(5, -1)	(1, -1)	0	1
(5, 1)	(2, -1)	1	1	(5, -1)	(A, B)	1	0
(6, 1)	(1, -1)	0	1	(5, -1)	(2, -1)	1	1
(7, 1)	(1, -1)	0	1	(6, -1)	(1, -1)	0	1
(8, 1)	(1, -1)	2	1	(7, -1)	(1, -1)	0	1
(9, 1)	(1, -1)	0	1	(8, -1)	(1, -1)	0	1
(10, 1)	(1, -1)	0	1	(9, -1)	(1, -1)	0	1
(11, 1)	(1, -1)	0	1	(10, -1)	(1, -1)	0	1
(12, 1)	(1, -1)	0	1	(11, -1)	(1, -1)	0	1
(13, 1)	(1, -1)	0	1	(12, -1)	(1, -1)	0	1
(14, 1)	(1, -1)	0	1	(13, -1)	(1, -1)	0	1
(15, 1)	(1, -1)	0	1	(14, -1)	(1, -1)	0	1
(16, 1)	(1, -1)	0	1	(15, -1)	(1, -1)	0	1
(17, 1)	(4, 1)	1	1	(16, -1)	(1, -1)	0	1
(17, 1)	(1, -1)	0	1	(17, -1)	(1, -1)	0	1

(17, 1)	(1, -1)	1	3		(17, -1)	(4, 1)	1	1
(17, 1)	(4, -1)	1	1		(17, -1)	(1, -1)	1	3
(18, 1)	(1, -1)	0	1		(17, -1)	(4, -1)	1	1
(19, 1)	(1, -1)	0	1		(18, -1)	(1, -1)	0	1
(20, 1)	(1, -1)	0	1		(19, -1)	(1, -1)	0	1
(1, -1)	(1, -1)	0	1		(20, -1)	(1, -1)	0	1

where $1 \leq A \leq 20$ and $B = \pm 1$.

Theorem 4. Let the sequence $\{U_n(P,Q)\}$ be nondegenerate with $1 \leq P \leq 20$ and $Q = \pm 1$. If p is a prime number such that $p = U_n(P,Q)$, then the set of the solutions (P,Q,x,n) with $x \geq 1$ and $n \geq 2$ of equation (7) is given by

$$(P,Q,x,n) \in \{(17,1,2,2),(1,-1,1,3),(2,-1,1,2),(4,-1,2,3),(16,-1,4,3),$$
$$(17,-1,2,2)\}.$$

Corollary 5. Suppose that $\{U_n\}$ is nondegenerate. If $p = U_n(P_1,Q_1)$ and $x = U_k(P_2,Q_2)$ such that $n \geq 2, k \geq 1, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1,1\}$, then the solutions of equation (7) are completely given by

$$((P_1,Q_1),(P_2,Q_2),n,k) \in \{((17,1),(1,-1),2,3),((17,1),(2,-1),2,2),$$
$$((1,-1),(A,B),3,1),((1,-1),(1,-1),3,2),((2,-1),(A,B),2,1),((4,$$
$$-1),(1,-1),3,3),((4,-1),(2,-1),3,2),((16,-1),(4,1),3,2),((16,$$
$$-1),(4,-1),3,2),((17,-1),(1,-1),2,3),((17,-1),(2,-1),2,2)\}$$

where $1 \leq A \leq 20$ and $B = \pm 1$.

Corollary 6. Assume that $\{U_n\}$ and $\{V_n\}$ are nondegenerate. If p is a prime number and x is a positive integer such that $p = U_n(P_1,Q_1)$ and $x = V_k(P_2,Q_2)$, where $n \geq 2, k \geq 0, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1,1\}$, then the set of solutions to equation (7) is as follows

$$((P_1,Q_1),(P_2,Q_2),n,k) \in \{((17,1),(A,B),2,0)((17,1),(2,-1),2,1),$$
$$((1,-1),(1,-1),3,1),((2,-1),(1,-1),2,1),((4,-1),(A,B),3,0),((4,$$
$$-1),(2,-1),3,1),((16,-1),(4,1),3,1),((16,-1),(1,-1),3,3),((16,$$
$$-1),(4,-1),3,1),((17,-1),(A,B),2,0),((17,-1),(2,-1),2,1)\}.$$

where $1 \leq A \leq 20$ and $B = \pm 1$.

