Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Analytical Solutions of Fractional Differential Equations via the Variational Iteration Method within the Modified Conformable Fractional Derivative

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| ABSTRACT
In this paper, we develop a variational iteration method for solving linear and nonlinear fractional differential equations within the modified conformable calculus framework. Rather than reducing the fractional system to integer-order derivatives, we construct the correction functional directly using the intrinsic conformable differential measure. Through fractional integration by parts and variational stationary conditions, we determine the optimal Lagrange multiplier. This derivation fundamentally confirms that the variational kernel inevitably yields the modified conformable fractional integral operator, while the recurrence formulation inherently satisfies the prescribed fractional initial conditions. The proposed algorithm circumvents computationally expensive convolution integrals and Gamma functions, relying exclusively on standard factorials. The proposed method is applied to different types of fractional problems, yielding closed-form series solutions that converge rapidly to the exact solutions. The results show the proposed algorithm's superior accuracy, simplicity, and computational efficiency.

| KEYWORDS
Modified conformable fractional derivative; Modified conformable fractional integral; Variational iteration method

| ARTICLE INFORMATION
ACCEPTED: 08 October 2026 **PUBLISHED:** 04 November 2026 **DOI:** 10.32996/jmss.2026.7.6.15

1. Introduction

In recent years, fractional differential equations (FDEs) have transformed mathematical modeling across scientific and engineering disciplines. By accurately representing memory, hereditary traits, and anomalous behaviors in physics, engineering, and mathematical biology, FDEs successfully overcome the inherent limitations of classical integer-order models that fail to adequately govern [12,15].

In the literature, there are many fractional differential operators, including Riemann–Liouville, Caputo, Grünwald- Letnikov, Hadamard, and Cossar derivatives [7,18]. Despite their wide applicability, these classical operators often lack standard computational properties such as the Leibniz product rule, the quotient rule, and the chain rule, and they frequently require computationally intensive evaluations of gamma functions and convolution integrals [15,13]. However, in the past few years, Khalil et al. [11] introduced a new concept of derivative termed the conformable fractional derivative (CFD), which depends on the basic limit definition of the derivative. They claimed that this definition is the simplest, most natural, and efficient approach to fractional derivatives, with properties that closely align with the classical derivative and satisfy the classical product, quotient, and chain rules, making it easier to solve fractional differential equations. Subsequently, researchers have devoted significant effort to advancing CFD calculus and investigating its properties. T. Abdeljawad [1] established the theoretical foundation of conformable calculus by developing fractional versions of integration by parts, exponential functions, and Taylor series expansions. Following these advances, researchers adapted numerous analytical and numerical techniques to conformable differential equations, including the differential transform method [19], the homotopy analysis method [4], the adomian decomposition method [3], and the residual power series method [16].

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Nonetheless, the CFD formulation and its associated conformable fractional integral (CFI) have limitations: differentiation and integration operators generally do not commute, and sequential derivatives and integrals are not equivalent to higher-order operators, creating mathematical ambiguity when constructing differential equations of order greater than one. Recently, El-Ajou [8] proposed a modified conformable fractional derivative (MCFD) and a modified conformable fractional integral (MCFI). With this modification, the operators regain the commutative property and establish complete equivalence between sequential and higher-order operations. In this context, Vinodbhai and Dubey [20] developed the invariant subspace method to obtain exact solutions for linear and nonlinear time-space fractional mixed partial differential equations. Additionally, Altalla et al. [6] utilized the MCFD framework to formulate a generalized Taylor's series and establish fractional power series solutions for various fractional differential equations.

The variational iteration method (VIM), proposed by He [9,10], is a prominent semi-analytical approach widely used to solve ordinary and partial differential equations across linear, nonlinear, homogeneous, and inhomogeneous cases. The method's primary advantage is its flexibility and ability to handle nonlinear equations directly, yielding convergent successive approximations toward the exact solution, provided it exists. For instance, Momani and Odibat [14] utilized this approach to handle nonlinear FDEs. As an extension to the conformable fractional derivative (CFD), Acan et al. [2] proposed the conformable VIM to solve FDEs. Furthermore, Ahmed and Hussain [5] extended it to linear and nonlinear systems of differential equations, while Wazwaz [21] used VIM to solve ordinary differential equations with variable coefficients.

This work aims to develop the VIM within the MCFD sense and to formulate the correction functional using the fundamental conformable differential measure $\mathrm{d}_\alpha\zeta = (\zeta - \mathrm{s})^{\alpha-1}\mathrm{d}$. We analytically derive the optimal Lagrange multiplier $\lambda(\zeta, \mathrm{t})$ analytically for arbitrary integer orders $\mathrm{m} \geq 1$ via fractional integration by parts, demonstrating that the resulting variational kernel naturally yields the exact MCFI operator $(\mathrm{J}_{\mathrm{s}}^{\mathrm{m}\alpha})$. To demonstrate the method's applicability, accuracy, and convergence, we investigate several challenging nonlinear initial-value problems, including a fractional delay differential equation, a sequential nonlinear problem, and a fourth-order nonlinear model with variable coefficients, and show that the series solutions converge rapidly to the exact solutions.

The remainder of this paper is structured as follows: Section 2 presents the theories and concepts underlying our findings. Section 3 presents the details of the mathematical derivation of the VIM within the MCFD framework. Section 4 presents three numerical examples to confirm the method's feasibility and superiority. Finally, Section 5 presents the conclusion of our work to confirm the method's feasibility.

2. Preliminaries

This section reviews the essential definitions and fundamental concepts related to modified conformable fractional calculus:

Definition 2.1. [1] The CFD of a function $f: [s, \infty) \to \mathbb{R}$ of order $\alpha \in (0,1]$is defined by:

$$T_s^\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-s)^{1-\alpha}) - f(t)}{\epsilon}, \qquad t > s \tag{1}$$

If $(T_s^\alpha f)$ exists on (s, w) then $(T_s^\alpha f)(s) = \lim 1_{t \to s^+} (T_s^\alpha f)(t)$

Definition 2.2. [8] Let $\langle \alpha: 0 < \alpha \leq 1 \rangle$ be a cyclic subgroup of $(R, +)$ generated by α. A real number $\beta \in (m-1, m]$ where $m \in N$ is said to belong to the class of α denoted by $\beta \in \chi_\alpha$, if:

$$\beta \in \chi_\alpha = \left\{\alpha: \frac{m-1}{m} < \alpha \leq 1,\ \alpha = \frac{\beta}{m}\right\} \tag{2}$$

Definition 2.3. [8] The MCFD of a function $f: [s, \infty) \to \mathbb{R}$ for order $\beta \in \chi_\alpha$ is defined by:

$$\mathcal{T}_s^\beta f(t) = \lim_{\epsilon \to 0} \frac{f^{((m-1)\alpha)}(t + \epsilon(t-s)^{1-\alpha}) - f^{((m-1)\alpha)}(t)}{\epsilon}, \quad t > s, \tag{3}$$

provided that $f(t)$ is $\big((m-1)\alpha\big)$-differentiable for $t > s$.

