Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

A New View on Length Module

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| ABSTRACT
This paper consists of several new results about Length property of the module M. F-length of any Module comes from several concepts like Neotherian module and Artinian modules (N_{eo} and A_{rt}) with comp- sition series. We proved that any N_{eo} -module has submodule T and N_{eo} -module is F-length. This implies that T also has F-length Property. Finally, some remarks, examples and definitions have been presented in this paper.

| KEYWORDS
Artinian module, Neotherian module, Simple module, Semisimple module, Finite length.

| ARTICLE INFORMATION
ACCEPTED: 01 April 2023 **PUBLISHED:** 03 April 2023 **DOI:** 10.32996/jmss.2023.4.2.2

1. Introduction
All rings in this paper are commutative with unity and all R- modules are unital. Let R be a ring . We say that N_{eo}-module (Neotherian module) if every submodule T of N_{eo} is finitely generated or N_{eo}-module satisfy ascending chain condition (A.C.C) [Kasch, 1982]. On the other hand, any A_{rt} -module (Artinian modules) means A_{rt}-module satisfy descending chain condition (D.C.C)[Kourki, 2018]. We definition of composition series between o and M_{eo} such that $M_{eo1} \subseteq M_{eo2} \subseteq .....$. Ascending chain condition means : $0 = M_{eo} \subseteq M_{eo1} \subseteq ... = M_{eo}$.

A definition of maximal submodule and more details about this concept in [Faith, 1995] . Note that descending chain condition and kurll ordinal in [2]. More information of finite length property was studied by [Spanier, 1995].
In this paper, we present the best way to find less conditions about finite length and we focus on the two concepts in module theory namely Artinian and Neotherian modules based on composition series .
In the context of group theory a sequence $M_0 \overset{f_1}{\to} M_1 \overset{f_2}{\to} M_2 \overset{f_3}{\to} ..... \overset{f_n}{\to} M_n$ of groups and group homomorphism is said to be exact at G_i if $im(f_i)= \ker(f_{i+1})$. The sequence is called exact if it is exact at each G_i for all $1\leq i < n$ I.e. kernel of the next equal image of all homomorphism [5]. Short exact sequences are exact sequences of the form $0 \to \Omega \overset{f}{\to} B \overset{g}{\to} C \to 0$. As established above, for any such short exact sequences , f is a monomorphism and g is an epimorphim. Furthermore, the image of f is equal to the Kernel of g. It is helpful to think of Ω as a sub object of B with f embedding Ω in to B , and of C as the corresponding factor object (or quotient) B/A ,with g inducing an isomorphism. $C \cong B/im(f) = B/\ker(g)$.

The short exact sequence $0 \to \Omega \overset{f}{\to} B \overset{g}{\to} C \to 0$ is called split. An exact seq: $0 \to \overset{f}{\to} L\, M$ is called split if Im(f) is a direct summand of M ($M= Im(f) \oplus A \ni A \hookrightarrow M$). Also, a short seq: $M \overset{f}{\to} N \to 0$ is called splits if ker(g) is a direct summand of M ($M=\ker(g) \oplus B \ni B \hookrightarrow M$). Also, a short exact: $0 \to L \overset{f}{\to} M \overset{f}{\to} N \to 0$ is called split if $Im(f) = \ker(g)$ is a direct summand of M($M = Im(f)\oplus A=\ker(g) + A \ni A \hookrightarrow M$. [Spanier, 1995].

2 : N_{eo} and A_{rt} Modules:

In this part , we study and present general result of two important modules namely N_o and A_{rt} module. Some Properties and more details can introduce it in this section.

2.1 Definition

If every sub module of M_o is F- generated then M_o is called N_o and denoted by (N_o) .

2.2 Remarks and Examples

1) M is N_{eo} if : $0 = M_{o\,0} \subseteq M_{o\,1} \subseteq \cdots \subseteq M_{o\,n} = M_o$ where $M_{o\,0}, M_{o\,1}, \ldots, M_{o\,n}$ are submodules of M_O.

I.e. Ascending Chain condition (A-C-C) is hold.

2) If any sub module of M_o is cyclic , then M_o is a N_{eo} - module , because every cyclic module gives F- generated module .

3) If every sub module of M_o is simple , so M_o is also, N_o Module, because every Simple sub module is cyclic.

4)If R is N_o- ring and R-module M_o have F- length, so M is F- generated.

5) The ring R is N_o if every Ascending Chain Condition (A.C.C) of an ideals hold.

i.e. $0=I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = R$

We can say all the details of the above information can be summarized by the following:

An R-Module M_0 is called N_{eo} if the family sub modules T_i in M_0 satisfies (A.C.C) and M_o is called A_{rt}- module if T_i achieved (D.C.C).

2.3 Definition

For every R is called N_{eo}- ring if it is a N_{eo} - R- module and R is called A_{rt} – ring if it is A_{rt} R-module.