Theorem 5. Let $\{V_n(P,Q)\}$ be a nondegenerate with $n \geq 0, 1 \leq P \leq 20$ and $Q \in \{-1,1\}$. If p is a prime number such that $p = V_n(P,Q)$, then the complete list of solutions of equation (7) is as follows (assuming that $x \geq 1$):

$$(P,Q,x,n) \in \{(3,1,1,0),(4,1,1,0),(5,1,1,0),(6,1,1,0),(7,1,1,0),$$
$$(8,1,1,0),(9,1,1,0),(10,1,1,0),(11,1,1,0),(12,1,1,0),(13,1,1,0),$$
$$(14,1,1,0),(15,1,1,0),(16,1,1,0),(17,1,1,0),(17,1,2,1),(18,1,1,$$
$$0),(19,1,1,0),(20,1,1,0),(1,-1,1,0),(2,-1,1,0),(2,-1,1,1),(3,$$
$$-1,1,0),(4,-1,1,0),(5,-1,1,0),(6,-1,1,0),(7,-1,1,0),(8,-1,$$

$1, 0), (9, -1, 1, 0), (10, -1, 1, 0), (11, -1, 1, 0), (12, -1, 1, 0), (13, -1, 1,$

$0), (14, -1, 1, 0), (15, -1, 1, 0), (16, -1, 1, 0), (17, -1, 1, 0), (17, -1, 2,$

$1), (18, -1, 1, 0), (19, -1, 1, 0), (20, -1, 1, 0)\}.$

Corollary 7. Suppose that $\{U_n\}$ and $\{V_n\}$ are nondegenerate. If $p = V_n(P_1, Q_1)$ and $x = U_k(P_2, Q_2)$ such that $n \geq 0, k \geq 1, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then all the solutions of equation (7) are given in the following table:

(P_1, Q_1)	(P_2, Q_2)	n	k
(3, 1)	(A, B)	0	1
(3, 1)	(1, -1)	0	2
(4, 1)	(A, B)	0	1
(4, 1)	(1, -1)	0	2
(5, 1)	(A, B)	0	1
(5, 1)	(1, -1)	0	2
(6, 1)	(A, B)	0	1
(6, 1)	(1, -1)	0	2
(7, 1)	(A, B)	0	1
(7, 1)	(1, -1)	0	2
(8, 1)	(A, B)	0	1
(8, 1)	(1, -1)	0	2
(9, 1)	(A, B)	0	1
(9, 1)	(1, -1)	0	2
(10. 1)	(A, B)	0	1
(10, 1)	(1, -1)	0	2
(11, 1)	(A, B)	0	1
(11, 1)	(1, -1)	0	2
(12, 1)	(A, B)	0	1
(12, 1)	(1, -1)	0	2
(13, 1)	(A, B)	0	1
(13, 1)	(1, -1)	0	2
(14, 1)	(A, B)	0	1
(14, 1)	(1, -1)	0	2
(15, 1)	(A, B)	0	1
(15, 1)	(1, -1)	0	2
(16, 1)	(A, B)	0	1
(16, 1)	(1, -1)	0	2
(17, 1)	(A, B)	0	1
(17, 1)	(1, -1)	0	2
(17, 1)	(1, -1)	1	3
(17, 1)	(2, -1)	1	2

