


## Fundamental Results on Determining Matrices for a Certain Class of Hereditary Systems

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### ABSTRACT

Three major tools are required to investigate the controllability of control systems, namely, determining matrices, index of control systems and controllability Grammian. Determining matrices are the preferred choice for autonomous control systems due to the fact that they are devoid of integral operators in their computations. This article developed the structure of certain parameter-ordered determining matrices of generic double time-delay linear autonomous functional differential control systems, with a view to obtaining the controllability matrix associated with the rank condition for Euclidean controllability of the system. Expressions for the relevant determining matrices were formulated and it was established that the determining matrices for double time-delay linear autonomous functional differential control systems do not exist if one of the time-delays is not an integer multiple of the other paving the way for the investigation of the Euclidean controllability of generic double time-delay control systems.

### 1. Introduction

Controllability results for specific types of hereditary systems with diversity in treatment approaches are quite prevalent in control literature. Angell (1980) discussed controllability of nonlinear hereditary systems, using a fixed point approach, (Balachandran, 1986) discussed controllability of nonlinear systems with delays in both state and control variables using a constructive control approach and an appeal to Arzela-Ascoli, and Schauder fixed point theorems to guarantee the existence and admissibility of such controls. Balachandran (1992) and Balachandran and Balasubramaniam (1993) studied the controllability of Volterra Integro-differential systems. Banks and Kent (1972) discussed controllability of functional differential equations of retarded and neutral types to targets in function space; Dauer and Gahl (1977) looked at controllability of nonlinear delay systems; Onwuatu (1984) studied null controllability in function space of nonlinear neutral differential systems with limited controls; Underwood R. G. and Chukwu (1988) investigated null controllability of nonlinear neutral differential equations, to mention just a few.

Chukwu (2001) further extended this research field by formulating differential models and neutral systems for controlling the wealth of nations. His work derives from the economic principles of the dynamics of national income, interest rate, employment, the value of the capital stock prices and cumulative balance of payments. Chukwu used a Volterra neutral integro-differential game of pursuit where the quarry control is government intervention in the form of taxation, control of money and supply tariffs. Other notable results focus on integro-differential equations and impulsive differential equations with finite and infinite delays (Chang & Chalishajar, 2008; Selvi & Mallika, 2012). These works and others appropriate relevant existence and uniqueness of solution theorems which this study relies on.

Gabasov and Kirillova (1976) first introduced the concept of determining matrices by formulating necessary and sufficient conditions for the Euclidean controllability of the single time-delay system:  $\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t)$  with

piecewise continuous controls using a sequence of matrices  $Q_k(s), k = 0, 1, \dots, s$  real. But the investigation of the dependence of  $Q_\infty(t_1)$  on  $Q_0(s), Q_1(s), \dots, Q_{n-1}(s)$  which is very crucial to establishing Euclidean controllability was not fully addressed.

Ukwu (1992) developed computational criteria for the Euclidean controllability of the above single time-delay system using the determining matrices with a very simple structure, effectively eliminating the afore-mentioned drawback. However, a major drawback of his major result is that it relied on (Manitius, 1978) for the necessary and sufficient conditions for the Euclidean controllability of the delay system, stated in terms of the transition matrices, which until (Ukwu, 1992) was a herculean or almost impossible task to obtain. Ukwu (2014) states that multiple summations and permutation of objects were used to derive the expressions for the determining matrices. By using the change of variables technique, deft application of mathematical induction principles and careful avoidance of some induction pitfalls, he was able to obtain the structure of the determining matrices for the single-delay neutral control model:  $\frac{d}{dt}[x(t) - A_{-1}x(t-h)] = A_0x(t) + A_1x(t-h)Bu(t), t \geq 0$ , without which the computational investigation of Euclidean controllability would be impossible.

Ukwu (2016a) pioneered the introduction of the least integer function in the statement and proof of the necessary and sufficient conditions for the Euclidean controllability of linear hereditary systems of double time-delay:  $\dot{x}(t) = A_0x(t) + A_1x(t-h) + A_2x(t-2h) + Bu(t); t \geq 0$  with an initial function:  $x(t) = \phi(t), t \in [-2h, 0], h > 0$ ; this made the controllability matrix in (Gabasov & Kirillova, 1976) quite computable and eliminates any ambiguity that could arise in its application, he also incorporated the characterization of Euclidean controllability in terms of the indices of control systems and appropriated Taylor's theorem as an indispensable tool. The article provided an illustrative example on the computation of the controllability matrices and stated the implication of the result for Euclidean controllability.

However, these works were all limited to single time-delay control systems and double time-delay control systems of non-generic delays in the state variable. This study will extend the research activity to a class of generic double time-delay control systems when the delay in the state variable is non-commensurate or generic with certain conditions imposed on the delay values to avoid ambiguity. The expected results will add to the already existing body of knowledge in this field and will have practical implications.

## 2. Methodology

Consider the generic double time-delay linear autonomous control system:

$$\dot{x}(t) = A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2) + Bu(t); t \geq 0 \quad (1)$$

$$x(t) = \phi(t), t \in [-h, 0], h_1 < h_2, h_1 \neq h_2, h = \max\{h_1, h_2\} > 0 \quad (2)$$

where  $A_0, A_1, A_2$  are  $n \times n$  constant matrices with real entries,  $B$  is an  $n \times m$  constant matrix with real entries. The initial function  $\phi$  is in  $C([-h, 0], \mathbf{R}^n)$ , the space of continuous functions from  $[-h, 0]$  into the real  $n$ -dimensional Euclidean space,  $\mathbf{R}^n$  with norm defined by  $\|\phi\| = \sup_{t \in [-h, 0]} |\phi(t)|$  (the sup norm). The control  $u$  is in the space  $L_\infty([0, t_1], \mathbf{R}^m)$ , the space of essentially bounded measurable functions taking  $[0, t_1]$  into  $\mathbf{R}^m$  with norm  $\|u\| = \text{ess sup}_{t \in [0, t_1]} |u(t)|$ . Any control  $u \in L_\infty([0, t_1], \mathbf{R}^m)$  will be referred to as an admissible control.

### 2.1 Preliminaries on the Partial Derivatives $\frac{\partial^k X(\tau, t)}{\partial \tau^k}, k = 0, 1, \dots$

Let  $t, \tau \in [0, t_1]$ . For fixed  $t$ , let  $\tau \rightarrow X(\tau, t)$  satisfy the matrix differential equation:

$$\frac{\partial}{\partial \tau} X(\tau, t) = -X(\tau, t)A_0 - X(\tau + h_1, t)A_1 - X(\tau + h_2, t)A_2 \quad (3)$$

for  $0 < \tau < t, \tau \neq t - kh, h = \max\{h_1, h_2\}, k = 0, 1, \dots$  where  $X(\tau - t) = \begin{cases} I_n, \tau = t \\ 0, \tau > t \end{cases}$

Chukwu in [7] has the properties of  $X(t, \tau)$  clearly stated; of particular importance is the fact that  $\tau \rightarrow X(\tau, t)$  is analytic on the intervals  $(t_1 - (j+1)h, t_1 - jh), j = 0, 1, \dots, t_1 - (j+1)h > 0$ . Any such  $\tau \in (t_1 - (j+1)h, t_1 - jh)$  is called a regular point of  $\tau \rightarrow X(\tau, t)$ .

Let  $X^k(\tau, t)$  denote  $\frac{\partial^k}{\partial \tau^k} X(\tau, t_1)$ , the  $k^{th}$  partial derivative of  $X(\tau, t_1)$  with respect to  $\tau$  where  $\tau \in (t_1 - (j+1)h, t_1 - jh); j = 0, 1, \dots, r$  for some integer  $r$  such that  $t_1 - (r+1)h > 0$ . Write  $X^{(k+1)}(\tau, t_1) = \frac{\partial}{\partial \tau} X^k(\tau, t_1)$  and define  $\Delta X^k(t_1 - jh, t_1) = X^k((t_1 - jh)^-, t_1) - X^k((t_1 - jh)^+, t_1)$  for  $k = 0, 1, \dots; j = 0, 1, \dots; t_1 - jh > 0$  where  $X^k((t_1 - jh)^-, t_1)$  and  $X^k((t_1 - jh)^+, t_1)$  denote respectively the left and right hand limits of  $X(\tau, t_1)$  at  $\tau = t_1 - jh$ .

## 2.2 Investigation of the Expressions and Structure of the Determining Matrices for (1)

Let  $Q_k(s)$  be an  $n \times n$  matrix function defined by:

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s - h_1) + A_2 Q_{k-1}(s - h_2) \quad (4)$$

for  $k = 1, 2, \dots; s > 0$ , with initial conditions:

$$Q_0(0) = I_n \quad (5)$$

$$Q_0(s) = 0; s \neq 0 \quad (6)$$

These initial conditions guarantee the unique solvability of (4) (Gabasov & Kirillova, 1976).