Definition 2.4. [8] The n-sequential MCFD of a function $f: [s, \infty) \to R$ of order $\alpha \in (0,1]$ is defined by:

$$\mathcal{T}_s^{n\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f^{((n-1)\alpha)}(t + \epsilon(t-s)^{1-\alpha}) - f^{((n-1)\alpha)}(t)}{\epsilon}, \quad t > s, \tag{4}$$

provided that $f(t)$ is $\big((n-1)\alpha\big)$-differentiable for $t > s$.

In operator form, the derivative satisfies the recurrence relation:

$$\mathcal{T}_s^{n\alpha} f(t) = \underbrace{\mathcal{T}_s^\alpha \circ \mathcal{T}_s^\alpha \circ \ldots \circ \mathcal{T}_s^\alpha}_{\text{n times}} f(\mathrm{t}) = (t-s)^{1-\alpha} \frac{d}{dt}\left(\mathcal{T}_s^{(n-1)\alpha} f(t)\right) \tag{5}$$

Definition 2.5. [8] A real function $f(t)$,for $t > s$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $\kappa > \mu$ such that $f(t) = (t-s)^\kappa f_1(t)$, where $f_1(t) \in C[s, \infty)$

Definition 2.6. [8] The MCFI of a function $f \in C_\mu, \mu \geq -\alpha$ starting at s of order $\beta \in \chi_\alpha$ is defined by:

$$J_s^\beta f(t) = \frac{1}{\alpha^{m-1}(m-1)!}\int_s^t \frac{((t-s)^\alpha - (\zeta-s)^\alpha)^{m-1}}{(\zeta-s)^{1-\alpha}} f(\zeta)\, d\zeta, \qquad t > s \tag{6}$$

Definition 2.7. [8] The n-sequential MCFI of a function $f \in C_\mu, \mu \geq -\alpha$ starting at s of order $\alpha \in (0,1]$ is defined by:

$$J_s^{n\alpha} f(t) = \frac{1}{\alpha^{n-1}(n-1)!}\int_s^t \frac{((t-s)^\alpha - (\zeta-s)^\alpha)^{n-1}}{(\zeta-s)^{1-\alpha}} f(\zeta)\, d\zeta, \qquad t > s \tag{7}$$

For $n = 1$, it simplifies to the fundamental conformable integral:

$$J_s^\alpha f(t) = \int_s^t (\zeta-s)^{\alpha-1} f(\zeta)\, d\zeta = \int_s^t f(\zeta)\, d_\alpha \zeta \tag{8}$$

Lemma 2.8. [1] Let $f, g: [a,b] \to \mathbb{R}$ be sufficiently conformable differentiable functions. Then :

$$\int_s^t f(t)\big(\mathcal{T}_s^\alpha g(t)\big)(t-s)^{\alpha-1}dt = [f(t)g(t)]_s^t - \int_s^t \big(\mathcal{T}_s^\alpha f(t)\big)\, g(t)(t-s)^{\alpha-1}dt \tag{9}$$

Lemma 2.9. [8] For $\beta \in \chi_\alpha, n \in \mathbb{N}, f \in C_\mu, \mu \geq -\alpha$ and $t > s$, then:

1. $J_s^\beta (t-s)^\zeta = \frac{(t-s)^{\zeta+\beta}}{\prod_{k=1}^m(\zeta+k\alpha)}$,
2. $J_s^\beta (C) = C\frac{(t-s)^\beta}{m!\alpha^m}$, $t > s$ and C is a constant.
3. $T_s^\beta J_s^\beta f(t) = f(t)$.
4. $J_s^\beta \mathcal{T}_s^\beta f(t) = f(t) - \sum_{r=0}^{m-1} f^{(r\alpha)}(s^+)\frac{(t-s)^{r\alpha}}{r!\alpha^r}$.

Lemma 2.10. [8] For $\alpha \in (0,1], n \in \mathbb{N}, f \in C_\mu, \mu \geq -\alpha$ and $t > s$, then

1. $J_s^{n\alpha}(t-s)^\zeta = \frac{(t-a)^{\zeta+n\alpha}}{\prod_{k=1}^n(\zeta+k\alpha)}$
2. $J_s^{n\alpha}(C) = C\frac{(t-s)^{n\alpha}}{n!\alpha^n}$, $t > s$ and C is a constant.
3. $\mathcal{T}_s^{n\alpha} J_s^{n\alpha} f(t) = f(t)$.
4. $J_s^{n\alpha}\mathcal{T}_s^{n\alpha} f(t) = f(t) - \sum_{r=0}^{n-1} f^{(r\alpha)}(s^+)\frac{(t-s)^{r\alpha}}{r!\alpha^r}$.

3. Variational Iteration Method for solving FDEs

In this section, we adapt the VIM to solve various fractional differential equations within the MCFD framework. By constructing the correction functional directly upon the conformable measure, we derive the optimal Lagrange multiplier via variational principles rather than classical reductions. This formulation satisfies the fractional initial conditions and produces rapidly convergent series solutions.

Consider the non-linear fractional differential equation of order $\beta \in \chi_\alpha$:

$$\mathcal{T}_s^\beta u(t) + L\big(u(t)\big) + N\big(u(t)\big) = g(t), \quad t > s, \tag{10}$$

subject to the fractional initial conditions:

$$\mathcal{T}_s^{r\alpha} u(s) = c_r, \quad r = 0,1,\dots,m-1, \tag{11}$$

where $\mathcal{T}_s^\beta$ is the MCFD operator, L is a lower-order linear differential operator, N is a non-linear operator, and $g(t)$ is a source term. According to the variational iteration method, the correction functional is constructed using the fundamental conformable differential measure $d_\alpha\zeta = (\zeta-s)^{\alpha-1}d$ with a general unknown Lagrange multiplier $\lambda(\zeta,t)$:

$$u_{k+1}(t) = u_k(t) + \int_s^t \lambda(\zeta,t)\Big[\mathcal{T}_s^\beta u_k(\zeta) + L\big(\widetilde{u_k}(\zeta)\big) + N\big(\widetilde{u_k}(\zeta)\big) - g(\zeta)\Big]\, d_\alpha\zeta \tag{12}$$

where $\widetilde{u_k}$ denotes a restricted variation (i.e., $\delta\widetilde{u_k} = 0$).