(P_1, Q_1)	(P_2, Q_2)	n	k
(2, -1)	(A, B)	1	1
(2, -1)	(1, -1)	0	2
(2, -1)	(1, -1)	1	2
(3,-1)	(A, B)	0	1
(3,-1)	(1, -1)	0	2
(4, -1)	(A, B)	0	1
(4, -1)	(1, -1)	0	2
(5, -1)	(A, B)	0	1
(5, -1)	(1, -1)	0	2
(6, -1)	(A, B)	0	1
(6, -1)	(1, -1)	0	2
(7, -1)	(A, B)	0	1
(7, -1)	(1, -1)	0	2
(8, -1)	(A, B)	0	1
(8, -1)	(1, -1)	0	2
(9, -1)	(A, B)	0	1
(9, -1)	(1, -1)	0	2
(10, -1)	(A, B)	0	1
(10, -1)	(1, -1)	0	2
(11, -1)	(A, B)	0	1
(11, -1)	(1, -1)	0	2
(12, -1)	(A, B)	0	1
(12, -1)	(1, -1)	0	2
(13, -1)	(A, B)	0	1
(13, -1)	(1, -1)	0	2
(14, -1)	(A, B)	0	1
(14, -1)	(1, -1)	0	2
(15, -1)	(A, B)	0	1
(15, -1)	(1, -1)	0	2
(16, -1)	(A, B)	0	1
(16, -1)	(1, -1)	0	2
(17, -1)	(A, B)	0	1

(18, 1)	(A, B)	0	1	(17, -1)	(1, -1)	0	2
(18, 1)	(1, -1)	0	2	(17, -1)	(1, -1)	1	3
(19, 1)	(A, B)	0	1	(17, -1)	(2, -1)	1	2
(19, 1)	(1, -1)	0	2	(18, -1)	(A, B)	0	1
(20, 1)	(A, B)	0	1	(18, -1)	(1, -1)	0	2
(20, 1)	(1, -1)	0	2	(19, -1)	(A, B)	0	1
(1, -1)	(A, B)	0	1	(19, -1)	(1, -1)	0	2
(1, -1)	(1, -1)	0	2	(20, -1)	(A, B)	0	1
(2, -1)	(A, B)	0	1	(20, -1)	(1, -1)	0	2

where $1 \leq A \leq 20$ and $B = \pm 1$.

Corollary 8. Assume that $\{V_n\}$ is nondegenerate. If $p = V_n(P_1, Q_1)$ and $x = V_k(P_2, Q_2)$ such that $n, k \geq 0, 1 \leq P_1, P_2 \leq 20$ and $Q_1, Q_2 \in \{-1, 1\}$, then the solutions to equation (7) are fully given by the following table:

(P_1, Q_1)	(P_2, Q_2)	*n*	*k*	(P_1, Q_1)	(P_2, Q_2)	*n*	*k*
(3, 1)	(1, -1)	0	1	(2, -1)	(1, -1)	1	1
(4, 1)	(1, -1)	0	1	(3, -1)	(1, -1)	0	1
(5, 1)	(1, -1)	0	1	(4, -1)	(1, -1)	0	1
(6, 1)	(1, -1)	0	1	(5, -1)	(1, -1)	0	1
(7, 1)	(1, -1)	0	1	(6, -1)	(1, -1)	0	1
(8, 1)	(1, -1)	0	1	(7, -1)	(1, -1)	0	1
(9, 1)	(1, -1)	0	1	(8, -1)	(1, -1)	0	1
(10, 1)	(1, -1)	0	1	(9, -1)	(1, -1)	0	1
(11, 1)	(1, -1)	0	1	(10, -1)	(1, -1)	0	1
(12, 1)	(1, -1)	0	1	(11, -1)	(1, -1)	0	1
(13, 1)	(1, -1)	0	1	(12, -1)	(1, -1)	0	1
(14, 1)	(1, -1)	0	1	(13, -1)	(1, -1)	0	1
(15, 1)	(1, -1)	0	1	(14, -1)	(1, -1)	0	1
(16, 1)	(1, -1)	0	1	(15, -1)	(1, -1)	0	1
(17, 1)	(1, -1)	0	1	(16, -1)	(1, -1)	0	1
(17, 1)	(A, B)	1	0	(17, -1)	(1, -1)	0	1
(17, 1)	(2, -1)	1	1	(17, -1)	(A, B)	1	0
(18, 1)	(1, -1)	0	1	(17, -1)	(2, -1)	1	1
(19, 1)	(1, -1)	0	1	(18, -1)	(1, -1)	0	1
(20, 1)	(1, -1)	0	1	(19, -1)	(1, -1)	0	1
(1, -1)	(1, -1)	0	1	(20, -1)	(1, -1)	0	1
(2, -1)	(1, -1)	0	1	-	-	-	-

where $1 \leq A \leq 20$ and $B = \pm 1$.