In other word, R is N_o-ring if and only if (I_i) is the family of an ideals of R satisfy (A.C.C) and R is Art- ring If (I_i) achieved (D.C.C).

2.4 Remarks

1) Let $M_{o\,1} \cong M_{o2}$. If M_o , achieved (A.C.C), so is $M_{o\,2}$.

2) Let $M_{o1} \cong M_{o\,2}$. If My achieved (D.C.C), so is $M_{o\,2}$.

3)If M_o achieved (A.C.C), so $T \leq M_O$ achieved (A.C.C)and if M_O achieved (D.C.C), so $T \leq M_o$ achieved (D.C.C).

4) If $T \leq M_o$. and M_o achieved (A.C.C) so M_o/T achieved (A.C.C) and If M_o achieved (D.C.C), so M_o/T .

A Recall that Let $T \leq M_o$ then M_o achieved (A.C.C) if and only D and M_o/N are achieved (A.C.C) .

2.5 Definition

Assume that M_o is an R-module. We define Composition series to M_o between 0 and M_o such that $M_{o\,i-1} \subseteq M_{o\,1} \subseteq M_{o\,I}$.

2.6 Remark

A composition series of M_O is a strict F- chain ($0 = M_{o\,0} \subseteq M_{o\,1} \subseteq \cdots \subseteq M_{o\,n} = M_o$) that is Max. "On the other hand if a F- composition series exists, this means M_o has F- length.

2.7 Lemma

If $M_{o\,1} \cong M_{o\,2}$ and $M_{o\,1}$ have F- length then $M_{o\,2}$ is also has F- length.

2.8 Proposition

Assume that M_o be an R-module and let $T \leq M_o$ If M_o has F- length then T that also F- length .

Proof:

Suppose that $0= M_{0\,0} \subseteq M_{0\,1} \subseteq \ldots \subseteq M_{o\,n} = M_{o\,1}$ is a composition series. Assume that $T_i=M_{o\,i} \cap T$. So $T_i \to M_{o\,i} \to M_{o\,i}/M_{i-1}$, $0 < i \leq n$, the Kernel is T .Heace $T_i/T_{i-1} \to M_{o\,i}/M_{oi-1}$ is an injective . *So* $N_i/N_{i-1} \cong Img\ M_i/M_{i-1}$ *But* M_i/M_{i-1} *is simple so* $T_i/T_{i-1} \cong M_{O\,i}/M_{O\,i-1}$ *and so simple or* $T_i/T_{i-1} \cong 0$ *if* $T_i = T_{i-1}$ Therefore $0=T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n=T$. Since $T_i/T_{i-1}$is simple, so it is composition series .Thus $f(t) \leq f(m_0)$

2.9 Corollary
Let M_o be an R-module has F- length. If Strict F- Chain of sub modules M_{0i}. of M_{0i}, $0 \subseteq M_{o\,1} \subseteq \ldots \subseteq M_{0\,k}$ then k less than equals $l(M_o)$.

Proof:
From proposition 2.8, $M_{o\,i}$ has F-length . Also $0 < f(M_{o\,1}) < \ldots < f(M_{o\,k}) \leq f(M_o)$. Thus $K \leq l(M_o)$.

2.10 Proposition
Let M_0 be an D-Module we have F-length with strict F- chain of submodule M_{0i} of M_0 . $0 \subseteq M_{o\,1} \subseteq \ldots \subseteq M_{o\,K}$.

Then K=f(T) in case $0 \subseteq M_{O\,1} \subseteq \ldots \subseteq M_{o\,k}$ is a composition series to M_0.

proof :
To show $K= l(M_0)$, suppose that the $0 \subseteq M_{o\,1} \subseteq \ldots \subseteq M_{o\,k}$ is a Composition series so by definition of length module, $K \geq l(M_o)$ but from proposition 2.10, $K \leq l\,(M_o)$. Thus $K= l(M_o)$.

2.11 Corollary
Assume that M_o is an R-Module has F- length with $0 \subseteq M_{o\,1} \subseteq \ldots \subseteq M_{o\,k}$ If $K=l(M_o)$, then $0 \subseteq M_{o\,1} \subseteq \ldots \subseteq M_{o\,k}$ is a composition series .

Proof:
Assume that $K=l(M_o)$. Suppose the Chain $0 \subseteq M_{01} \subseteq \ldots \subseteq M_{0k}$ is not composing Series. So we have longer Strick Chain has length K+1, but this contradiction with Proposition 2.10. Thus it is composition Series .

2.12 Proposition
Assume M_o is an R-Module If M_o satisfies (A.C.C) and (D.C.C) then M_o has F- length.

Proof:
Assume that M_o achieved (A.C.C) and (D.C.C) for partial ordered set (p o s) C. Let Ω , B in C with $\Omega \leq B$. So we get composition series between Ω and B. Let $C_0 = \Omega$. If $C_i < B$, SO C_{i+1} is a min- H in C such that $g_i < H \leq B$. Now $\forall i$ we have $g_i = B$. Then there is a composition Series between Ω and B.