Note: Since $\beta \in \chi_\alpha$ we can rewrite $\beta = m\alpha$.

Now, we take the first variation δ with respect to u_k on both sides of Equation (12).

$$\delta u_{k+1}(t) = \delta u_k(t) + \delta \int_s^t \lambda(\zeta,t)\left[\mathcal{J}_s^{m\alpha} u_k(\zeta) + L\left(\widetilde{u_k}(\zeta)\right) + N\left(\widetilde{u_k}(\zeta)\right) - g(\zeta)\right] d_\alpha$$

Since $\delta L(\widetilde{u_k}) = \delta N(\widetilde{u_k}) = 0$ by the restricted variation, and $\delta g(\zeta) = 0$ as $g(\zeta)$ does not depend on u_k, So we obtain:

$$\delta u_{k+1}(t) = \delta u_k(t) + \int_s^t \lambda(\zeta,t)\left(\mathcal{J}_s^{m\alpha}\delta u_k(\zeta)\right) d_\alpha\,\zeta \tag{13}$$

To evaluation of the Lagrange Multiplier $\lambda(\zeta, t)$

Case 1: For $m = 1$, Equation (13) becomes:

$$\delta u_{k+1}(t) = \delta u_k(t) + \int_s^t \lambda(\zeta,t)\left(\mathcal{J}_s^{\alpha}\delta u_k(\zeta)\right) d_\alpha$$

Applying fractional integration by parts once and using the initial boundary condition $\delta u_k(s) = 0$:

$$\begin{aligned}\delta u_{k+1}(t) &= \delta u_k(t) + [\lambda(\zeta,t)\delta u_k(\zeta)]_s^t - \int_s^t \left(\mathcal{J}_s^{\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_\alpha\zeta \\ &= \left(1 + \lambda(\zeta,t)|_{\zeta=t}\right)\delta u_k(t) - \int_s^t \left(\mathcal{J}_s^{\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_\alpha\zeta\end{aligned}$$

To obtain the optimal multiplier, we impose the stationary condition $\delta u_{k+1}(t) = 0$, which yields the Euler-Lagrange dual system:

$$\begin{cases}\mathcal{J}_s^{\alpha}\lambda(\zeta,t) = 0, & \zeta < t \\ 1 + \lambda(\zeta,t)|_{\zeta=t} = 0 \Longrightarrow \lambda(\zeta,t)|_{\zeta=t} = -1\end{cases}$$

Solving this system directly gives:

$$\lambda(\zeta,t) = -1. \tag{14}$$

Therefore, we obtain the following iteration formula:

$$u_{k+1}(t) = u_k(t) - \int_s^t \left[\mathcal{J}_s^{\beta} u_k(\zeta) + L\left(u_k(\zeta)\right) + N\left(u_k(\zeta)\right) - g(\zeta)\right] d_\alpha\zeta \tag{15}$$

In this case, we begin with the initial approximation: $u_0(t) = u(0)$

Case 2: For $m = 2$, Equation (13) becomes:

$$\delta u_{k+1}(t) = \delta u_k(t) + \int_s^t \lambda(\zeta,t)\left(\mathcal{J}_s^{m\alpha}\delta u_k(\zeta)\right) d_\alpha\,\zeta \tag{16}$$

Since $\mathcal{J}_s^{2\alpha} = \mathcal{J}_s^{\alpha}(\mathcal{J}_s^{\alpha})$,by integrating by parts twice, we obtain:

$$\begin{aligned}\delta u_{k+1}(t) &= \delta u_k(t) + \int_s^t \lambda\,(\zeta,t)\left[\mathcal{J}_s^{\alpha}\left(\mathcal{J}_s^{\alpha}\delta u_k(\zeta)\right)\right](\zeta - s)^{\alpha-1} d\zeta \\ &= \delta u_k(t) + [\lambda(\zeta,t)\mathcal{J}_s^{\alpha}\delta u_k(\zeta)]_s^t - [\left(\mathcal{J}_s^{\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)]_s^t + \int_s^t \left(\mathcal{J}_s^{2\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_\alpha\zeta\end{aligned}$$

Applying the conditions $\delta u_k(s) = 0$ and $\mathcal{J}_s^{\alpha}\delta u_k(\zeta)|_{\zeta=s} = 0$ we obtain:

$$\begin{aligned}\delta u_{k+1}(t) &= \delta u_k(t) + \lambda(\zeta,t)|_{\zeta=t}\mathcal{J}_s^{\alpha}\delta u_k(\zeta)|_{\zeta=t} - \mathcal{J}_s^{\alpha}\lambda(\zeta,t)|_{\zeta=t}\delta u_k(t) + \int_s^t \left(\mathcal{J}_s^{2\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_\alpha\zeta \\ &= \left(1 - \mathcal{J}_s^{\alpha}\lambda(\zeta,t)|_{\zeta=t}\right)\delta u_k(t) + \lambda(\zeta,t)|_{\zeta=t}\mathcal{J}_s^{\alpha}\delta u_k(\zeta)|_{\zeta=t} + \int_s^t \left(\mathcal{J}_s^{2\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_\alpha\zeta\end{aligned}$$

Setting $\delta u_{k+1}(t) = 0$ provides the stationary conditions:

$$\begin{cases}\mathcal{J}_s^{2\alpha}\lambda(\zeta,t) = 0, & \zeta < t \\ \lambda(\zeta,t)|_{\zeta=t} = 0 \\ 1 - \mathcal{J}_s^{\alpha}\lambda(\zeta,t)|_{\zeta=t} = 0 \Longrightarrow \mathcal{J}_s^{\alpha}\lambda(\zeta,t)|_{\zeta=t} = 1\end{cases} \tag{17}$$

The general solution to $\mathcal{J}_s^{2\alpha}\lambda(\zeta,t) = 0$ centered at $\zeta = t$ is:

$$\lambda(\zeta,t) = a_0 + a_1\left(\frac{(\zeta - s)^{\alpha} - (t - s)^{\alpha}}{\alpha}\right) \tag{18}$$