### 2.2.1 Preliminary Lemma on Determining Matrices for (1) $Q_k(s), s \in \mathbf{R}$

- i.  $Q_k(0) = A_0^k$
- ii.  $Q_k(s) = 0$  if  $s < 0$
- iii.  $Q_k(s) = 0$  if  $s \neq rh_1, s \neq rh_2, s \neq rh_1 + qh_2$  for any integer  $r, q \neq 0$
- iv.  $Q_k(h_1) = \sum_{(v_1, \dots, v_k) \in P_{0(k-1), 1(1), 2(0)}} \prod_{i=1}^k A_{v_i}; k \geq 1$
- v.  $Q_k(h_1 - rh_2) = 0$  for all  $k \geq 1, r \neq 0$  integer
- vi.  $Q_1(jh_1) = \begin{cases} A_0 & \text{if } j = 0 \\ A_1 & \text{if } j = 1 \\ A_2 & \text{if } h_2 = jh_1, j = \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}; j \geq 2 \\ 0, & \text{otherwise} \end{cases}$
- vii.  $Q_k(jh_1) = 0$  if  $h_2 \neq rh_1$  for any integer  $j, r > 1$
- viii.  $X^{(k)}(t_1^-, t_1) = (-A_0)^k = (-1)^k A_0^k, k = 1, 2, 3, \dots$

$$\text{ix. } X^{(k)}(t_1^+, t_1) = 0$$

$$\text{x. } \Delta X(t_1 - jh, t_1) = \begin{cases} I_n & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases}$$

**Proof:**

$$\text{i. } Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h_1) + A_2 Q_{k-1}(s-h_2)$$

For  $k=1$ :

$$Q_1(0) = A_0 Q_0(0) + A_1 Q_0(-h_1) + A_2 Q_0(-h_2) = A_0$$

(Since  $Q_0(0) = I_n$ ,  $Q_0(-h_1) = 0$ , if  $-h_1 \neq 0$  and  $Q_0(-h_2) = 0$ , if  $-h_2 = 0$ )

So, (i) is true for  $k=1$ ; we next assume (i) is true for  $1 < k \leq n$  for any integer  $n$  such that  $Q_n(0) = A_0^n$ , then

$$\begin{aligned} Q_{n+1}(0) &= A_0 Q_n(0) + A_1 Q_n(-h_1) + A_2 Q_n(-h_2) \\ &= A_0 A_0^n + A_1 Q_n(-h_1) + A_2 Q_n(-h_2) \end{aligned}$$

**Assertion 1:**  $Q_k(s) = 0$  if  $s < 0$

**Proof:** For  $k=1$

$$Q_1(s) = A_0 Q_0(s) + A_1 Q_0(s-h_1) + A_2 Q_0(s-h_2) = 0$$

(Since  $s \neq 0$  and  $h_1, h_2 > 0$ , it implies that  $s \neq h_1$  and  $s \neq h_2$ ; and from the initial condition)

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that  $Q_n(s) = 0$ , then

$$Q_{n+1}(s) = A_0 Q_n(s) + A_1 Q_n(s-h_1) + A_2 Q_n(s-h_2) = 0$$

(Since  $Q_n(s) = 0$  and  $s \neq h_1, h_2$  implies  $s-h_1, s-h_2 < 0$  thus  $Q_n(s-h_1), Q_n(s-h_2) = 0$ )

Hence by the induction hypothesis:  $Q_k(s) = 0$  if  $s < 0$  which proves the assertion; we proceed to conclude the proof of (i):

$$Q_{n+1}(0) = A_0 A_0^n + A_1 Q_n(-h_1) + A_2 Q_n(-h_2) = A_0^{n+1} \text{ (Since } Q_k(s) = 0 \text{ if } s < 0)$$

Hence by the induction hypothesis:  $Q_k(0) = A_0^k$

$$\text{ii. } Q_k(s) = 0 \text{ if } s < 0 \text{ (The proof is embedded in (i) above)}$$

iii. Let  $k=1$  and  $s \neq rh_1$  or  $s \neq qh_2$ ,  $r, q$  integers, then

$$Q_1(s) = A_0 Q_0(s) + A_1 Q_0(s-h_1) + A_2 Q_0(s-h_2) = 0$$

(Since  $s \neq 0$  and  $h_1, h_2 > 0$ , it implies that  $s \neq h_1$  and  $s \neq h_2$ ; and from the initial condition)

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that  $Q_n(s) = 0$  if  $s \neq rh_1, s \neq qh_2, s \neq rh_1 + qh_2$  then

$$Q_{n+1}(s) = A_0 Q_n(s) + A_1 Q_n(s-h_1) + A_2 Q_n(s-h_2)$$

**Assertion 2:**  $Q_k(s-h_1) = 0$  if  $s \neq rh_1, s \neq rh_1 + qh_2$ ,  $r, q$  integers

**Proof:** Let  $k=1$ , then

$$Q_1(s-h_1) = A_0 Q_0(s-h_1) + A_1 Q_0(s-2h_1) + A_2 Q_0(s-h_1-h_2) = 0$$

(Since  $s \neq h_1, s \neq 2h_1, s \neq h_1 + h_2$  implies  $Q_0(s-h_1) = Q_0(s-2h_1) = Q_0(s-h_1-h_2) = 0$ )

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that  $Q_n(s-h_1)=0$  if  $s \neq rh_1, s \neq rh_1+qh_2, r, q$  integers, then

$$Q_{n+1}(s-h_1) = A_0Q_n(s-h_1) + A_1Q_n(s-2h_1) + A_2Q_n(s-h_1-h_2) = 0$$

(Since  $s \neq h_1, s \neq 2h_1, s \neq h_1+h_2$  implies  $Q_n(s-h_1) = Q_n(s-2h_1) = Q_n(s-h_1-h_2) = 0$ )

Hence by the induction hypothesis:  $Q_k(s-h_1) = 0$  if  $s \neq rh_1, s \neq rh_1+qh_2, r, q$  integers which proves the assertion.

Similarly,

**Assertion 3:**  $Q_k(s-h_2) = 0$  if  $s \neq qh_2, s \neq rh_1+qh_2, r, q$  integers

**Proof:** Let  $k=1$  then

$$Q_1(s-h_2) = A_0Q_0(s-h_2) + A_1Q_0(s-h_2-h_1) + A_2Q_0(s-2h_2) = 0$$

(Since  $s \neq h_2, s \neq 2h_2, s \neq h_1+h_2$  implies  $Q_0(s-h_2) = Q_0(s-2h_2) = Q_0(s-h_1-h_2) = 0$ )

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that  $Q_n(s-h_2) = 0$  if  $s \neq qh_2, s \neq rh_1+qh_2, r, q$  integers, then

$$Q_{n+1}(s-h_2) = A_0Q_n(s-h_2) + A_1Q_n(s-h_2-h_1) + A_2Q_n(s-2h_2) = 0$$

(Since  $s \neq h_2, s \neq 2h_2, s \neq h_1+h_2$  implies  $Q_n(s-h_2) = Q_n(s-2h_2) = Q_n(s-h_1-h_2) = 0$ )

Hence by induction hypothesis:  $Q_k(s-h_2) = 0$  if  $s \neq qh_2, s \neq rh_1+qh_2, r, q$  integers which proves the assertion.

Combining the Assertions 2 and 3 will yield:

$$Q_{n+1}(s) = A_0Q_n(s) + A_1Q_n(s-h_1) + A_2Q_n(s-h_2) = 0$$

And thus by the induction hypothesis:

$Q_k(s) = 0$  if  $s \neq rh_1, s \neq rh_2, s \neq rh_1+qh_2$  for any integer  $r, q \neq 0$ , which proves (iii).

iv. Let  $k=1$ , then

$$Q_1(h_1) = A_0Q_0(h_1) + A_1Q_0(0) + A_2Q_0(h_1-h_2) = A_1 = \sum_{v_1 \in P_{(0),1(1),2(0)}} A_{v_1}$$

(Since  $h_1 \neq 0, h_2 \neq 0$  implies  $Q_0(h_1) = Q_0(h_1-h_2) = 0$  and  $Q_0(0) = I_n$ )

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that

$Q_n(h_1) = \sum_{(v_1, \dots, v_n) \in P_{(n-1),1(1),2(0)}} \prod_{i=1}^n A_{v_i}$ , then

$$\begin{aligned} Q_{n+1}(h_1) &= A_0Q_n(h_1) + A_1Q_n(0) + A_2Q_n(h_1-h_2) \\ &= A_0 \sum_{(v_1, \dots, v_n) \in P_{(n-1),1(1),2(0)}} \prod_{i=1}^n A_{v_i} + A_1A_0^n + A_2Q_n(h_1-h_2) \end{aligned}$$

**Assertion 4:**  $Q_k(h_1-rh_2) = 0$  for all  $k \geq 1, r \neq 0$  integer

**Proof:** Let  $k=1$  then

$$Q_1(h_1-rh_2) = A_0Q_0(h_1-rh_2) + A_1Q_0(-rh_2) + A_2Q_0(h_1-(r+1)h_2) = 0$$

(Since  $h_2 > h_1$  so  $h_1 \neq rh_2$  and  $h_1 \neq (r+1)h_2$ , thus  $Q_0(h_1-rh_2) = Q_0(h_1-(r+1)h_2) = 0 = Q_0(-rh_2)$ )

So the assertion is true for  $k=1$ , we assume it is true for  $1 < k \leq n$  for some integer  $n$  such that  $Q_n(h_1-rh_2) = 0, r$  integer, then

$$Q_{n+1}(h_1-rh_2) = A_0Q_n(h_1-rh_2) + A_1Q_n(-rh_2) + A_2Q_n(h_1-(r+1)h_2) = 0$$

(Since  $Q_n(h_1 - rh_2) = Q_n(h_1 - (r+1)h_2) = 0$  and  $Q_n(-rh_2) = 0$ )

Hence by the induction hypothesis:  $Q_k(h_1 - rh_2) = 0$  for all  $k \geq 1$ ,  $r \neq 0$  integer which proves the assertion.

