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| RESEARCH ARTICLE

Type of Wigner Transform

Mohammed Kh. Abdullah
Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq.
Corresponding Author: ***Mohammed Kh. Abdullah***, **E-mail**: mohammed.abdulah@uobasrah.edu.iq

| ABSTRACT

Let $\varphi: H \longrightarrow H$ be a bijective continuous homomorphism from a locally compact abelian group H onto itself whose inverse is also continuous such that $\varphi(z^2) = z$ for all . This paper introduces$z \in H$ a type of Wigner transform, which we term the Migner transform on the Hilbert space $L^2(H)$ associated with φ and presents some important results concerning this transform.

| KEYWORDS

Locally compact abelian group, dual group, Wigner transform, Fourier-Wigner transform.

| ARTICLE INFORMATION

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1. INTRODUCTION[1]

Recently, time-frequency analysis methods [6,8] have been employed and have played a fundamental role in the development and analysis of partial differential equations. This prominent field of harmonic analysis [5] focuses on analyzing a given function, or studying the behavior of a distribution, in both its time and frequency variables simultaneously. The aim of such analysis is to identify possible localizations of the frequencies. Time-frequency analysis is more closely related to signal theory, specifically exploring the local frequencies of functions or distributions through time–frequency representations of signals, for example, the short-time Fourier transform or its discrete counterpart [2,3,4,5]. For a detailed exploration of the applications of time-frequency analysis to the study of pseudodifferential and Fourier integral operator theory, see [7,9,10,11].

Throughout this paper, H denotes a locally compact abelian topological group and $\widehat{H}$ is its dual group, φ is a continuous topological automorphism on H, satisfying $\varphi(z^2) = z$ for all $z \in H$. $C_c(H)$, is the dense subspace of $L^2(H)$ which contains all compactly supported complex-valued functions defined on H. The Fourier transform of h is $\mathcal{F}(h)(\gamma) = \hat{h}(\gamma) = \int_H \overline{\gamma(z)} h(z)\, dz$ and its inverse is given by:

$$\mathcal{F}^{-1}(h)(z) = \check{h}(z) = \int_{\widehat{H}} \gamma(z) h(\gamma)\, d\gamma$$

for $h \in L^2(H)$ [6]. The Fourier transform of $h \in L^2(H \times H)$ with respect to the second variable is defined as:

$$(\mathcal{F}_2 h)(t, \gamma) = \int_H \overline{\gamma(z)}\, h(t, z) dz$$

and is denoted by $\mathcal{F}_2$. The Fourier transform on $L^1(H) \cap L^2(H)$ can be extended from $L^2(H)$ into $L^2(H)$ as a unitary isomorphism known as the Plancherel transform [4,5]. Let $F \in L^2(H \times H)$, the linear operator T^{φ} on F which is called the twisted operator with respect to φ is defined as:

$$(T^{\varphi} F)(t, z) = F\big(t\varphi(z), t^{-1}\varphi(z)\big), \quad t, z \in H$$

and the inverse of the twisted operator is given by:

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$$\left(T^{\varphi^{-1}}F\right)(t,z) = F(\varphi(tz^{-1}),tz), \qquad t,z \in H$$

For any function h the translation L_t (sometimes denoted by T_t) and the modulation M_γ operators are defined by $L_t h(z) = h(t^{-1}z)$ and $M_\gamma h(t) = \gamma(t)h(t)$ respectively for any $t,z \in H$ and $\gamma \in \widehat{H}$.

Let k be an open subgroup of H and let $\Phi: H \longrightarrow k$ be a topological isomorphism; for all $h,q \in L^2(H)$, the Fourier-Wigner transform on $L^2(H)$ is given by [8],

$$V(h,q)(\eta,z) = \int_H^{\square} \eta(t)h(t\Phi(z))\overline{q\left(t\Phi(z^{-1})\right)}dt,$$

The Wigner transform on $L^2(H)$ is

$$W(h,q)(z,\eta) = \int_H^{\square} \overline{\eta(y)}h(z\Phi(y))\overline{q\left(z\Phi(y^{-1})\right)}dy,$$

For more details, see [1].

This paper is organized as follows. In Section 2, we first present the definition of a unitary operator $\psi(\gamma,t): L^2(H) \longrightarrow L^2(H)$ for all $(\gamma,t) \in \widehat{H} \times H$, and then define the Fourier-Migner transform $\mathcal{N}$ and then show that it is equal to the value of $\langle \psi(\gamma,t)h,q\rangle$, $(\gamma,t) \in \widehat{H} \times H$.

In Section 3, we introduce the Migner transform $\mathcal{M}$ and some harmonic analysis results concerning this transform.