Applying the initial conditions at $\zeta = t$ from Equation (17):

$$\lambda(\zeta,t)|_{\zeta=t} = a_0 = 0 \qquad \text{and} \qquad \mathcal{T}_s^{\alpha}\lambda(\zeta,t)|_{\zeta=t} = a_1 = 1.$$

Hence, the Lagrange multiplier is:

$$\lambda(\zeta,t) = \frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha} \tag{19}$$

Therefore, we obtain the following iteration formula:

$$u_{k+1}(t) = u_k(t) + \int_s^t \left(\frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha}\right)\left[\mathcal{T}_s^{2\alpha}u_k(\zeta) + L\big(u_k(\zeta)\big) + N\big(u_k(\zeta)\big) - g(\zeta)\right]d_{\alpha}\zeta \tag{20}$$

In this case, we begin with the initial approximation:

$$u_0(t) = \sum_{r=0}^{1} u^{r\alpha}(0)\frac{(t-s)^{r\alpha}}{\alpha^r r!} \tag{21}$$

Case 3: Generalization for Any Integer $\mathrm{m} \geq 1$

In fact, if we apply the fractional integration by parts for $m$times to Equation (13) yields:

$$\begin{aligned}\delta u_{k+1}(t) = \delta u_k(t) \quad &+\sum_{j=1}^{m}(-1)^{j-1}\left[\mathcal{T}_s^{(j-1)\alpha}\lambda(\zeta,t)\cdot\mathcal{T}_s^{(m-j)\alpha}\delta u_k(\zeta)\right]_s^t \\ &+(-1)^m\int_s^t\left(\mathcal{T}_s^{m\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_{\alpha}\zeta\end{aligned} \tag{22}$$

Because all initial fractional variations vanish at $\zeta = s$ (i.e., $\mathcal{T}_s^{r\alpha}\delta u_k(\zeta)|_{\zeta=s} = 0$ for $r = 0,1,\dots,m-1$), the boundary terms at $\zeta = t$ give:

$$\begin{aligned}\delta u_{k+1}(t) \quad &= \delta u_k(t) + (-1)^{m-1}\mathcal{T}_s^{(m-1)\alpha}\lambda(\zeta,t)\Big|_{\zeta=t}\delta u_k(t) \\ &+\sum_{j=1}^{m-1}(-1)^{j-1}\mathcal{T}_s^{(j-1)\alpha}\lambda(\zeta,t)\Big|_{\zeta=t}\left(\mathcal{T}_s^{(m-j)\alpha}\delta u_k(\zeta)\Big|_{\zeta=t}\right) \\ &+(-1)^m\int_s^t\left(\mathcal{T}_s^{m\alpha}\lambda(\zeta,t)\right)\delta u_k(\zeta)\, d_{\alpha}\zeta\end{aligned}$$

Setting $\delta u_{k+1}(t) = 0$ leads the following generalized Euler-Lagrange dual system :

$$\begin{cases}\mathcal{T}_s^{m\alpha}\lambda(\zeta,t) = 0, & \zeta < t \\ \mathcal{T}_s^{r\alpha}\lambda(\zeta,t)|_{\zeta=t} = 0, & r = 0,1,\dots,m-2 \\ 1+(-1)^{m-1}\mathcal{T}_s^{(m-1)\alpha}\lambda(\zeta,t)\Big|_{\zeta=t} = 0 \Rightarrow \mathcal{T}_s^{(m-1)\alpha}\lambda(\zeta,t)\Big|_{\zeta=t} = (-1)^m\end{cases}$$

Expanding $\lambda(\zeta,t)$ using the MCFD Taylor series centered at $\zeta = t$:

$$\lambda(\zeta,t) = \sum_{n=0}^{m-1}\frac{1}{n!}\left(\frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha}\right)^n\mathcal{T}_s^{n\alpha}\lambda(\zeta,t)|_{\zeta=t} \tag{23}$$

Substituting the boundary conditions into the series, all terms vanish except for the $(m-1)$ −th term:

$$\lambda(\zeta,t) = \frac{(-1)^m}{(m-1)!}\left(\frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha}\right)^{m-1} \tag{24}$$

Therefore, in general, we can obtain the following iteration formula:

$$u_{k+1}(t) = u_k(t) + \frac{(-1)^m}{(m-1)!}\int_s^t\left(\frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha}\right)^{m-1}\left[\mathcal{T}_s^{m\alpha}u_k(\zeta) + L\big(u_k(\zeta)\big) + N\big(u_k(\zeta)\big) - g(\zeta)\right]d_{\alpha}\zeta \tag{25}$$

In this case, we begin with the initial approximation:

$$u_0(t) = \sum_{r=0}^{m-1} u^{r\alpha}(0)\frac{(t-s)^{r\alpha}}{\alpha^r r!} \tag{26}$$

In fact, the solution of Equation (10) is considered as the fixed point of the following functional by starting with the initial condition $u_0(t)$ given in (26).

Now, for the nonlinear mapping associated with the iterative formula (25):

$$\mathcal{A}[u](t) = u(t) + \frac{(-1)^m}{(m-1)!}\int_s^t\left(\frac{(\zeta-s)^{\alpha}-(t-s)^{\alpha}}{\alpha}\right)^{m-1}\left[\mathcal{T}_s^{m\alpha}u(\zeta) + L\big(u(\zeta)\big) + N\big(u(\zeta)\big) - g(\zeta)\right]d_{\alpha}\zeta \tag{27}$$

A sufficient condition for the convergence of the proposed VIM is that the mapping $\mathcal{A}$ is a contraction. Under this condition, the sequence generated by $u_{k+1} = \mathcal{A}[u_k]$ converges to the fixed point of $\mathcal{A}$, which is also the exact solution of Equation (10).

Theorem 3.1. [12] (Banach's Fixed Point Theorem). Assume that X is a Banach space and $\mathcal{A}: X \to X$ is a nonlinear mapping, and suppose that:

$$\| \mathcal{A}[u] - \mathcal{A}[v] \| \leq \gamma \| u - v \|, \quad u, v \in X \tag{28}$$

for some constant $\gamma < 1$. Then $\mathcal{A}$ has a unique fixed point, and the sequence $u_{k+1} = A[u_k]$, with an arbitrary choice of $u_0 \in X$, converges to the fixed point of $\mathcal{A}$.