5. Proofs of Results

Proof of Theorem 2. Here, we directly follow the approach presented in the proof of Theorem 1. Since we are interested in the values of x and n satisfying equation (6) at every pair (P, Q) with $1 \leq P \leq 20$ and $Q \in \{-1, 1\}$ such that the sequence $\{U_n(P, Q)\}$ is nondegenerate containing prime terms; the first step is determining the values of x derived from the integral points (x, y) of the genus 1 curves presented by (9). In the following table, we provide the details of computations noting that the triples $[A, B, C]$ representing the coefficients of the elliptic curves $y^2 = Ax^4 + Bx^2 + C$ corresponds to the curves in (9) for every pair (P, Q).

(P, Q)	[A, B, C]	{x}	(P, Q)	[A, B, C]	{x}
(3, 1)	[5, 10, 9]	{0}	(2, -1)	[8, 16, 4] [8, 16, 12]	{-2, 0, 2} {-1, 1}
(4, 1)	[12, 24 16]	{0}	(3, -1)	[13, 26, 9] [13, 26, 17]	{-3, 0, 3} {}
(5, 1)	[21, 42, 25]	{-2, 0, 2}	(4, -1)	[20, 40, 16] [20, 40, 24]	{-4, 0, 4} {}
(6, 1)	[32, 64, 36]	{0}	(5, -1)	[29, 58, 25] [29, 58, 33]	{-5, 0, 5} {-2, 2}
(7, 1)	[45, 90, 49]	{0}	(6, -1)	[40, 80, 36] [40, 80, 44]	{-6, 0, 6} {}
(8, 1)	[60, 120, 64]	{0}	(7, -1)	[53, 106, 49] [53, 106, 57]	{-7, 0, 7} {}
(9, 1)	[77, 154, 81]	{0}	(8, -1)	[68, 136, 64] [68, 136, 72]	{-8, 0, 8} {}
(10, 1)	[96, 192, 100]	{-3, 0, 3}	(9, -1)	[85, 170, 81] [85, 170, 89]	{-9, 0, 9} {}
(11, 1)	[117, 234, 121]	{0}	(10, -1)	[104, 208, 100] [104, 208, 108]	{-10, 0, 10} {-3, 3}
(12, 1)	[140, 280, 144]	{0}	(11, -1)	[125, 250, 121] [125, 250, 129]	{-11, 0, 11} {}
(13, 1)	[165, 330, 169]	{0}	(12, -1)	[148, 296, 144] [148, 296, 152]	{-12, 0, 12} {}
(14, 1)	[192, 384, 196]	{0}	(13, -1)	[173, 346, 169] [173, 346, 177]	{-13, 0, 13} {}
(15, 1)	[221, 442, 225]	{0}	(14, -1)	[200, 400, 196] [200, 400, 204]	{-14, 0, 14} {}
(16, 1)	[252, 504, 256]	{0}	(15, -1)	[229, 458, 225] [229, 458, 233]	{-15, 0, 15} {}
(17, 1)	[285, 570, 289]	{-4, 0, 4}	(16, -1)	[260, 520, 256] [260, 520, 264]	{-16, 0, 16} {}
(18, 1)	[320, 640, 324]	{0}	(17, -1)	[293, 586, 289]	{-17, 0, 17}

				[293, 586, 297]	{-4, 4}
(19, 1)	[357, 714, 361]	{0}	(18, -1)	[328, 656, 324] [328, 656, 332]	{-18, 0, 18} {}
(20, 1)	[396, 792, 400]	{0}	(19, -1)	[365, 730, 361] [365, 730, 369]	{-19, 0, 19} {}
(1, -1)	[5, 10, 1] [5,10, 9]	{-2, -1, 0, 1, 2} {0}	(20, -1)	[404, 808, 400] [404, 808, 408]	{-20, 0, 20} {}