So to conclude the proof, we have:

$$Q_{n+1}(h_1) = A_0 \sum_{(v_1, \dots, v_n) \in P_{0(n-1), 1(1), 2(0)}} \prod_{i=1}^n A_{v_i} + A_1 A_0^n + A_2 \cdot 0$$

$$= \sum_{(v_1, \dots, v_{n+1}) \in P_{0(n), 1(1), 2(0)}} \prod_{i=1}^{n+1} A_{v_i} \text{ (with a leading } A_0) + \sum_{(v_1, \dots, v_{n+1}) \in P_{0(n), 1(1), 2(0)}} \prod_{i=1}^{n+1} A_{v_i} \text{ (with a leading } A_1)$$

Thus:  $Q_{n+1}(h_1) = \sum_{(v_1, \dots, v_{n+1}) \in P_{0(n), 1(1), 2(0)}} \prod_{i=1}^{n+1} A_{v_i}$

Hence by the induction hypothesis:  $Q_k(h_1) = \sum_{(v_1, \dots, v_k) \in P_{0(k-1), 1(1), 2(0)}} \prod_{i=1}^k A_{v_i}$ ;  $k \geq 1$  which proves (iv).

v. (The proof is embedded in (iv) above)

vi.  $Q_1(0) = A_0$  and  $Q_1(h_1) = A_1$  by Lemma (i) and (vi) respectively; then for  $j \geq 2$

$$Q_1(jh_1) = A_0 Q_0(jh_1) + A_1 Q_0([j-1]h_1) + A_2 Q_0(jh_1 - h_2) = A_2 Q_0(jh_1 - h_2)$$

(Since  $j, j-1 > 0$  then  $Q_0(jh_1) = Q_0([j-1]h_1) = 0$  from the initial condition) and

$$Q_0(jh_1 - h_2) = \begin{cases} I_n \text{ if } h_2 = jh_1 \Rightarrow j = \begin{bmatrix} h_2 \\ h_1 \end{bmatrix} \\ 0 \text{ otherwise} \end{cases}$$

Hence,  $Q_1(jh_1) = \begin{cases} A_2 \text{ if } h_2 = jh_1 \Rightarrow j = \begin{bmatrix} h_2 \\ h_1 \end{bmatrix}; \text{ which proves the lemma.} \\ 0 \text{ otherwise} \end{cases}$

vii. Let  $k = 1$  and let  $h_2 \neq rh_1$  for any integer  $r > 1$ . Then

$$Q_1(jh_1) = A_0 Q_0(jh_1) + A_1 Q_0([j-1]h_1) + A_2 Q_0(jh_1 - h_2)$$

From the initial condition; since  $h_2 \neq rh_1$  and  $j, r > 1$ , then

$Q_0(jh_1) = 0$  if  $jh_1 \neq 0$ ,  $Q_0([j-1]h_1) = 0$  if  $[j-1]h_1 \neq 0$ , and  $Q_0(jh_1 - h_2) = 0$  if  $h_2 \neq jh_1$ ; thus

$$Q_1(jh_1) = 0 \text{ for all } j, r > 1, h_2 \neq rh_1$$

Assume that this is true for  $2 \leq k \leq n$  for some integer  $n$ . Then

$$Q_{n+1}(jh_1) = A_0 Q_n(jh_1) + A_1 Q_n([j-1]h_1) + A_2 Q_n(jh_1 - h_2) = 0$$

by the induction hypothesis. Hence  $Q_k(jh_1) = 0$  if  $h_2 \neq rh_1$  for any integer  $j, r > 1$ . This result establishes that the determining matrices become zero if the larger delay term  $h_2$  is not an integer multiple of the smaller delay term  $h_1$ .

viii.  $X^{(k)}(t_1^-, t_1) = \frac{\partial}{\partial \tau} X^{(k-1)}(\tau, t_1) \Big|_{\tau=t_1^-}$

For  $k = 1$ , this yields

$$X^{(1)}(t_1^-, t_1) = -X(t_1^-, t_1)A_0 - X((t_1 + h_1)^-, t_1)A_1 - X((t_1 + h_2)^-, t_1)A_2$$

$$= -A_0 - 0 \cdot A_1 - 0 \cdot A_2 = -A_0 \text{ (Since } t_1 + h_1 > t_1 \text{ and } t_1 + h_2 > t_1)$$

Therefore  $X^{(1)}(t_1^-, t_1) = -A_0 = (-1)A_0$ . Thus (ix) is true for  $k = 1$

Note that for  $\tau$  sufficiently close to  $t_1$ ,  $\tau + h_1 > t_1$  and  $\tau + h_2 > t_1$

For  $k = 2$ :

$$\begin{aligned} X^{(2)}(t_1^-, t_1) &= -\left[\frac{\partial}{\partial \tau} X^{(1)}(\tau, t_1)\right]_{\tau=t_1^-} = X^{(1)}(t_1^-, t_1)A_0 - X^{(1)}((t_1 + h_1)^-, t_1)A_1 - X^{(1)}((t_1 + h_2)^-, t_1)A_2 \\ &= -(-A_0)A_0 - \left[-X((t_1 + h_1)^-, t_1)A_0 - X((t_1 + 2h_1)^-, t_1)A_1 - X((t_1 + h_1 + h_2)^-, t_1)A_2\right]A_1 \\ &\quad - \left[-X((t_1 + h_2)^-, t_1)A_0 - X((t_1 + h_2 + h_1)^-, t_1)A_1 - X((t_1 + 2h_2)^-, t_1)A_2\right]A_2 \\ &= A_0^2 - [-0.A_0 - 0.A_1 - 0.A_2]A_1 - [-0.A_0 - 0.A_1 - 0.A_2]A_2 = A_0^2 = (-1)^2 A_0^2 \end{aligned}$$

Thus (viii) is true for  $k = 2$ . We assume that  $X^{(k)}(t_1^-, t_1) = (-1)^k A_0^k$  for  $2 < k \leq n$ , for some integer  $n$ . Then

$$\begin{aligned} X^{(n+1)}(t_1^-, t_1) &= -\left[\frac{\partial}{\partial \tau} X^{(n)}(\tau, t_1)\right]_{\tau=t_1^-} = X^{(n)}(t_1^-, t_1)A_0 - X^{(n)}((t_1 + h_1)^-, t_1)A_1 - X^{(n)}((t_1 + h_2)^-, t_1)A_2 \\ &= (-1)(-1)^n A_0^n A_0 - 0.A_1 - 0.A_2 = (-1)^{n+1} A_0^{n+1} \end{aligned}$$

Therefore  $X^{(k)}(t_1^-, t_1) = (-1)^k A_0^k$ , proving lemma (viii).

ix. 
$$X^{(k)}(t_1^+, t_1) = \lim_{t_1 < \tau < t_1 + h_2} X^{(k)}(\tau, t_1) = 0 \text{ (Since } \tau > t_1 \text{)}$$

Therefore  $X^{(k)}(t_1^+, t_1) = 0$  proving lemma (ix).

x. 
$$\frac{\partial}{\partial \tau} X(\tau, t_1) = -X(\tau, t_1)A_0 - X(\tau + h_1, t_1)A_1 - X(\tau + h_2, t_1)A_2$$

for  $0 < \tau < t_1, \tau \neq t_1 - jh; h = \max(h_1, h_2), j = 0, 1, \dots$ , where

$$X(\tau, t) = \begin{cases} I_n, & \tau = t \\ 0, & \tau > t \end{cases}$$

Let  $j$  be a non-negative number such that  $t_1 - jh > 0$ . Then we integrate the system  $\frac{\partial}{\partial \tau} X(\tau, t_1)$ , apply the above initial matrix function condition and the fundamental theorem of calculus (F.T.C) to get:

$$\begin{aligned} \int_0^{(t_1 - jh)^-} \frac{\partial}{\partial \tau} [X(\tau, t_1)] d\tau &= X((t_1 - jh)^-, t_1) - X(0, t_1) \text{ (by the F.T.C)} \\ &= -\int_0^{(t_1 - jh)^-} [X(\tau, t_1)A_0 + X(\tau + h_1, t_1)A_1 + X(\tau + h_2, t_1)A_2] d\tau \end{aligned}$$

Similarly,

$$X((t_1 - jh)^+, t_1) - X(0, t_1) = -\int_0^{(t_1 - jh)^+} [X(\tau, t_1)A_0 + X(\tau + h_1, t_1)A_1 + X(\tau + h_2, t_1)A_2] d\tau$$

Therefore

$$\begin{aligned} &X((t_1 - jh)^-, t_1) - X((t_1 - jh)^+, t_1) \\ &= -\int_{(t_1 - jh)^-}^{(t_1 - jh)^+} [X(\tau, t_1)A_0 + X(\tau + h_1, t_1)A_1 + X(\tau + h_2, t_1)A_2] d\tau = 0 \end{aligned}$$

Since  $\tau \rightarrow X(\tau, t_1)A_0 + X(\tau + h_1, t_1)A_1 + X(\tau + h_2, t_1)A_2$  is bounded and integrable (being of bounded variation) and the fact that  $\lim_{\varepsilon \rightarrow 0} \int_{a-\varepsilon}^{a+\varepsilon} f(t) dt = 0$  for any bounded integrable function  $f$ . Therefore  $\Delta X(t_1 - jh, t_1) = 0$ , for  $j \neq 0$ .