2. The Fourier-Migner Transform

The Fourier–Migner transform,$\mathcal{N}$ with respect to φ (a topological automorphism on satisfying$H$ $\varphi(z^2) = z$ for all $z \in H$) is defined in this section. Let $\widehat{H}$ be the dual group of H, and h be any measurable function on H so we can define

$$(\psi(\gamma,z)h)(t) = \gamma\left(t^{-1}\varphi(z)\right)h(t^{-1}z), \qquad (2,1)$$

for all $(\gamma,z) \in \widehat{H} \times H$

Also, one can write (2,1) as

$$(\psi(\gamma,z)h)(t) = \gamma(\varphi(z))M_\gamma L_{z^{-1}}h(t^{-1}),$$

where $L_z h(\mathrm{t}) = h(\mathrm{t}z^{-1})$ and $M_\gamma h(t) = \gamma(t)h(t)$.

In the following proposition, we show that $\psi(\gamma,z)$ is a unitary operator on $L^2(H)$ for all $(\gamma,z) \in \widehat{H} \times H$.

Proposition 2.1.

for any $(\gamma,z) \in \widehat{H} \times H$, $\psi(\gamma,z): L^2(H) \longrightarrow L^2(H)$ is a unitary operator

Proof:

First, we show that $\|\psi(\gamma,z)h\| = \|h\|$ for all $h \in L^2(G)$

$$\|\psi(\gamma,z)h\|^2 = \int_H^{\square} |\psi(\gamma,z)h(t)|^2\,dt = \int_H^{\square} \left|\gamma(t^{-1})\gamma\left(\varphi(z)\right)h(t^{-1}z)\right|^2\,dt = \int_H^{\square} |h(t^{-1}z)|^2\,dt = \int_H^{\square} |h(t^{-1})|^2\,dt = \|h\|^2.$$

Finally, we show that $\psi(\gamma,z)$ is surjective for any $(\gamma,z) \in \widehat{H} \times H$, to this end, let f be any function in $L^2(H)$ and assume that the function h on H is defined by

$$h(t) = \gamma(t^{-1})\gamma(\varphi(z))f(t^{-1}z), \qquad (2,2)$$

Clearly, the function h is in $L^2(H)$. Then, by (2,1) , (2,2) and $\varphi(z^2) = z$ for all $z \in H$, we find that

$$\begin{aligned}\psi(\gamma,z)h(t) &= \gamma(t^{-1})\gamma\left(\varphi(z)\right)h(t^{-1}z) = \gamma(t^{-1})\gamma\left(\varphi(z)\right)\gamma(tz^{-1})\gamma\left(\varphi(z)\right)f(t) = \gamma(t^{-1})\gamma(t)\gamma\left(\varphi(z)\right)\gamma(z^{-1})\gamma\left(\varphi(z)\right)f(t)\\ &= \gamma\left(\varphi(z^2)\right)\gamma(z^{-1})f(t) = \gamma(z)\gamma(z^{-1})f(t) = f(t).\end{aligned}$$

Hence $\psi(\gamma,z)$ is a unitary operator for all $(\gamma,z) \in \widehat{H} \times H$.

□

Now, let h and q be two measurable functions in $L^2(H)$. We then define the Fourier–Migner transform $\mathcal{N}$ with respect to φ by

$$\mathcal{N}(h,q)(\gamma,z) = \int_H^{\square} \gamma(t)h(t\varphi(z))\overline{q\left(t^{-1}\varphi(z)\right)}dt,$$

It is clear that for any h and q in $L^2(H)$, and all $(\gamma,z) \in \widehat{H} \times H$ we have

$$\mathcal{N}(h,q)(\gamma,z) = \langle \psi(\gamma,z)h,q\rangle, \qquad (2,3)$$

where the notation $\langle,\rangle$ denotes the inner product on the Hilbert space $L^2(H)$. To show this, we use the definition of the inner product and replacing t by $\varphi(z)t$ and then replacing t by t^{-1} , we obtain

$$\begin{aligned}\langle \psi(\gamma,z)h,q\rangle &= \int_H^{\square} \psi(\gamma,z)h(t)\,\overline{q(t)}dt = \int_H^{\square} \gamma(t^{-1})\gamma\left(\varphi(z)\right)h(t^{-1}z)\,\overline{q(t)}dt = \int_H^{\square} \gamma(t^{-1})h\left(t^{-1}\varphi(z)\right)\overline{q\left(t\varphi(z)\right)}dt\\ &= \int_H^{\square} \xi(t)h\left(t\varphi(z)\right)\overline{q\left(t^{-1}\varphi(z)\right)}dt = \mathcal{N}(h,q)(\gamma,z).\end{aligned}$$

3. Migner Transformation

In this section, we define the Migner transformation $\mathcal{M}$ on $L^2(H)$ (Hilbert space), prove that it is the Fourier transform of the Fourier–Migner transformation, and study its essential properties.