Next, we only consider the values of x in the above table (with $x \geq 1$), for which $x^2 + 1 = p$ is a prime number and is a term in the sequence $\{U_n(P,Q)\}$ for every corresponding value of P and Q. Finally, for the obtained values of x satisfying the above conditions, we determine the values of n such that $U_n(P,Q) = x^2 + 1 = p$. Again, we summarize the details of computations in the following table:

(P, Q)	x	p	n
(5, 1)	2	5	2
(17, 1)	4	17	2
(1, -1)	1 2	2 5	3 5
(2, -1)	1 2	2 5	2 3
(4, -1)	4	17	3
(5, -1)	2	5	2

(P, Q)	x	p	n
(6, -1)	6	37	3
(10, -1)	10	101	3
(14, -1)	14	197	3
(16, -1)	16	257	3
(17, -1)	4	17	2
(20, -1)	20	401	3

Hence, Theorem 2 is completely proven.

Proof Corollary 1. From the result of Theorem 2, it is clear that $(P_1, Q_1, n) \in \{(5,1,2), (17,1,2), (1,-1,\{3,5\}), (2,-1,\{2,3\}), (4,-1,3), (5,-1,2), (6,-1,3), (10,-1,3), (14,-1,3), (16,-1,3), (17,-1,2), (20,-1,3)\}$. Hence, it remains to find the values of $k \geq 1$ with which the corresponding values of x satisfy

$$U_k(P_2, Q_2) = x,$$

where $1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. For instance, let's consider the $(P_1, Q_1, n) = (5,1,2)$. Here, we have $U_2(5,1) = p = 5 = 22 + 1$ in which we are seeking the values of P_2, Q_2 and k such that $U_k(P_2, Q_2) = 2$ holds with $1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. This clearly implies that $(P_2, Q_2, k) = (1,-1,3)$ or $(2,-1,2)$. The remaining cases are handled similarly, and the results are summarized in the following table:

(P_1, Q_1)	(P_2, Q_2)	p	n	k
(17, 1)	(4, 1)	17	2	2
(17, 1)	(4, -1)	17	2	2
(1, -1)	(A, B)	2	3	1
(1, -1)	(1, -1)	5	5	3
(1, -1)	(2, -1)	5	5	2
(2, -1)	(A, B)	2	2	1
(4, -1)	(4, 1)	17	3	2

(P_1, Q_1)	(P_2, Q_2)	p	n	k
(6, -1)	(6, -1)	37	3	2
(10, -1)	(10, 1)	101	3	2
(10, -1)	(10, -1)	101	3	2
(14, -1)	(14, 1)	17	3	2
(14, -1)	(14, -1)	197	3	2
(16, -1)	(16, 1)	257	3	2
(16, -1)	(16, -1)	257	3	2

(4, -1)	(4, -1)	17	3	2	(17, -1)	(4, 1)	17	2	2
(5, -1)	(1, -1)	5	2	3	(17, -1)	(4, -1)	197	2	2
(5, -1)	(2, -1)	5	2	2	(20, -1)	(20, 1)	401	3	2
(6, -1)	(6,1)	37	3	2	(20, -1)	(20, -1)	401	3	2

for all $1 \leq A \leq 20$ and $B = \pm 1$. The above table proves the result of Corollary 1.

Proof Corollary 2. The proof of this corollary similarly follows from the result of Theorem 2, following the same approach used in the proof of Corollary 1. For instance, if we again consider $(P_1, Q_1, x, n) = (5, 1, 2, 2)$. Here, we are looking for the values of k, P_2, Q_2 for which

$$5 = p = U_2(5,1) = x^2 + 1 = 2^2 + 1 = V_k(P_2, Q_2)^2 + 1,$$

within $1 \leq P_2 \leq 20, Q_2 = \pm 1$ and $k \geq 0$. Since $V_0(P_2, Q_2) = 2$ with all $1 \leq P_2 \leq 20, Q_2 = \pm 1$, thus the latter equation holds at all the values $(P_2, Q_2, k) = (A, B, 0)$ such that $A = P_2$ and $B = Q_2$. Moreover, it holds when $(P_2, Q_2, k) = (2, -1, 1)$. Hence, the first two solutions given in Corollary 2 are obtained, and the others can be found similarly. Therefore, we omit the details of computations in the proof.