For  $j = 0$ , we have:  $\Delta X(t_1, t_1) = X(t_1^-, t_1) - X(t_1^+, t_1) = I_n - 0 = I_n$ , which proves lemma (x).

### 3. Results

#### 3.1 Illustration of the Computation of the Determining Matrice $Q_k(jh_1); 0 \leq j \leq k, k \neq 0$ and $Q_k(jh_1); j > k \geq 1$ with respect to (1)

For notational optimality, we define:  $\text{SumProdPerm}[A_0(r_0), A_1(r_1), A_2(r_2)]$  to be the sum of the products of the permutations of  $A_0, A_1$  and  $A_2$  in which  $A_i$  appears  $r_i$  times,  $i \in \{0, 1, 2\}$

##### 3.1.1 Determining Matrices for $j = 2, k \geq 2$

For  $j = 2, k = 2$

$$Q_2(2h_1) = A_0Q_1(2h_1) + A_1Q_1(h_1) + A_2Q_1(2h_1 - h_2)$$

$$= \begin{cases} A_0A_2 + A_2A_0 + A_1^2; h_2 = 2h_1 \\ A_1^2; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \text{SumProdPerm}[A_0(1), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(2), A_2(0)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(0), A_1(2), A_2(0)]; \text{ otherwise} \end{cases}$$

For  $j = 2, k = 3$

$$Q_3(2h_1) = A_0Q_2(2h_1) + A_1Q_2(h_1) + A_2Q_2(2h_1 - h_2)$$

$$= \begin{cases} A_0^2A_2 + A_0A_1^2 + A_0A_2A_0 + A_1A_0A_1 + A_1^2A_0 + A_2A_0^2; h_2 = 2h_1 \\ A_0A_1^2 + A_1A_0A_1 + A_1^2A_0; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \text{SumProdPerm}[A_0(2), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(2), A_2(0)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(1), A_1(2), A_2(0)]; \text{ otherwise} \end{cases}$$

For  $j = 2, k = 4$

$$Q_4(2h_1) = A_0Q_3(2h_1) + A_1Q_3(h_1) + A_2Q_3(2h_1 - h_2)$$

$$= \begin{cases} A_0^3A_2 + A_0^2A_1^2 + A_0^2A_2A_0 + A_0A_1A_0A_1 + A_0A_1^2A_0 + A_0A_2A_0^2 + A_1A_0^2A_1 + A_1A_0A_1A_0 + A_1^2A_0^2 + A_2A_0^3; h_2 = 2h_1 \\ A_0^2A_1^2 + A_0A_1A_0A_1 + A_0A_1^2A_0 + A_1A_0^2A_1 + A_1A_0A_1A_0 + A_1^2A_0^2; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \text{SumProdPerm}[A_0(3), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(2), A_1(2), A_2(0)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(2), A_1(2), A_2(0)]; \text{ otherwise} \end{cases}$$

For  $j = 2, k = 5$

$$Q_5(2h_1) = A_0Q_4(2h_1) + A_1Q_4(h_1) + A_2Q_4(2h_1 - h_2)$$



$$\begin{aligned}
 & \left\{ \begin{aligned} & A_0^4 A_2 + A_0^3 A_1^2 + A_0^3 A_2 A_0 + A_0^2 A_1 A_0 A_1 + A_0^2 A_1^2 A_0 + A_0^2 A_2 A_0^2 + A_0 A_1 A_0^2 A_1 + A_0 A_1 A_0 A_1 A_0 \\ & + A_0 A_1^2 A_0^2 + A_0 A_2 A_0^3 + A_1 A_0^3 A_1 + A_1 A_0^2 A_1 A_0 + A_1 A_0 A_1 A_0^2 + A_1^2 A_0^3 + A_2 A_0^4; h_2 = 2h_1 \end{aligned} \right. \\
 = & \left\{ \begin{aligned} & A_0^3 A_1^2 + A_0^2 A_1 A_0 A_1 + A_0^2 A_1^2 A_0 + A_0 A_1 A_0^2 A_1 + A_0 A_1 A_0 A_1 A_0 + A_0 A_1^2 A_0^2 + A_1 A_0^3 A_1 + A_1 A_0^2 A_1 A_0 \\ & + A_1 A_0 A_1 A_0^2 + A_1^2 A_0^3; \text{ otherwise} \end{aligned} \right. \\
 = & \left\{ \begin{aligned} & \text{SumProdPerm} [A_0(4), A_1(0), A_2(1)] + \text{SumProdPerm} [A_0(3), A_1(2), A_2(0)]; h_2 = 2h_1 \\ & \text{SumProdPerm} [A_0(3), A_1(2), A_2(0)]; \text{ otherwise} \end{aligned} \right.
 \end{aligned}$$

Emerging pattern of  $Q_k(jh_1)$  for  $j = 2, k \geq 2$ :

$$= \left\{ \begin{aligned} & \sum_{r=0}^1 \text{SumProdPerm} [A_0(r+k-2), A_1(2-2r), A_2(r)]; h_2 = 2h_1 \\ & \text{SumProdPerm} [A_0(k-2), A_1(2), A_2(0)]; \text{ otherwise} \end{aligned} \right.$$

### 3.1.2 Determining Matrices for $j = 3, k \geq 2$

For  $j = 3, k = 2$

$$\begin{aligned}
 & Q_2(3h_1) = A_0 Q_1(3h_1) + A_1 Q_1(2h_1) + A_2 Q_1(3h_1 - h_2) \\
 = & \left\{ \begin{aligned} & A_1 A_2 + A_2 A_1; h_2 = 2h_1 \\ & A_0 A_2 + A_2 A_0; h_2 = 3h_1 \\ & 0; \text{ otherwise} \end{aligned} \right. \\
 = & \left\{ \begin{aligned} & \text{SumProdPerm} [A_0(0), A_1(1), A_2(1)]; h_2 = 2h_1 \\ & \text{SumProdPerm} [A_0(1), A_1(0), A_2(1)]; h_2 = 3h_1 \\ & 0; \text{ otherwise} \end{aligned} \right.
 \end{aligned}$$

For  $j = 3, k = 3$

$$\begin{aligned}
 & Q_3(3h_1) = A_0 Q_2(3h_1) + A_1 Q_2(2h_1) + A_2 Q_2(3h_1 - h_2) \\
 = & \left\{ \begin{aligned} & A_0 A_1 A_2 + A_0 A_2 A_1 + A_1 A_0 A_2 + A_1^3 + A_1 A_2 A_0 + A_2 A_0 A_1 + A_2 A_1 A_0; h_2 = 2h_1 \\ & A_0^2 A_2 + A_0 A_2 A_0 + A_1^3 + A_2 A_0^2; h_2 = 3h_1 \\ & A_1^3 \end{aligned} \right.
 \end{aligned}$$

$$= \begin{cases} \text{SumProdPerm}[A_0(1), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(3), A_2(0)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(2), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(3), A_2(0)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(0), A_1(3), A_2(0)]; \text{ otherwise} \end{cases}$$

For  $j = 3, k = 4$

$$Q_4(3h_1) = A_0Q_3(3h_1) + A_1Q_3(2h_1) + A_2Q_3(3h_1 - h_2)$$

$$= \begin{cases} A_0^2A_1A_2 + A_0^2A_2A_1 + A_0A_1A_0A_2 + A_0A_1^3 + A_0A_1A_2A_0 + A_0A_2A_0A_1 + A_0A_2A_1A_0 + A_1A_0^2A_2 + A_1A_0A_1^2 \\ + A_1A_0A_2A_0 + A_1^2A_0A_1 + A_1^3A_0 + A_1A_2A_0^2 + A_2A_0^2A_1 + A_2A_0A_1A_0 + A_2A_1A_0^2; h_2 = 2h_1 \\ A_0^3A_2 + A_0^2A_2A_0 + A_0A_1^3 + A_0A_2A_0^2 + A_1A_0A_1^2 + A_1^2A_0A_1 + A_1^3A_0 + A_2A_0^3; h_2 = 3h_1 \\ A_0A_1^3 + A_1A_0A_1^2 + A_1^2A_0A_1 + A_1^3A_0; \text{ otherwise} \end{cases}$$

$$= \begin{cases} \text{SumProdPerm}[A_0(2), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(3), A_2(0)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(3), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(3), A_2(0)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(1), A_1(3), A_2(0)]; \text{ otherwise} \end{cases}$$