Theorem 3.2. Let $X = (C[s,T], \|\cdot\|_\infty)$ be a Banach space equipped with the norm $\| u \|_\infty = \max_{t\in[s,T]}|u(t)|$. Assume that the operators L and N satisfy the following Lipschitz conditions:

$$\| L(u) - L(v) \|_\infty \leq L_1 \| u - v \|_\infty, \quad \| N(u) - N(v) \|_\infty \leq L_2 \| u - v \|_\infty \tag{29}$$

where L_1 and L_2 are positive Lipschitz constants. If:

$$\gamma = \frac{(L_1+L_2)(T-s)^{m\alpha}}{m!\alpha^m} < 1 \tag{30}$$

then the sequence $\{u_k\}_{k=0}^{\infty}$ generated by (27) converges to the unique exact solution of Equation 10, which is obtained by:

$$u(t) = \lim_{k\to\infty} u_k(t) \tag{31}$$

Proof. Using the MCFI operator, the integral term in the mapping can be expressed using the $J_s^{m\alpha}$ operator as follows:

$$\mathcal{A}[u](t) = u(t) - J_s^{m\alpha}\left[\mathcal{T}_s^{m\alpha}u(t) + L\big(u(t)\big) + N\big(u(t)\big) - g(t)\right] \tag{32}$$

Applying the fundamental inverse identity from Lemma 2.9, we have:

$$J_s^{m\alpha}[\mathcal{T}_s^{m\alpha}u(t)] = u(t) - \sum_{r=0}^{m-1} \mathcal{T}_s^{r\alpha}\, u(s^+)\frac{(t-s)^{r\alpha}}{r!\,\alpha^r} = u(t) - u_0(t) \tag{33}$$

Substituting this identity into the definition of $\mathcal{A}[u](t)$, the term $u(t)$ cancels:

$$\mathcal{A}[u](t) = u_0(t) - J_s^{m\alpha}\left[L\big(u(t)\big) + N\big(u(t)\big) - g(t)\right] \tag{34}$$

Now, for any $u, v \in X$, we evaluate the difference:

$$\begin{aligned} |\mathcal{A}[u](t) - \mathcal{A}[v](t)| &= \left|-J_s^{m\alpha}\left(\left[L\big(u(t)\big) - L\big(v(t)\big)\right] + \left[N\big(u(t)\big) - N\big(v(t)\big)\right]\right)\right| \\ &\leq J_s^{m\alpha}\left(\left|L\big(u(\zeta)\big) - L\big(v(\zeta)\big)\right| + \left|N\big(u(\zeta)\big) - N\big(v(\zeta)\big)\right|\right) \\ &\leq (L_1 + L_2) \| u - v \|_\infty J_s^{m\alpha}(1). \end{aligned}$$

Using Lemma 2.9, we have $J_s^{m\alpha}(1) = \frac{(t-s)^{m\alpha}}{m!\alpha^m}$. Taking the maximum over $t \in [s,T]$ of both sides, we obtain:

$$\| \mathcal{A}[u] - \mathcal{A}[v] \|_\infty \leq \frac{(L_1 + L_2)(T-s)^{m\alpha}}{m!\,\alpha^m} \| u - v \|_\infty = \gamma \| u - v \|_\infty \tag{35}$$

Since $\gamma < 1$, the operator $\mathcal{A}$ is a contraction mapping on X. Therefore, by Banach's fixed point theorem, the sequence $\{u_k\}_{k=0}^{\infty}$ converges to the unique fixed point $u(t) \in X$, and the exact solution is given by $u(t) = \lim_{k\to\infty} u_k(t)$.

4.Numerical examples

To clarify the steps outlined in the previous section and demonstrate the ease and efficiency of VIM within the MCFD framework, provide ease and efficiency in finding either an exact or an accurate approximate solution to nonlinear fractional differential equations, we present three distinct examples in this section. In the first example, we solve a nonlinear fractional delay differential equation. The second example addresses a sequential nonlinear fractional initial value problem, while the third illustrates the application of the fourth-order nonlinear fractional differential equation with variable coefficients.

Example 4.1: Consider the non-linear fractional delay differential equation [17]:

$$\mathcal{T}^{3\alpha}u(t) = -1 + 2u^2\left(\frac{t}{2}\right), \quad 0 < \alpha \leq 1, \quad t \geq 0 \tag{36}$$

subject to the fractional initial conditions:

$$u(0) = 0, \quad u^{\alpha}(0) = 1, \quad u^{2\alpha}(0) = 0. \tag{37}$$

Since $n = 3$, and according to the formula (23), the iteration formula for Equation (36) is given by:

$$u_{k+1}(t) = u_k(t) - \frac{1}{2}\int_0^t \left(\frac{t^\alpha - \zeta^\alpha}{\alpha}\right)^2 \left[\mathcal{T}^{3\alpha}u_k(\zeta) + 1 - 2u_k^2\left(\frac{\zeta}{2}\right)\right] d_\alpha\zeta \tag{38}$$

According to formula (26), the variational iteration formula begins with:

$$u_0(t) = u(0) + u^\alpha(0)\frac{t^\alpha}{\alpha} + u^{2\alpha}(0)\frac{t^{2\alpha}}{2\alpha^2} = \frac{t^\alpha}{\alpha}. \tag{39}$$

The following approximations are obtained:

$$\begin{aligned}
u_0(t) &= \frac{t^\alpha}{\alpha} \\
u_1(t) &= u_0(t) - \frac{1}{2\alpha^2}\int_0^t (t^\alpha - \zeta^\alpha)^2 \left[\mathcal{T}^{3\alpha}u_0(\zeta) + 1 - 2u_0^2\left(\frac{\zeta}{2}\right)\right] d_\alpha\zeta \\
&= \frac{t^\alpha}{\alpha} - \frac{t^{3\alpha}}{6\alpha^3} + \frac{t^{5\alpha}}{30(4^\alpha\alpha^5)} \\
u_2(t) &= u_1(t) - \frac{1}{2\alpha^2}\int_0^t (t^\alpha - \zeta^\alpha)^2 \left[\mathcal{T}^{3\alpha}u_1(\zeta) + 1 - 2u_1^2\left(\frac{\zeta}{2}\right)\right] d_\alpha\zeta \\
&= \frac{t^\alpha}{\alpha} - \frac{t^{3\alpha}}{6\alpha^3} + \frac{t^{5\alpha}}{30\cdot 4^\alpha\alpha^5} - \frac{t^{7\alpha}}{315\cdot 16^\alpha\alpha^7} + \left(\frac{1}{9072 * 64^\alpha\alpha^9} + \frac{1}{3780 * 256^\alpha\alpha^9}\right)t^{9\alpha} \\
&\quad - \frac{t^{11\alpha}}{44550\cdot 1024^\alpha\alpha^{11}} + \frac{t^{13\alpha}}{772200 * 16384^\alpha\alpha^{13}}
\end{aligned}$$

As special case when $\alpha = 1$, we have

$$u(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!} + \frac{t^{13}}{13!} + \cdots = sin(t)$$

which is the exact solution of problem (36) – (37) at $\alpha = 1$.