Proof of Theorem 3. The proof of this theorem is achieved by following the same approach used in the proof of Theorem 2 with the use of equation (10) in order to determine all the values of x with $1 \leq P \leq 20$ and $Q \in \{-1, 1\}$. Therefore, we omit the details of the proof. Indeed, in the end, we obtain the values of x and n with $x \geq 1$ and $n \geq 0$, for which $V_n(P, Q) = x^2 + 1 = p$ as follows:

(P, Q)	x	p	n	(P, Q)	x	p	n
(3, 1)	1	2	0	(2, -1)	1	2	{0, 1}
(4, 1)	1	2	0	(3, -1)	1	2	0
(5, 1)	1 2	2 5	0 1	(4, -1)	1	2	0
(6, 1)	1	2	0	(5, -1)	1 2	2 5	0 1
(7, 1)	1	2	0	(6, -1)	1	2	0
(8, 1)	1	2	0	(7, -1)	1	2	0
(9, 1)	1	2	0	(8, -1)	1	2	0
(10, 1)	1	2	0	(9, -1)	1	2	0
(11, 1)	1	2	0	(10, -1)	1	2	0
(12, 1)	1	2	0	(11, -1)	1	2	0
(13, 1)	1	2	0	(12, -1)	1	2	0
(14, 1)	1	2	0	(13, -1)	1	2	0
(15, 1)	1	2	0	(14, -1)	1	2	0
(16, 1)	1	2	0	(15, -1)	1	2	0
(17, 1)	1 4	2 17	0 1	(16, -1)	1	2	0
(18, 1)	1	2	0	(17, -1)	1 4	2 17	0 1
(19, 1)	1	2	0	(18, -1)	1	2	0

(20, 1)	1	2	0		(19, -1)	1	2	0
(1, -1)	1	2	0		(20, -1)	1	2	0

Hence, the desired results are obtained, and the proof of Theorem 3 is completed.

Proof Corollary 3. From the result of Theorem 3, it is clear that

$$(P_1, Q_1, n) \in \{(3,1,0),(4,1,0),(5,1,\{0,1\}),(6,1,0),(7,1,0),(8,1,0),(9,1,0),(10,1,0),(11,1,0),(12,1,0),(13,1,0),(14,1,0),(15,1,0),(16,1,0),(17,1,\{0,1\}),(18,1,0),(19,1,0),(20,1,0),(1,-1,0),(2,-1,\{0,1\}),(3,-1,0),(4,-1,0),(5,-1,\{0,1\}),(6,-1,0),(7,-1,0),(8,-1,0),(9,-1,0),(10,-1,0),(11,-1,0),(12,-1,0),(13,-1,0),(14,-1,0),(15,-1,0),(16,-1,0),(17,-1,\{0,1\}),(18,-1,0),(19,-1,0),(20,-1,0)\}.$$

Therefore, it remains to get the values of (P_2, Q_2, k) such that the corresponding values of x satisfy the equation

$$U_k(P_2, Q_2) = x,$$

where $k \geq 1, 1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. We consider a couple of cases; the first case is when $(P_1, Q_1, n) = (3,1,0)$ with $x = 1$ then $2 = V_0(3,1) = U_k^2(P_2, Q_2) + 1$. The second case is when $(P_1, Q_1, n) = (20,-1,0)$ and $x = 1$, which lead to the equation $2 = V_0(20,-1) = U_k^2\,(P_2, Q_2) + 1$. These equations hold whenever $(P_2, Q_2, k) = (1,-1,2)$ and $(A, B, 1)$, where $1 \leq A \leq 20$ and $B \in \{-1,1\}$. The same approach is applied to the rest of the cases. Hence, the proof of the result of Corollary 3 is achieved.