Emerging pattern of  $Q_k(jh_1)$  for  $j = 3, k \geq 3$

$$= \begin{cases} \sum_{r=0}^1 \text{SumProdPerm}[A_0(r+k-3), A_1(3-2r), A_2(r)]; h_2 = 2h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(2r+k-3), A_1(3-3r), A_2(r)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(k-3), A_1(3), A_2(0)]; \text{ otherwise} \end{cases}$$

**3.1.3 Determining Matrices for  $j = 4, k \geq 2$**

For  $j = 4, k = 2$

$$Q_2(4h_1) = A_0Q_1(4h_1) + A_1Q_1(3h_1) + A_2Q_1(4h_1 - h_2)$$

$$= \left\{ \begin{array}{l} A_2^2; h_2 = 2h_1 \\ A_1A_2 + A_2A_1; h_2 = 3h_1 \\ A_0A_2 + A_2A_0; h_2 = 4h_1 \\ 0; \text{ otherwise} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm}[A_0(0), A_1(0), A_2(2)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(0), A_1(1), A_2(1)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(1), A_1(0), A_2(1)]; h_2 = 4h_1 \\ 0; \text{ otherwise} \end{array} \right.$$

For  $j = 4, k = 3$

$$Q_3(4h_1) = A_0Q_2(4h_1) + A_1Q_2(3h_1) + A_2Q_2(4h_1 - h_2)$$

$$= \left\{ \begin{array}{l} A_0A_2^2 + A_1^2A_2 + A_1A_2A_1 + A_2A_0A_2 + A_2A_1^2 + A_2^2A_0; h_2 = 2h_1 \\ A_0A_1A_2 + A_0A_2A_1 + A_1A_0A_2 + A_1A_2A_0 + A_2A_0A_1 + A_2A_1A_0; h_2 = 3h_1 \\ A_0^2A_2 + A_0A_2A_0 + A_2A_0^2; h_2 = 4h_1 \\ 0; \text{ otherwise} \\ \text{SumProdPerm}[A_0(1), A_1(0), A_2(2)] + \text{SumProdPerm}[A_0(0), A_1(2), A_2(1)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(1), A_1(1), A_2(1)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(2), A_1(0), A_2(1)]; h_2 = 4h_1 \\ 0; \text{ otherwise} \end{array} \right.$$

For  $j = 4, k = 4$

$$Q_4(4h_1) = A_0Q_3(4h_1) + A_1Q_3(3h_1) + A_2Q_3(4h_1 - h_2)$$

$$\begin{cases}
A_0^2 A_2^2 + A_0 A_1^2 A_2 + A_0 A_1 A_2 A_1 + A_0 A_2 A_0 A_2 + A_0 A_2 A_1^2 + A_0 A_2^2 A_0 + A_1 A_0 A_1 A_2 + A_1^2 A_0 A_2 \\
+ A_1^4 + A_1^2 A_2 A_0 + A_1 A_2 A_0 A_1 + A_1 A_2 A_1 A_0 + A_2 A_0^2 A_2 + A_2 A_0 A_1^2 + A_2 A_0 A_2 A_0 + A_2 A_1 A_0 A_1 \\
+ A_2 A_1^2 A_0 + A_2^2 A_0^2; h_2 = 2h_1 \\
\\
A_0^2 A_1 A_2 + A_0^2 A_2 A_1 + A_0 A_1 A_0 A_2 + A_0 A_1 A_2 A_0 + A_0 A_2 A_0 A_1 + A_0 A_2 A_1 A_0 + A_1 A_0^2 A_2 + A_1 A_0 A_2 A_0 \\
+ A_1^4 + A_1 A_2 A_0^2 + A_2 A_0^2 A_1 + A_2 A_0 A_1 A_0 + A_2 A_1 A_0^2; h_2 = 3h_1 \\
\\
A_0^3 A_2 + A_0^2 A_2 A_0 + A_0 A_2 A_0^2 + A_2 A_0^3 + A_1^4; h_2 = 4h_1 \\
\\
A_1^4; \text{ otherwise} \\
\\
\text{SumProdPerm}[A_0(2), A_1(0), A_2(2)] + \text{SumProdPerm}[A_0(1), A_1(2), A_2(1)] \\
+ \text{SumProdPerm}[A_0(0), A_1(4), A_2(0)]; h_2 = 2h_1 \\
\\
\text{SumProdPerm}[A_0(2), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(4), A_2(0)]; h_2 = 3h_1 \\
\\
\text{SumProdPerm}[A_0(3), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(4), A_2(0)]; h_2 = 4h_1 \\
\\
\text{SumProdPerm}[A_0(0), A_1(4), A_2(0)]; \text{ otherwise}
\end{cases}$$

For  $j = 4, k = 5$

$$Q_5(4h_1) = A_0 Q_4(4h_1) + A_1 Q_4(3h_1) + A_2 Q_4(4h_1 - h_2)$$

$$\begin{aligned}
 & \left\{ \begin{aligned}
 & A_0^3 A_2^2 + A_0^2 A_1^2 A_2 + A_0^2 A_1 A_2 A_1 + A_0^2 A_2 A_0 A_2 + A_0^2 A_2 A_1^2 + A_0^2 A_2^2 A_0 + A_0 A_1 A_0 A_1 A_2 \\
 & + A_0 A_1 A_0 A_2 A_1 + A_0 A_1^2 A_0 A_2 + A_0 A_1^4 + A_0 A_1^2 A_2 A_0 + A_0 A_1 A_2 A_0 A_1 + A_0 A_1 A_2 A_1 A_0 + A_0 A_2 A_0^2 A_2 \\
 & + A_0 A_2 A_0 A_1^2 + A_0 A_2 A_0 A_2 A_0 + A_0 A_2 A_1 A_0 A_1 + A_0 A_2 A_1^2 A_0 + A_0 A_2^2 A_0^2 + A_1 A_0^2 A_1 A_2 + A_1 A_0^2 A_2 A_1 \\
 & + A_1 A_0 A_1 A_0 A_2 + A_1 A_0 A_1^3 + A_1 A_0 A_1 A_2 A_0 + A_1 A_0 A_2 A_0 A_1 + A_1 A_0 A_2 A_1 A_0 + A_1^2 A_0^2 A_2 + A_1^2 A_0 A_1^2 \\
 & + A_1^2 A_0 A_2 A_0 + A_1^3 A_0 A_1 + A_1^4 A_0 + A_1^2 A_2 A_0^2 + A_1 A_2 A_0^2 A_1 + A_1 A_2 A_0 A_1 A_0 + A_1 A_2 A_1 A_0^2 + A_2 A_0^3 A_2 \\
 & + A_2 A_0^2 A_1^2 + A_2 A_0^2 A_2 A_0 + A_2 A_0 A_1 A_0 A_1 + A_2 A_0 A_1^2 A_0 + A_2 A_0 A_2 A_0^2 + A_2 A_1 A_0^2 A_1 + A_2 A_1 A_0 A_1 A_0 \\
 & + A_2 A_1^2 A_0 + A_2^2 A_0^3; h_2 = 2h_1
 \end{aligned} \right. \\
 = & \left\{ \begin{aligned}
 & A_0^3 A_1 A_2 + A_0^3 A_2 A_1 + A_0^2 A_1 A_0 A_2 + A_0^2 A_1 A_2 A_0 + A_0^2 A_2 A_0 A_1 + A_0^2 A_2 A_1 A_0 + A_0 A_1 A_0^2 A_2 + A_0 A_1 A_0 A_2 A_0 \\
 & + A_0 A_1^4 + A_0 A_1 A_2 A_0^2 + A_0 A_2 A_0^2 A_1 + A_0 A_2 A_0 A_1 A_0 + A_0 A_2 A_1 A_0^2 + A_1 A_0^3 A_2 + A_1 A_0^2 A_2 A_0 + A_1 A_0 A_1^3 \\
 & + A_1 A_0 A_2 A_0^2 + A_1^2 A_0 A_1^2 + A_1^3 A_0 A_1 + A_1^4 A_0 + A_1 A_2 A_0^3 + A_2 A_0^3 A_1 + A_2 A_0^2 A_1 A_0 + A_2 A_0 A_1 A_0^2 \\
 & + A_2 A_1 A_0^3; h_2 = 3h_1
 \end{aligned} \right. \\
 & \left\{ \begin{aligned}
 & A_0^4 A_2 + A_0^3 A_2 A_0 + A_0^2 A_2 A_0^2 + A_0 A_2 A_0^3 + A_1 A_0 A_1^3 A_1^2 A_0 A_1^2 + A_1^3 A_0 A_1 + A_1^4 A_0 + A_2 A_0^4 + A_0 A_1^4; h_2 = 4h_1 \\
 & A_0 A_1^4 + A_1 A_0 A_1^3 + A_1^2 A_0 A_1^2 + A_1^3 A_0 A_1 + A_1^4 A_0; \text{ otherwise}
 \end{aligned} \right. \\
 & \left\{ \begin{aligned}
 & \text{SumProdPerm}[A_0(3), A_1(0), A_2(2)] + \text{SumProdPerm}[A_0(2), A_1(2), A_2(1)] \\
 & + \text{SumProdPerm}[A_0(1), A_1(4), A_2(0)]; h_2 = 2h_1
 \end{aligned} \right. \\
 = & \left\{ \begin{aligned}
 & \text{SumProdPerm}[A_0(3), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(4), A_2(0)]; h_2 = 3h_1 \\
 & \text{SumProdPerm}[A_0(4), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(4), A_2(0)]; h_2 = 4h_1 \\
 & \text{SumProdPerm}[A_0(1), A_1(4), A_2(0)]; \text{ otherwise}
 \end{aligned} \right.
 \end{aligned}$$