Table 1: Exact (at $\alpha = 1$) and approximate solutions of Example 4.1 for various values of α, together with the absolute error at $\alpha = 1$.

t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Absolute Error
0.0	*0.000000*	*0.000000*	*0.000000*	*0.000000*	*0.*
0.1	*0.196819*	*0.139425*	*0.099833*	*0.099833*	*0.*
0.2	*0.338146*	*0.258074*	*0.198669*	*0.198669*	5.55×10^{-17}
0.3	*0.459268*	*0.367195*	*0.295520*	*0.295520*	5.55×10^{-15}
0.4	*0.565310*	*0.468092*	*0.389418*	*0.389418*	1.31×10^{-13}
0.5	*0.658326*	*0.560956*	*0.479426*	*0.479426*	1.52×10^{-12}
0.6	*0.739400*	*0.645651*	*0.564642*	*0.564642*	1.13×10^{-11}
0.7	*0.809238*	*0.721945*	*0.644218*	*0.644218*	6.11×10^{-11}
0.8	*0.868379*	*0.789594*	*0.717356*	*0.717356*	2.65×10^{-10}
0.9	*0.917295*	*0.848386*	*0.783327*	*0.783327*	9.62×10^{-10}
1.0	*0.956441*	*0.898162*	*0.841471*	*0.841471*	3.05×10^{-9}

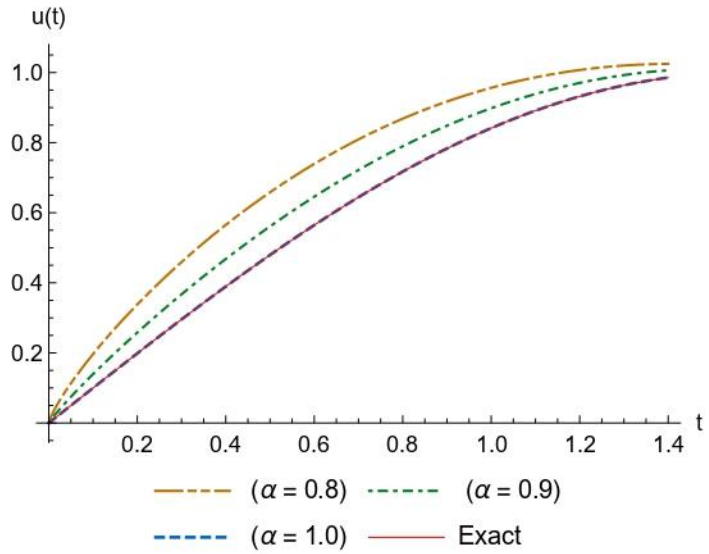


Figure 1: Comparison of the approximate and exact solutions of $u(t)$ for different values of α in Example 1

Table 1 displays the exact and approximate solutions of Equation (36) for different values of ($\alpha = 0.8, 0.9$, and1.0). It is evident that, for $\alpha = 1.0$, the approximate solution shows excellent agreement with the exact solution, with an absolute error on the order of 10^{-12} for $t \leq 0.5$, remaining below 10^{-9} up to $t = 1.0$. It is worth noting that only the second iteration $u_2(t)$ was utilized to compute these approximations, demonstrating rapid convergence toward the exact analytical solution $u(t) = \sin(t)$. Furthermore, Figure 1 illustrates the profiles of the fractional solutions for varying values of α, showing how they smoothly approach the classical exact solution as $\alpha \to 1$.

Example 4.2. Consider the nonlinear fractional differential equation [14]:

$$\mathcal{T}^{\beta}u(t) + u'^{(t)} + u^3(t) = 0, \quad 1 < \beta \leq 2, \quad t \geq 0 \tag{40}$$

subject to the fractional initial conditions:

$$u(0) = 1, \quad u^{\alpha}(0) = 0 \tag{41}$$

For $\beta \in (1,2]$, i.e.$\beta \in \chi_{\frac{\beta}{2}}$, we set $m = 2$ and define the auxiliary fractional parameter $\alpha = \frac{\beta}{2}$, with the corresponding operator given by $\mathcal{T}_0^{\beta} = \mathcal{T}_0^{2\alpha}$.

According to the formula (20), the iteration formula for Equation (40) is given by:

$$u_{k+1}(t) = u_k(t) + \int_0^t \left(\frac{\zeta^{\alpha} - t^{\alpha}}{\alpha}\right)\left[\mathcal{T}_0^{2\alpha}u_k(\zeta) + u'_k(\zeta) + u_k^3(\zeta)\right]d_{\alpha}\zeta \tag{42}$$

According to formula (21), we obtain the initial approximation as:

$$u_0(t) = u(0) + u^{\alpha}(0)\frac{t^{\alpha}}{\alpha} = 1 \tag{43}$$

We can obtain the following approximations:

$$
\begin{aligned}
u_0(t) &= 1 \\
u_1(t) &= u_0(t) + \frac{1}{\alpha}\int_0^t \frac{(t^\alpha - \zeta^\alpha)}{\zeta^{1-\alpha}}[\mathcal{I}_0^{2\alpha}(1) + (1)' + 1^3]d\zeta \\
&= 1 - \frac{t^{2\alpha}}{2\alpha^2} \qquad (44) \\
u_2(t) &= u_1(t) + \frac{1}{\alpha}\int_0^t \frac{(t^\alpha - \zeta^\alpha)}{\zeta^{1-\alpha}}[\mathcal{I}_0^{2\alpha}u_1(\zeta) + u'_1(\zeta) + u_1^3(\zeta)]\, d\zeta \\
&= 1 - \frac{t^{2\alpha}}{2\alpha^2} + \frac{t^{4\alpha-1}}{\alpha(3\alpha-1)(4\alpha-1)} + \frac{t^{4\alpha}}{8\alpha^4} - \frac{t^{6\alpha}}{40\alpha^6} + \frac{t^{8\alpha}}{448\alpha^8} \qquad (45)
\end{aligned}
$$