2.13 Corollary
For any module M_o has F- length then M_o achieved (A.C.C) and (D.C.C).

Proof: clear

2.14 Remark
If for $t \in T$, there exist fi chain of sub modules: $0 = \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_n$ .then any module M_o has no F- length.

Proposition 2.15.
Let M_o be an noetherian an module. Then M_o F- length where Ω and M_0/Ω are F-length where $\Omega \leq M_o$.

proof:
Suppose Ω and M_0/Ω both have F- length. To show M_o has F-length. From Remark: 2.4,(3), corollary 2.13 M_0 has F- length if Ω and M_0/T have F- length . Suppose that $0 = \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_n = \Omega$ is a composition series of Ω . Assume that $0 = B_0/\Omega \subseteq \frac{B_1}{\Omega} \subseteq \cdots \frac{B_n}{\Omega} = \frac{M_o}{\Omega}$ IS a composition series of $\frac{M_o}{\Omega}$.

Hence $0= \Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_n = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n = M_o$ is a composition series of M_o .So length of this series is the length of M_o .

2.16 Remark
Let M_o be an R-module. If M_o has only trivial sub modules, then M_o has F- length $l(M_o)=1$

2.17 Example
Assume R is a discrete valuation ring (DVR)/R. Then $\forall d \in R, l(\frac{R}{d}) = R(d)$ .It is clear that there is a strong exact relation between sequence concept and F- length property. But before sequence presents this relation, we need to define exact sequence by the following:

2.18 Propositions

Assume that $0 \to M_{o1} \to Mo \to M_{o2} \to 0$ is a short exact sequence of an R-module M_o. If M_o has F- length property, then M_{o1}, and M_{o2} have F- length ($l(M_o)=l(M_{o1}) + l(M_{o2})$).

Proof:

Suppose that M_{o1}, and M_{o2} have F- length . So there exists a composition series of. M_o by connecting the composition series of M_o, and M_{o2} Hence $l(M_o) = l(M_{o1}) + l(M_{o2})$.

2.19 Corollary

Let $0 \to M_{o1} \to Mo \to M_{o2} \to 0$ be an exact sequence of an D-Module M_o. If $l(M_o) = l(M_o)+l(M_{o2})$ then M_o has F-length Property. Recall that if $M_o = \Omega_0 \supseteq \Omega_1 \supseteq \ldots$ is a submodules sequence, so M_o is F- length means M_o is satisfies (D.C.C), because : $L(\Omega_0) > l(\Omega_1) > \ldots$ but this is contradiction.

2.20 Prososition

Let M_o be an R-Module If

1-R satisfies (A.C.C) a series ideals,

2-M_o has composition series (f. length), then $M_o = \sum d_i x_i$, $d_i \in R$, $x_i \in M_o$.

Proof:

M_o is F-length, (M_o has composition series) $0=\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \Omega_r = M_o$

such that $\Omega_l / \Omega_{i-1} = R/X_i$, $1 \le i \le k$, x_i is a max-ideal of R . Also Ω_i / Ω_{i-1} is f-generated. Then M_o is also f generated.

As a result to proposition we present the following result :

2.2.1 *Corollary*

Let R achieve (A.C.C.) and let $M_o = \sum d_i \ X_i$,$d_i \in R$, $X_i \in M$ So M has f- length.

Proof:

It is clear that if $M_o = \sum d_i x_i$ and R achieved . (A.C.C) for Ω_i ideas in R. So $0=\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_{k=1}$

We have $\frac{\Omega_i}{\Omega_{i-1}} = \frac{\Omega 1}{Pi}$ $1 \le i \le k$ and Pi is a prim ideal of R. Suppose that the dimention $(\frac{B}{ann_R(Mo))})=0$. Hence any prime ideal of $\frac{B}{ann_R(Mo)}$ is Max –ideal .

But $ann_R(M) \subseteq Pi \ \forall i$.

So $\frac{\Omega i}{\Omega i-1}$ has no proper sub modules (simple module).

Thus M_o has l. length property.

2.2.2 Example

(R,X) achieved (A.C.C) and a uniqe Max ideal .

Then M_o has f- length when

$M_o = \sum d_i \ x_i$,$d_i \in$ has R ,$x_i \in M_o$.

3. Conclusion

The finite length property of the modules plays a vital role in the module theory, special carry when we rely on two basic concepts: The aritinian and Neotherian modules.If we have the exact sequence and composition series of submodules in module M, so $l(M)= l(M_{eo1})+l(M_{eo2})$. Also, we proved that if M is a finite length.

Funding: This research received no external funding.
Conflicts of Interest: The authors declare no conflict of interest.

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