Emerging pattern for  $j = 4, k \geq 4$

$$Q_k(4h_1) = \begin{cases} \sum_{r=0}^2 \text{SumProdPerm}[A_0(r+k-4), A_1(4-2r), A_2(r)]; h_2 = 2h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(2r+k-4), A_1(4-3r), A_2(r)]; h_2 = 3h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(3r+k-4), A_1(4-4r), A_2(r)]; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(k-4), A_1(4), A_2(0)]; \text{ otherwise} \end{cases}$$

**3.1.4 Determining Matrices for  $j = 5, k \geq 2$**

For  $j = 5, k = 2$

$$Q_2(5h_1) = A_0Q_1(5h_1) + A_1Q_1(4h_1) + A_2Q_1(5h_1 - h_2)$$

$$= \begin{cases} 0; h_2 = 2h_1 \\ 0; h_2 = 3h_1 \\ A_1A_2 + A_2A_1; h_2 = 4h_1 \\ A_0A_2 + A_2A_0; h_2 = 5h_1 \\ 0; \text{ otherwise} \\ 0; h_2 = 2h_1 \\ 0; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(0), A_1(1), A_2(1)]; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(1), A_1(0), A_2(1)]; h_2 = 5h_1 \\ 0; \text{ otherwise} \end{cases}$$

For  $j = 5, k = 3$

$$Q_3(5h_1) = A_0Q_2(5h_1) + A_1Q_2(4h_1) + A_2Q_2(5h_1 - h_2)$$

$$\begin{aligned}
 & \left\{ \begin{aligned} & A_1 A_2^2 + A_2 A_1 A_2 + A_2^2 A_1; h_2 = 2h_1 \\ & A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2; h_2 = 3h_1 \\ & A_0 A_1 A_2 + A_0 A_2 A_1 + A_1 A_0 A_2 + A_1 A_2 A_0 + A_2 A_0 A_1 + A_2 A_1 A_0; h_2 = 4h_1 \\ & A_0^2 A_2 + A_0 A_2 A_0 + A_2 A_0^2; h_2 = 5h_1 \\ & 0; \text{ otherwise} \end{aligned} \right. \\
 & \left\{ \begin{aligned} & \text{SumProdPerm}[A_0(0), A_1(1), A_2(2)]; h_2 = 2h_1 \\ & \text{SumProdPerm}[A_0(0), A_1(2), A_2(1)]; h_2 = 3h_1 \\ & \text{SumProdPerm}[A_0(1), A_1(1), A_2(1)]; h_2 = 4h_1 \\ & \text{SumProdPerm}[A_0(2), A_1(0), A_2(1)]; h_2 = 5h_1 \\ & 0; \text{ otherwise} \end{aligned} \right.
 \end{aligned}$$

For  $j = 5, k = 4$

$$Q_4(5h_1) = A_0 Q_3(5h_1) + A_1 Q_3(4h_1) + A_2 Q_3(5h_1 - h_2)$$

$$\begin{aligned}
 & \left\{ \begin{aligned} & A_0 A_1 A_2^2 + A_0 A_2 A_1 A_2 + A_0 A_2^2 A_1 + A_1 A_0 A_2^2 + A_1^3 A_2 + A_1^2 A_2 A_1 + A_1 A_2 A_0 A_2 + A_1 A_2 A_1^2 + A_1 A_2^2 A_0 \\ & + A_2 A_0 A_1 A_2 + A_2 A_0 A_2 A_1 + A_2 A_1 A_0 A_1 + A_2 A_1^2 A_0 + A_2^2 A_0 A_2 + A_2^2 A_1 A_0; h_2 = 2h_1 \\ & A_0 A_1^2 A_2 + A_0 A_1 A_2 A_1 + A_0 A_2 A_1^2 + A_1 A_0 A_1 A_2 + A_1 A_0 A_2 A_1 + A_1^2 A_0 A_2 + A_1^2 A_2 A_0 + A_1 A_2 A_0 A_1 + A_1 A_2 A_1 A_0 \\ & + A_2 A_0 A_1^2 + A_2 A_1 A_0 A_1 + A_2 A_1^2 A_0; h_2 = 3h_1 \\ & A_0^2 A_1 A_2 + A_0^2 A_2 A_1 + A_0 A_1 A_0 A_2 + A_0 A_1 A_2 A_0 + A_0 A_2 A_0 A_1 + A_0 A_2 A_1 A_0 + A_1 A_0^2 A_2 + A_1 A_0 A_2 A_0 + A_1 A_2 A_0^2 \\ & + A_2 A_0^2 A_1 + A_2 A_0 A_1 A_0 + A_2 A_1 A_0^2; h_2 = 4h_1 \\ & A_0^3 A_2 + A_0^2 A_2 A_0 + A_0 A_2 A_0^2 + A_2 A_0^3; h_2 = 5h_1 \\ & 0; \text{ otherwise} \end{aligned} \right.
 \end{aligned}$$

$$\left\{ \begin{aligned}
 & \text{SumProdPerm}[A_0(1), A_1(1), A_2(2)] + \text{SumProdPerm}[A_0(0), A_1(3), A_2(1)] \\
 & + \text{SumProdPerm}[A_0(1), A_1(2), A_2(1)]; h_2 = 2h_1 \\
 \\
 & \text{SumProdPerm}[A_0(1), A_1(2), A_2(1)]; h_2 = 3h_1 \\
 \\
 & \text{SumProdPerm}[A_0(2), A_1(1), A_2(1)]; h_2 = 4h_1 \\
 \\
 & \text{SumProdPerm}[A_0(3), A_1(0), A_2(1)]; h_2 = 5h_1 \\
 \\
 & 0; \text{ otherwise}
 \end{aligned} \right.$$

For  $j = 5, k = 5$

$$Q_5(5h_1) = A_0Q_4(5h_1) + A_1Q_4(4h_1) + A_2Q_4(5h_1 - h_2)$$

$$\left\{ \begin{aligned}
 & A_0^2A_1A_2^2 + A_0^2A_2A_1A_2 + A_0^2A_2^2A_1 + A_0A_1A_0A_2^2 + A_0A_1^3A_2 + A_0A_1^2A_2A_1 + A_0A_1A_2A_0A_2 + A_0A_1A_2A_1^2 \\
 & + A_0A_1A_2^2A_0 + A_0A_2A_0A_1A_2 + A_0A_2A_0A_2A_1 + A_0A_2A_1A_0A_1 + A_0A_2A_1^2A_0 + A_0A_2^2A_0A_2 + A_0A_2^2A_1A_0 \\
 & + A_1A_0^2A_2^2 + A_1A_0A_1^2A_2 + A_1A_0A_1A_2A_1 + A_1A_0A_2A_0A_2 + A_1A_0A_2A_1^2 + A_1A_0A_2^2A_0 + A_1^2A_0A_1A_2 + A_1^2A_0A_2A_1 \\
 & + A_1^3A_0A_2 + A_1^5 + A_1^3A_2A_0 + A_1^2A_2A_0A_1 + A_1^2A_2A_1A_0 + A_1A_2A_0^2A_2 + A_1A_2A_0A_1^2 + A_1A_2A_0A_2A_0 \\
 & + A_1A_2A_1A_0A_1 + A_1A_2A_1^2A_0 + A_1A_2^2A_0^2 + A_2A_0^2A_1A_2 + A_2A_0^2A_2A_1 + A_2A_0A_1A_0A_2 + A_2A_0A_1^3 + A_2A_0A_1A_2A_0 \\
 & + A_2A_0A_2A_0A_2 + A_2A_0A_2A_1A_0 + A_2A_1A_0^2A_2 + A_2A_1A_0A_1^2 + A_2A_1A_0A_2A_0 + A_2A_1^2A_0A_1 + A_2A_1^3A_0 + A_2A_1A_2A_0^2 \\
 & + A_2^2A_0^2A_2 + A_2^2A_0A_1A_0 + A_2^2A_1A_0^2; h_2 = 2h_1 \\
 \\
 & A_0^2A_1^2A_2 + A_0^2A_1A_2A_1 + A_0^2A_2A_1^2 + A_0A_1A_0A_1A_2 + A_0A_1^2A_0A_2 + A_0A_1^2A_2A_0 + A_0A_1A_2A_0A_1 + A_0A_1A_2A_1A_0 \\
 & + A_0A_2A_0A_1^2 + A_0A_2A_1A_0A_1 + A_0A_2A_1^2A_0 + A_1A_0^2A_1A_2 + A_1A_0^2A_2A_1 + A_1A_0A_1A_0A_2 + A_1A_0A_1A_2A_0 \\
 & + A_1A_0A_2A_0A_1 + A_1A_0A_2A_1A_0 + A_1^2A_0^2A_2 + A_1^2A_0A_2A_0 + A_1^5 + A_1^2A_2A_0^2 + A_1A_2A_0^2A_1 + A_1A_2A_0A_1A_0 \\
 & + A_1A_2A_1A_0^2 + A_2A_0^2A_1^2 + A_2A_0A_1A_0A_1 + A_2A_0A_1^2A_0 + A_2A_1A_0^2A_1 + A_2A_1A_0A_1A_0 + A_2A_1^2A_0^2; h_2 = 3h_1 \\
 \\
 & A_0^3A_1A_2 + A_0^3A_2A_1 + A_0^2A_1A_0A_2 + A_0^2A_1A_2A_0 + A_0^2A_2A_0A_1 + A_0^2A_2A_0A_1 + A_0A_1A_0^2A_2 + A_0A_1A_0A_2A_0 \\
 & + A_0A_1A_2A_0^2 + A_0A_2A_0^2A_1 + A_0A_2A_0A_1A_0 + A_0A_2A_1A_0^2 + A_1A_0^3A_2 + A_1A_0^2A_2A_0 + A_1A_0A_2A_0^2 + A_1A_2A_0^3 \\
 & + A_1^5 + A_2A_0^3A_1 + A_2A_0^2A_1A_0 + A_2A_0A_1A_0^2 + A_2A_1A_0^3; h_2 = 4h_1 \\
 \\
 & A_0^4A_2 + A_0^3A_2A_0 + A_0^2A_2A_0^2 + A_0A_2A_0^3 + A_2A_0^4 + A_1^5; h_2 = 5h_1 \\
 \\
 & A_1^5; \text{ otherwise}
 \end{aligned} \right.$$