Proof Corollary 4. The proof of this corollary also follows from the result of Theorem 3 and follows the same method used in the proof of Corollary 3. Therefore, we omit the details of the proof.

Proof of Theorem 4. Since we are here seeking the values of P, Q, x and n that satisfy equation (7), namely

$$U_n(P, Q) = p = x^4 + 1 = (x^2)^2 + 1$$

with $n \geq 2, x \geq 1, 1 \leq P \leq 20$ and $Q \in \{-1,1\}$, the proof of this theorem can be followed directly from the result of Theorem 2 by determining the values of x that have a positive integer square root. Hence, the values of x satisfying the above conditions with the corresponding values of P, Q and n are summarized in the following table:

(P, Q)	x	p	n		(P, Q)	x	p	n
(17, 1)	2	17	2		(4, -1)	2	17	3
(1, -1)	1	2	3		(16, -1)	4	257	3
(2, -1)	1	2	2		(17, -1)	2	17	2

Therefore, the results in the above table give the complete set of solutions to equation (7), and that completes the proof of Theorem 4.

Proof Corollary 5. From the result of Theorem 4, it is clear that $(P_1, Q_1, n) \in \{(17,1,2),(1,-1,3),(2,-1,2),(4,-1,3),(16,-1,3),(17,-1,2)\}$. Therefore, it remains to find the values of $k \geq 1$ with which the corresponding values of *x* satisfy

$$U_k(P_2, Q_2) = x,$$

where $1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. For example, if we consider $(P_1, Q_1, n) = (17,1,2)$. Here, we have $U_2(17,1) = p = 17 = 2^4 + 1$, in which we are seeking the values of P_2, Q_2 and k such that $U_k(P_2, Q_2) = 2$ holds with $1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. This clearly implies that $(P_2, Q_2, k) = (1,-1,3)$ *or* $(2,-1,2)$. In a similar way the remaining cases are handled similarly, and the results are as follows:

(P_1, Q_1)	(P_2, Q_2)	p	n	k		(P_1, Q_1)	(P_2, Q_2)	p	n	k
(1, -1)	(A, B)	2	3	1		(16, -1)	(4, 1)	257	3	2
(1, -1)	(1, -1)	2	3	2		(16, -1)	(4, 1)	257	3	2
(2, -1)	(A, B)	2	2	1		(17, -1)	(1, -1)	17	2	3

(4, -1)	(1, -1)	17	3	3	(17, -1)	(2, -1)	17	2	2
(4, -1)	(2, -1)	17	3	2	-	-	-	-	-

for all $1 \leq A \leq 20$ and $B = \pm 1$. The above table proves the result of Corollary 5.

Proof Corollary 6. Again, the proof of this corollary follows from the result of Theorem 4 by following the same approach used in the proof of Corollary 5. For example, let's again consider $(P_1, Q_1, x, n) = (17, 1, 2, 2)$. Here, we want to find the values of k, P_2, Q_2 in which

$$17 = p = U_2(17,1) = x^4 + 1 = 2^4 + 1 = V_k(P_2, Q_2)^4 + 1,$$

where $1 \leq P_2 \leq 20, Q_2 = \pm 1$ and $k \geq 0$. Since $V_0(P_2, Q_2) = 2$ with all $1 \leq P_2 \leq 20, Q_2 = \pm 1$, thus the latter equation holds at all the values $(P_2, Q_2, k) = (A, B, 0)$ such that $A = P_2$ and $B = Q_2$. Moreover, it holds in the case of $(P_2, Q_2, k) = (2, -1, 1)$. Hence, the first two solutions given in Corollary 6 are found, and the others can be obtained similarly. Therefore, we omit the details of the computations.