Now, transforming each term in equations (44) and (45) by $\alpha = \frac{\beta}{2}$, yield:

$$
\begin{aligned}
u_0 &= 1 \\
u_1(t) &= 1 - \frac{2t^\beta}{\beta^2} \\
u_2(t) &= 1 - \frac{2t^\beta}{\beta^2} + \frac{16\, t^{2\beta-1}}{\beta(3\beta-2)(4\beta-2)} + \frac{2t^{2\beta}}{\beta^4} - \frac{8t^{3\beta}}{5\beta^6} + \frac{4t^{4\beta}}{7\beta^8} \qquad (46)
\end{aligned}
$$

For $\beta = 2$ the solution (46) becomes as follows:

$$u(t) = 1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{8}t^4 - \frac{1}{40}t^6 + \frac{1}{448}t^8$$

Table 2: Exact (at $\beta = 2$) and approximate solutions of Example 4.2 for various values of β, together with the absolute error at $\beta = 2$.

t	$\beta = 1.8$	$\beta = 1.9$	$\beta = 2.0$	Exact	Absolute Error
0.0	1.000000	1.000000	1.000000	1.000000	0.
0.1	0.990896	0.993372	0.995179	0.995179	1.11×10^{-16}
0.2	0.970333	0.976548	0.981532	0.981532	0.
0.3	0.942733	0.952286	0.960494	0.960494	1.11×10^{-16}
0.4	0.911291	0.923011	0.933766	0.933766	1.11×10^{-16}
0.5	0.878828	0.891114	0.903264	0.903264	1.11×10^{-16}
0.6	0.847921	0.858965	0.871071	0.871071	0.
0.7	0.820914	0.828872	0.839367	0.839367	1.11×10^{-16}
0.8	0.799896	0.803034	0.810354	0.810354	0.
0.9	0.786676	0.783479	0.786187	0.786187	1.11×10^{-16}
1.0	0.782762	0.772017	0.768899	0.768899	1.11×10^{-16}

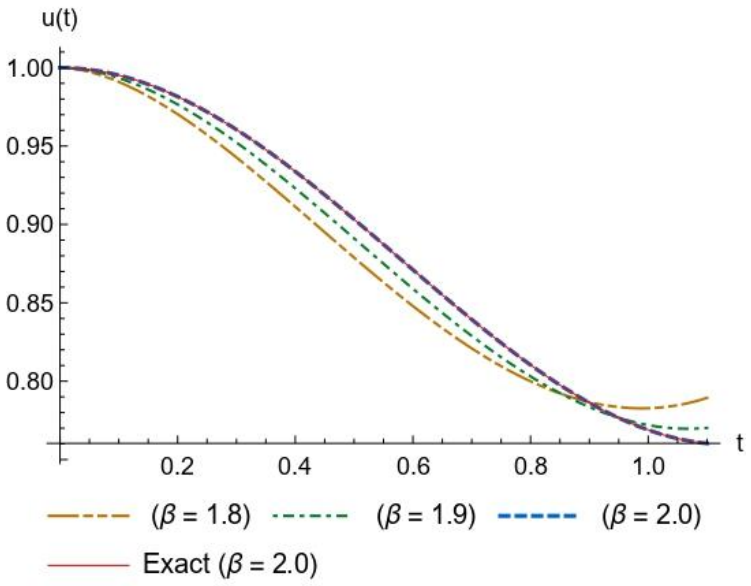


Figure 2: Comparison of the approximate and exact solutions of $u(t)$ for different values of β in Example 4.2

Table 2 presents a comparison between the exact solution with the approximate solutions obtained via the proposed method for Example 4.2 at various fractional orders, specifically at $\beta = 1.8$,1.9, and 2.0. For the classical case $\beta = 2$ ($\alpha = 1$), the results demonstrate exceptional accuracy, with the approximate solution $u_2(t)$ exhibiting near-perfect agreement with the exact series solution, yielding an absolute error on the order of 10^{-16} across the entire domain $t \in [0,1]$. Furthermore, Figure 2 illustrates the behavior of the approximate solution $u(t)$ for different fractional values of β ($\beta = 1.8$,1.9, and 2.0) along with the exact solution. The figure clearly shows that the approximate solution for $\beta = 2.0$ converges to the exact solution.

Example 4.3. Consider the nonlinear fractional differential equation [21]:

$$\mathcal{T}^{4\alpha}u(t) - sin\left(\frac{t^\alpha}{\alpha}\right)\mathcal{T}^{2\alpha}u(t) + u(t)\mathcal{T}^{\alpha}u(t) - u(t) = -1 - sin\left(\frac{t^\alpha}{\alpha}\right), 0 < \alpha \leq 1,\ t \geq 0 \tag{47}$$

subject to the fractional initial conditions:

$$u(0) = 2, \quad u^{(\alpha)}(0) = 0, \quad u^{(2\alpha)}(0) = -1, \quad u^{(3\alpha)}(0) = 0 \tag{48}$$

Since $m = 4$, according to general formula (24) the optimal Lagrange multiplier is given by:

$$\lambda(\zeta, t) = \frac{(-1)^4}{(4-1)!}\left(\frac{(\zeta - s)^\alpha - (t-s)^\alpha}{\alpha}\right)^{4-1} = -\frac{1}{6}\left(\frac{t^\alpha - \zeta^\alpha}{\alpha}\right)^3 \tag{49}$$

Consequently, the variational iteration formula for Equation (47) takes the form:

$$u_{k+1}(t) = u_k(t) - \frac{1}{6\alpha^3}\int_0^t (t^\alpha - \zeta^\alpha)^3 \times \left[\mathcal{T}^{4\alpha}u_k(\zeta) - \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\mathcal{T}^{2\alpha}u_k(\zeta) + u_k(\zeta)\mathcal{T}^{\alpha}u_k(\zeta) - u_k(\zeta) + 1 + \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\right] d_\alpha\zeta$$

According to formula (26), we defined the initial trial approximation as:

$$u_0(t) = u(0) + u^{(\alpha)}(0)\frac{t^\alpha}{\alpha} + u^{(2\alpha)}(0)\frac{t^{2\alpha}}{2!\,\alpha^2} + u^{(3\alpha)}(0)\frac{t^{3\alpha}}{3!\,\alpha^3} = 2 - \frac{t^{2\alpha}}{2\alpha^2} \tag{50}$$