$$\left. \begin{aligned}
 & \text{SumProdPerm} [A_0(2), A_1(1), A_2(2)] + \text{SumProdPerm} [A_0(1), A_1(3), A_2(1)] \\
 & + \text{SumProdPerm} [A_0(0), A_1(5), A_2(0)]; h_2 = 2h_1 \\
 \\
 & \text{SumProdPerm} [A_0(2), A_1(2), A_2(1)] + \text{SumProdPerm} [A_0(0), A_1(5), A_2(0)]; h_2 = 3h_1 \\
 \\
 & \text{SumProdPerm} [A_0(3), A_1(1), A_2(1)] + \text{SumProdPerm} [A_0(0), A_1(5), A_2(0)]; h_2 = 4h_1 \\
 \\
 & \text{SumProdPerm} [A_0(4), A_1(0), A_2(1)] + \text{SumProdPerm} [A_0(0), A_1(5), A_2(0)]; h_2 = 5h_1 \\
 \\
 & \text{SumProdPerm} [A_0(0), A_1(5), A_2(0)]; \text{ otherwise}
 \end{aligned} \right\} =$$

For  $j = 5, k = 6$

$$Q_6(5h_1) = A_0 Q_5(5h_1) + A_1 Q_5(4h_1) + A_2 Q_5(5h_1 - h_2)$$

$$\left. \begin{aligned}
 & \text{SumProdPerm} [A_0(3), A_1(1), A_2(2)] + \text{SumProdPerm} [A_0(2), A_1(3), A_2(1)] \\
 & + \text{SumProdPerm} [A_0(1), A_1(5), A_2(0)]; h_2 = 2h_1 \\
 \\
 & \text{SumProdPerm} [A_0(3), A_1(2), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(5), A_2(0)]; h_2 = 3h_1 \\
 \\
 & \text{SumProdPerm} [A_0(4), A_1(1), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(5), A_2(0)]; h_2 = 4h_1 \\
 \\
 & \text{SumProdPerm} [A_0(5), A_1(0), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(5), A_2(0)]; h_2 = 5h_1 \\
 \\
 & \text{SumProdPerm} [A_0(1), A_1(5), A_2(0)]; \text{ otherwise}
 \end{aligned} \right\} =$$

Emerging pattern for  $j = 5, k \geq 5$

$$Q_k(5h_1) = \begin{cases} \sum_{r=0}^2 \text{SumProdPerm}[A_0(r+k-5), A_1(5-2r), A_2(r)]; h_2 = 2h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(2r+k-5), A_1(5-3r), A_2(r)]; h_2 = 3h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(3r+k-5), A_1(5-4r), A_2(r)]; h_2 = 4h_1 \\ \sum_{r=0}^1 \text{SumProdPerm}[A_0(4r+k-5), A_1(5-5r), A_2(r)]; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(k-5), A_1(5), A_2(0)]; \text{ otherwise} \end{cases}$$

**3.1.5 Determining Matrices for  $j = 6, k \geq 2$**

For  $j = 6, k = 2$

$$Q_2(6h_1) = A_0Q_1(6h_1) + A_1Q_1(5h_1) + A_2Q_1(6h_1 - h_2)$$

$$= \begin{cases} 0; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(0), A_1(0), A_2(2)]; h_2 = 3h_1 \\ 0; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(0), A_1(1), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(1), A_1(0), A_2(1)]; h_2 = 6h_1 \\ 0; \text{ otherwise} \end{cases}$$

For  $j = 6, k = 3$

$$Q_3(6h_1) = A_0Q_2(6h_1) + A_1Q_2(5h_1) + A_2Q_2(6h_1 - h_2)$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm} [A_0(0), A_1(0), A_2(3)]; h_2 = 2h_1 \\ \text{SumProdPerm} [A_0(1), A_1(0), A_2(2)]; h_2 = 3h_1 \\ \text{SumProdPerm} [A_0(0), A_1(2), A_2(1)]; h_2 = 4h_1 \\ \text{SumProdPerm} [A_0(1), A_1(1), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm} [A_0(2), A_1(0), A_2(1)]; h_2 = 6h_1 \\ 0; \text{ otherwise} \end{array} \right.$$

For  $j = 6, k = 4$

$$Q_4(6h_1) = A_0Q_3(6h_1) + A_1Q_3(5h_1) + A_2Q_3(6h_1 - h_2)$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm} [A_0(1), A_1(0), A_2(3)] + \text{SumProdPerm} [A_0(0), A_1(2), A_2(2)]; h_2 = 2h_1 \\ \text{SumProdPerm} [A_0(2), A_1(0), A_2(2)] + \text{SumProdPerm} [A_0(0), A_1(3), A_2(1)]; h_2 = 3h_1 \\ \text{SumProdPerm} [A_0(1), A_1(2), A_2(1)]; h_2 = 4h_1 \\ \text{SumProdPerm} [A_0(2), A_1(1), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm} [A_0(3), A_1(0), A_2(1)]; h_2 = 6h_1 \\ 0; \text{ otherwise} \end{array} \right.$$

For  $j = 6, k = 5$

$$Q_5(6h_1) = A_0Q_4(6h_1) + A_1Q_4(5h_1) + A_2Q_4(6h_1 - h_2)$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm}[A_0(2), A_1(0), A_2(3)] + \text{SumProdPerm}[A_0(1), A_1(2), A_2(2)] \\ + \text{SumProdPerm}[A_0(0), A_1(4), A_2(1)]; h_2 = 2h_1 \\ \\ \text{SumProdPerm}[A_0(3), A_1(0), A_2(2)] + \text{SumProdPerm}[A_0(1), A_1(3), A_2(1)]; h_2 = 3h_1 \\ \\ \text{SumProdPerm}[A_0(2), A_1(2), A_2(1)]; h_2 = 4h_1 \\ \\ \text{SumProdPerm}[A_0(3), A_1(1), A_2(1)]; h_2 = 5h_1 \\ \\ \text{SumProdPerm}[A_0(4), A_1(0), A_2(1)]; h_2 = 6h_1 \\ \\ 0; \text{ otherwise} \end{array} \right.$$

For  $j = 6, k = 6$

$$Q_6(6h_1) = A_0Q_5(6h_1) + A_1Q_5(5h_1) + A_2Q_5(6h_1 - h_2)$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm}[A_0(3), A_1(0), A_2(3)] + \text{SumProdPerm}[A_0(2), A_1(2), A_2(2)] \\ + \text{SumProdPerm}[A_0(1), A_1(4), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; h_2 = 2h_1 \\ \\ \text{SumProdPerm}[A_0(4), A_1(0), A_2(2)] + \text{SumProdPerm}[A_0(2), A_1(3), A_2(1)] \\ + \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; h_2 = 3h_1 \\ \\ \text{SumProdPerm}[A_0(3), A_1(2), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; h_2 = 4h_1 \\ \\ \text{SumProdPerm}[A_0(4), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; h_2 = 5h_1 \\ \\ \text{SumProdPerm}[A_0(5), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; h_2 = 6h_1 \\ \\ \text{SumProdPerm}[A_0(0), A_1(6), A_2(0)]; \text{ otherwise} \end{array} \right.$$