Proof of Theorem 5. Since we want to find the values of P, Q, x and n that satisfy the equation

$$V_n(P,Q) = p = x^4 + 1 = (x^2)^2 + 1,$$

where $n \geq 0, x \geq 1, 1 \leq P \leq 20$ and $Q = \pm 1$, the proof of this theorem can be achieved directly from the result of Theorem 3 by finding the values of x that have a positive integer square root. Such values with their corresponding values of P, Q and n are summarized in the following table:

(P, Q)	x	p	n	(P, Q)	x	p	n
(3, 1)	1	2	0	(2, -1)	1	2	1
(4, 1)	1	2	0	(3, -1)	1	2	0
(5, 1)	1	2	0	(4, -1)	1	2	0
(6, 1)	1	2	0	(5, -1)	1	2	0
(7, 1)	1	2	0	(6, -1)	1	2	0
(8, 1)	1	2	0	(7, -1)	1	2	0
(9, 1)	1	2	0	(8, -1)	1	2	0
(10, 1)	1	2	0	(9, -1)	1	2	0
(11, 1)	1	2	0	(10, -1)	1	2	0
(12, 1)	1	2	0	(11, -1)	1	2	0
(13, 1)	1	2	0	(12, -1)	1	2	0
(14, 1)	1	2	0	(13, -1)	1	2	0
(15, 1)	1	2	0	(14, -1)	1	2	0
(16, 1)	1	2	0	(15, -1)	1	2	0
(17, 1)	1	2	0	(16, -1)	1	2	0
(17, 1)	2	17	1	(17, -1)	1	2	0
(18, 1)	1	2	0	(17, -1)	2	17	1
(19, 1)	1	2	0	(18, -1)	1	2	0
(20, 1)	1	2	0	(19, -1)	1	2	0
(1, -1)	1	2	0	(20, -1)	1	2	0
(2, -1)	1	2	0	-	-	-	-

So, the results in the above table provide the complete set of solutions to equation (7), and Theorem 5 is completely proven.

Proof Corollary 7. From Theorem 5, we have that

$$(P_1, Q_1, n) \in \{(3,1,0),(4,1,0),(5,1,0,),(6,1,0),(7,1,0),(8,1,0),(9,1,0),(10,1,0),(11,1,0),(12,1,0),(13,1,0),(14,1,0),(15,1,0),(16,1,0),(17,1,\{0,1\}),(18,1,0),(19,1,0),(20,1,0),(1,-1,0),(2,-1,\{0,1\}),(3,-1,0),(4,-1,0),(5,-1,0),(6,-1,0),(7,-1,0),(8,-1,0),(9,-1,0),(10,-1,0),(11,-1,0),(12,-1,0),(13,-1,0),(14,-1,0),(15,-1,0),(16,-1,0),(17,-1,\{0,1\}),(18,-1,0),(19,-1,0),(20,-1,0)\}.$$

Next, we determine the values of (P_2, Q_2, k) such that the corresponding values of x satisfy the equation

$$U_k(P_2, Q_2) = x,$$

where $k \geq 1, 1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. We may look at two cases; the first case is when $(P_1, Q_1, n) = (4, 1, 0)$ with $x = 1$. Then, we have that $2 = V_0(4,1) = U_k^4(P_2, Q_2) + 1$, that is satisfied when $(P_2, Q_2, k) = (1, -1, 2)$ and $(A, B, 1)$, where $1 \leq A \leq 20$ and $B \in \{-1, 1\}$. The second case is when $(P_1, Q_1, n) = (17, -1, \{0, 1\})$. In case of $n = 0$, we have $x = 1 = U_k(P_2, Q_2)$, which is satisfied if $(P_2, Q_2, k) = (1, -1, 2)$ and $(A, B, 1)$ with $1 \leq A \leq 20$ and $B \in \{-1, 1\}$. On the other hand, if $n = 1$ then $x = 2$ that implies that $(P_2, Q_2, k) = (1, -1, 3)$ and $(2, -1, 2)$. Indeed, the same approach is applied to the rest of the cases. So, the proof for Corollary 7 is completed.

Proof Corollary 8. This Corollary is confirmed by the results of Theorem 5 by following the same technique used to prove Corollary 7. Therefore, we omit the details of the proof.

5. Conclusion

We conclude that the equations $p = x^2 + 1$ and $p = x^4 + 1$, which have infinitely many solutions over rational integers, have only finitely many solutions (p, x) where p and x are Lucas numbers of the first or second kind.

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