Therefore, we obtain the iteration approximations:

$$
\begin{aligned}
u_0(t) &= 2 - \frac{t^{2\alpha}}{2\alpha^2} \\
u_1(t) &= u_0(t) - \frac{1}{6\alpha^3}\int_0^t (t^\alpha - \zeta^\alpha)^3 \\
&\quad \times \left(\mathcal{T}^{4\alpha} u_0(\zeta) - \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\mathcal{T}^{2\alpha} u_0(\zeta) + u_0(\zeta)\mathcal{T}^{\alpha} u_0(\zeta) - u_0(\zeta) + 1 + \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\right) d_\alpha \zeta \\
&= 2 - \frac{t^{2\alpha}}{2\alpha^2} + \frac{t^{4\alpha}}{4!\,\alpha^4} - \frac{t^{6\alpha}}{6!\,\alpha^6} - \frac{t^{7\alpha}}{7!\,\alpha^7} + \cdots \\
u_2(t) &= u_1(t) - \frac{1}{6\alpha^3}\int_0^t (t^\alpha - \zeta^\alpha)^3 \\
&\quad \times \left(\mathcal{T}^{4\alpha} u_1(\zeta) - \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\mathcal{T}^{2\alpha} u_1(\zeta) + u_1(\zeta)\mathcal{T}^{\alpha} u_1(\zeta) - u_1(\zeta) + 1 + \sin\left(\frac{\zeta^\alpha}{\alpha}\right)\right) d_\alpha \zeta \\
&= 2 - \frac{t^{2\alpha}}{2!\,\alpha^2} + \frac{t^{4\alpha}}{4!\,\alpha^4} - \frac{t^{6\alpha}}{6!\,\alpha^6} + \frac{t^{8\alpha}}{8!\,\alpha^8} + \cdots
\end{aligned}
$$

As a special case when $\alpha = 1$, we have:

$$u(t) = 1 + \left(1 - \tfrac{t^2}{2!} + \tfrac{t^4}{4!} - \tfrac{t^6}{6!} + \tfrac{t^8}{8!} - \cdots\right) = 1 + cos(t)$$

which is the exact solution of problem (47) – (48) at $\alpha = 1$.

Table 3: Exact (at $\alpha = 1$) and approximate solutions of Example 4.3 for various values of α, together with the absolute error at $\alpha = 1$

t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	**Exact**	**Absolute Error**
0.0	2.000000	2.000000	2.000000	2.000000	0.
0.1	1.980440	1.990233	1.995004	1.995004	5.55×10^{-15}
0.2	1.941098	1.966126	1.980067	1.980067	2.69×10^{-12}
0.3	1.888332	1.930148	1.955336	1.955336	1.01×10^{-10}
0.4	1.825018	1.883697	1.921061	1.921061	1.3×10^{-9}
0.5	1.753165	1.827908	1.877583	1.877583	9.35×10^{-9}
0.6	1.674378	1.763806	1.825336	1.825336	4.64×10^{-8}
0.7	1.590034	1.692376	1.764842	1.764842	1.78×10^{-7}
0.8	1.501354	1.614579	1.696707	1.696707	5.64×10^{-7}
0.9	1.409451	1.531364	1.621612	1.621610	1.54×10^{-6}
1.0	1.315346	1.443675	1.540306	1.540302	3.76×10^{-6}

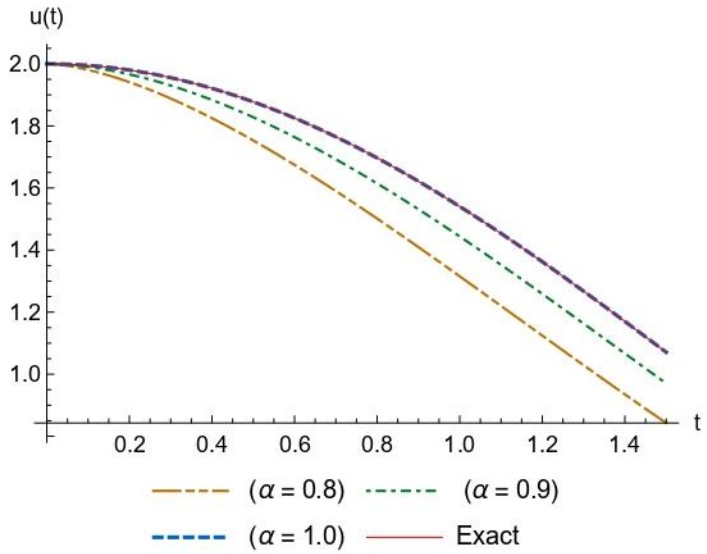


Figure 3: Comparison of the approximate and exact solutions of $u(t)$ for different values of α in Example 4.3

Table 3 illustrates the numerical results of Example 4.3, comparing the approximate solutions obtained via the proposed method for different fractional orders ($\alpha = 0.8$,0.9, and 1.0) with the exact analytical solution $u(t) = 1 + \cos(t)$ at $\alpha = 1.0$. It is evident that the second iteration $u_2(t)$ achieves remarkable precision, yielding an absolute error of only 5.55×10^{-15} at $t = 0.1$, which remains as small as 3.76×10^{-6} even at $t = 1.0$. Furthermore, Figure 3 depicts the dynamic profiles of $u(t)$ across the interval $t \in [0,1.5]$ for various values of α. The graphical comparison confirms an excellent, overlapping agreement between the approximate solution at $\alpha = 1.0$ and the exact solution. In addition, as the fractional parameter α increases towards 1, the fractional response curves continuously and smoothly merge into the classical solution $1 + \cos(t)$, validating both the fast convergence rate and the high analytical accuracy of the proposed method when solving higher-order nonlinear fractional differential equations.

5.Conclusion

In this work, we successfully adapted the VIM to solve linear and nonlinear fractional differential equations governed by the MCFD. By formulating the correction functional using the fundamental conformable differential measure $d_\alpha\zeta = (\zeta - s)^{\alpha-1}d\zeta$, we rigorously derived the optimal Lagrange multiplier $\lambda(\zeta, t)$ via fractional variational theory. More importantly, the mathematical derivation confirmed that the resulting kernel inherently recovers the exact MCFI operator. The computational efficiency, robustness, and precision of the scheme were verified across several challenging nonlinear models. In all cases, only two or three iterations were needed to produce series solutions that smoothly converge to the exact classical solutions as $\alpha \to 1$, achieving absolute precision on the order of 10^{-16} without fractional Gamma functions or computationally intensive convolutions. These findings show that this framework offers an accurate, straightforward alternative for fractional modeling, paving the way for future extensions to coupled systems and higher-dimensional fractional partial differential equations.

Funding: This research received no external funding.
Conflicts of Interest: The authors declare no conflict of interest.
ORCID iD: 0009-0004-0860-6303
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