For  $j = 6, k = 7$

$$Q_7(6h_1) = A_0Q_6(6h_1) + A_1Q_6(5h_1) + A_2Q_6(6h_1 - h_2)$$

$$= \left\{ \begin{array}{l} \text{SumProdPerm} [A_0(4), A_1(0), A_2(3)] + \text{SumProdPerm} [A_0(3), A_1(2), A_2(2)] \\ + \text{SumProdPerm} [A_0(2), A_1(4), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; h_2 = 2h_1 \\ \\ \text{SumProdPerm} [A_0(5), A_1(0), A_2(2)] + \text{SumProdPerm} [A_0(3), A_1(3), A_2(1)] \\ + \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; h_2 = 3h_1 \\ \\ \text{SumProdPerm} [A_0(4), A_1(2), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; h_2 = 4h_1 \\ \\ \text{SumProdPerm} [A_0(5), A_1(1), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; h_2 = 5h_1 \\ \\ \text{SumProdPerm} [A_0(6), A_1(0), A_2(1)] + \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; h_2 = 6h_1 \\ \\ \text{SumProdPerm} [A_0(1), A_1(6), A_2(0)]; \text{ otherwise} \end{array} \right.$$

Emerging pattern for  $j = 6, k \geq 6$

$$Q_k(6h_1) = \left\{ \begin{array}{l} \sum_{r=0}^3 \text{SumProdPerm} [A_0(r+k-6), A_1(6-2r), A_2(r)]; h_2 = 2h_1 \\ \\ \sum_{r=0}^2 \text{SumProdPerm} [A_0(2r+k-6), A_1(6-3r), A_2(r)]; h_2 = 3h_1 \\ \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(3r+k-6), A_1(6-4r), A_2(r)]; h_2 = 4h_1 \\ \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(4r+k-6), A_1(6-5r), A_2(r)]; h_2 = 5h_1 \\ \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(5r+k-6), A_1(6-6r), A_2(r)]; h_2 = 6h_1 \\ \\ \text{SumProdPerm} [A_0(k-6), A_1(6), A_2(0)]; \text{ otherwise} \end{array} \right.$$

### 3.1.6 Determining Matrices for $j = 7, k \geq 2$

For  $j = 7, k = 2$

$$Q_2(7h_1) = A_0Q_1(7h_1) + A_1Q_1(6h_1) + A_2Q_1(7h_1 - h_2)$$

$$= \left\{ \begin{array}{l} 0; h_2 = 2h_1 \\ 0; h_2 = 3h_1 \\ 0; h_2 = 4h_1 \\ 0; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(0), A_1(1), A_2(1)]; h_2 = 6h_1 \\ \text{SumProdPerm}[A_0(1), A_1(0), A_2(1)]; h_2 = 7h_1 \\ 0; \text{otherwise} \end{array} \right.$$

For  $j = 7, k = 3$

$$Q_3(7h_1) = A_0 Q_2(7h_1) + A_1 Q_2(6h_1) + A_2 Q_2(7h_1 - h_2)$$

$$= \left\{ \begin{array}{l} 0; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(0), A_1(1), A_2(2)]; h_2 = 3h_1 \\ 0; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(0), A_1(2), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(1), A_1(1), A_2(1)]; h_2 = 6h_1 \\ \text{SumProdPerm}[A_0(2), A_1(0), A_2(1)]; h_2 = 7h_1 \\ 0; \text{otherwise} \end{array} \right.$$

For  $j = 7, k = 4$

$$Q_4(7h_1) = A_0 Q_3(7h_1) + A_1 Q_3(6h_1) + A_2 Q_3(7h_1 - h_2)$$

$$= \begin{cases} \text{SumProdPerm}[A_0(0), A_1(1), A_2(3)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(1), A_1(1), A_2(2)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(0), A_1(3), A_2(1)]; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(1), A_1(2), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(2), A_1(1), A_2(1)]; h_2 = 6h_1 \\ \text{SumProdPerm}[A_0(3), A_1(0), A_2(1)]; h_2 = 7h_1 \\ 0; \text{ otherwise} \end{cases}$$

For  $j = 7, k = 5$

$$Q_5(7h_1) = A_0Q_4(7h_1) + A_1Q_4(6h_1) + A_2Q_4(7h_1 - h_2)$$

$$= \begin{cases} \text{SumProdPerm}[A_0(1), A_1(1), A_2(3)] + \text{SumProdPerm}[A_0(0), A_1(3), A_2(2)]; h_2 = 2h_1 \\ \text{SumProdPerm}[A_0(2), A_1(1), A_2(2)] + \text{SumProdPerm}[A_0(0), A_1(4), A_2(1)]; h_2 = 3h_1 \\ \text{SumProdPerm}[A_0(1), A_1(3), A_2(1)]; h_2 = 4h_1 \\ \text{SumProdPerm}[A_0(2), A_1(2), A_2(1)]; h_2 = 5h_1 \\ \text{SumProdPerm}[A_0(3), A_1(1), A_2(1)]; h_2 = 6h_1 \\ \text{SumProdPerm}[A_0(4), A_1(0), A_2(1)]; h_2 = 7h_1 \\ 0; \text{ otherwise} \end{cases}$$

For  $j = 7, k = 7$

$$Q_7(7h_1) = A_0Q_6(7h_1) + A_1Q_6(6h_1) + A_2Q_6(7h_1 - h_2)$$

$$\begin{aligned}
& \left\{ \begin{aligned}
& \text{SumProdPerm}[A_0(3), A_1(1), A_2(3)] + \text{SumProdPerm}[A_0(2), A_1(3), A_2(2)] \\
& + \text{SumProdPerm}[A_0(1), A_1(5), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 2h_1 \\
& \\
& \text{SumProdPerm}[A_0(4), A_1(1), A_2(2)] + \text{SumProdPerm}[A_0(2), A_1(4), A_2(1)] \\
& + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 3h_1 \\
& \\
& \text{SumProdPerm}[A_0(3), A_1(3), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 4h_1 \\
& \\
& \text{SumProdPerm}[A_0(4), A_1(2), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 5h_1 \\
& \\
& \text{SumProdPerm}[A_0(5), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 6h_1 \\
& \\
& \text{SumProdPerm}[A_0(6), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; h_2 = 7h_1 \\
& \\
& \text{SumProdPerm}[A_0(0), A_1(7), A_2(0)]; \text{ otherwise}
\end{aligned} \right.
\end{aligned}$$

For  $j = 7, k = 8$

$$Q_8(7h_1) = A_0Q_7(7h_1) + A_1Q_7(6h_1) + A_2Q_7(7h_1 - h_2)$$

$$\begin{aligned}
& \left\{ \begin{aligned}
& \text{SumProdPerm}[A_0(4), A_1(1), A_2(3)] + \text{SumProdPerm}[A_0(3), A_1(3), A_2(2)] \\
& + \text{SumProdPerm}[A_0(2), A_1(5), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 2h_1 \\
& \\
& \text{SumProdPerm}[A_0(5), A_1(1), A_2(2)] + \text{SumProdPerm}[A_0(3), A_1(4), A_2(1)] \\
& + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 3h_1 \\
& \\
& \text{SumProdPerm}[A_0(4), A_1(3), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 4h_1 \\
& \\
& \text{SumProdPerm}[A_0(5), A_1(2), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 5h_1 \\
& \\
& \text{SumProdPerm}[A_0(6), A_1(1), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 6h_1 \\
& \\
& \text{SumProdPerm}[A_0(7), A_1(0), A_2(1)] + \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; h_2 = 7h_1 \\
& \\
& \text{SumProdPerm}[A_0(1), A_1(7), A_2(0)]; \text{ otherwise}
\end{aligned} \right.
\end{aligned}$$

Emerging pattern for  $j = 7, k \geq 7$



$$Q_k(7h_1) = \left\{ \begin{array}{l} \sum_{r=0}^3 \text{SumProdPerm} [A_0(r+k-7), A_1(7-2r), A_2(r)]; h_2 = 2h_1 \\ \sum_{r=0}^2 \text{SumProdPerm} [A_0(2r+k-7), A_1(7-3r), A_2(r)]; h_2 = 3h_1 \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(3r+k-7), A_1(7-4r), A_2(r)]; h_2 = 4h_1 \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(4r+k-7), A_1(7-5r), A_2(r)]; h_2 = 5h_1 \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(5r+k-7), A_1(7-6r), A_2(r)]; h_2 = 6h_1 \\ \sum_{r=0}^1 \text{SumProdPerm} [A_0(6r+k-7), A_1(7-7r), A_2(r)]; h_2 = 7h_1 \\ \text{SumProdPerm} [A_0(k-7), A_1(7), A_2(0)]; \text{ otherwise} \end{array} \right.$$

From these expressions of the determining matrices for particular cases, a general expression for the determining matrices  $Q_k(jh_1); 0 \leq j \leq k, k \neq 0$  and  $Q_k(jh_1); j > k \geq 1$  can be formulated and proved.

**4. Conclusion**

The results on determining matrices derived here greatly extended the published works of Ukwu (2016) and Ukwu (2014). Moreover, the results here went further to examine the structure and computing complexity of the generic case of the double time-delay control systems that previous authors did not examine in this field in simpler systems. This work has successfully prosecuted a number of tasks; by exploiting relevant facts about the permutation of matrices already established in Ukwu (2014a) and deft application of mathematical induction principles, it is established that determining matrices for double time-delay linear autonomous functional differential control systems do not exist if one of the time-delays is an integer multiple of the other and the structures of the determining matrices for particular cases of the system in consideration follow certain patterns which can be further investigated to determine the Euclidean controllability of the system.

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