For any h and q in $L^2(H)$ the Migner transform $\mathcal{M}(h,q)$ on $H \times \hat{H}$ is defined by

$$\mathcal{M}(h,q)(t,\gamma) = \int_H^{\square} \overline{\gamma(z)} h(t\varphi(z)) \overline{q\big(t^{-1}\varphi(z)\big)} dz, \tag{3,1}$$

for all $(t,\gamma) \in H \times \hat{H}$.

Theorem 3.1.
For all h_1, h_2, q_1 and q_2 in $C_C(H)$, we have

$$\langle \mathcal{M}(h_1,q_1), \mathcal{M}(h_2,q_2) \rangle = \langle h_1, h_2 \rangle \overline{\langle q_1, q_2 \rangle}$$

Proof:
For any $P \in C_C(H \times H)$ we define $\widetilde{\mathcal{M}}P$ for all $(t,\gamma) \in H \times \hat{H}$ by

$$\big(\widetilde{\mathcal{M}}P\big)(t,\gamma) = \int_H^{\square} \overline{\gamma(z)}\, P\big(t\varphi(z), t^{-1}\varphi(z)\big) dz \tag{3,2}$$

Then, by (3,2)

$$\big(\widetilde{\mathcal{M}}P\big)(t,\gamma) = \int_H^{\square} \overline{\gamma(z)}\, P\big(t\varphi(z), t^{-1}\varphi(z)\big) dz = \int_H^{\square} \overline{\gamma(z)}\, T^{\varphi}P(t,z) dz = (\mathcal{F}_2 T^{\varphi}P)(t,\gamma),$$

so, by the Plancherel theorem and replacing t by $t\varphi(z)$ and z by $t^{-1}z$ it follows that

$$\begin{aligned}
\langle \widetilde{\mathcal{M}}P_1, \widetilde{\mathcal{M}}P_2 \rangle = \langle \mathcal{F}_2 T^{\varphi}P_1, \mathcal{F}_2 T^{\varphi}P_2 \rangle = \langle T^{\varphi}P_1, T^{\varphi}P_2 \rangle &= \int_H^{\square} \{ \int_H^{\square} T^{\varphi}P_1(t,z) \overline{T^{\varphi}P_2(t,z)}\, dz \} dt \\
&= \int_H^{\square} \int_H^{\square} P_1\big(t\varphi(z), t^{-1}\varphi(z)\big) \overline{P_2\big(t\varphi(z), t^{-1}\varphi(z)\big)}\, dz dt = \int_H^{\square} \int_H^{\square} P_1\big(t\varphi(z), t^{-1}\varphi(z)\big) \overline{P_2\big(t\varphi(z), t^{-1}\varphi(z)\big)}\, dt dz \\
&= \int_H^{\square} \int_H^{\square} P_1(tz, t^{-1}) \overline{P_2(tz, t^{-1})}\, dt dz = \int_H^{\square} \int_H^{\square} P_1(z,t) \overline{P_2(z,t)}\, dz dt
\end{aligned} \tag{3,3}$$

Hence

$$\langle \widetilde{\mathcal{M}}P_1, \widetilde{\mathcal{M}}P_2 \rangle = \langle P_1, P_2 \rangle, \tag{3,4}$$

Now, let h_1, h_2, q_1 and q_2 be in $C_C(H)$ with:

$$P_1(z,t) = h_1 \otimes \overline{q_1}(z,t),\ P_2(z,t) = h_2 \otimes \overline{q_2}(z,t) \tag{3,5}$$

where

$$h_1 \otimes \overline{q_1}(z,t) = h_1(z)\overline{q_1}(t), \quad \text{and} \quad h_2 \otimes \overline{q_2}(z,t) = h_2(z)\overline{q_2}(t).$$

Then, by (3,1), (3,2), (3,3), (3,4) and (3,5),

$$\begin{aligned}
\langle \mathcal{M}(h_1,q_1), \mathcal{M}(h_2,q_2) \rangle = \langle \widetilde{\mathcal{M}}P_1, \widetilde{\mathcal{M}}P_2 \rangle = \langle P_1, P_2 \rangle &= \int_H^{\square} \int_H^{\square} P_1(z,t) \overline{P_2(z,t)}\, dz\ dt = \int_H^{\square} \int_H^{\square} h_1(z) \overline{q_1(t)} \overline{h_2(z) \overline{q_2(t)}}\, dz\ dt \\
&= \int_H^{\square} \int_H^{\square} h_1(z) \overline{q_1(t) h_2(z)}\, q_2(t)\ dz dt = \int_H^{\square} h_1(z) \overline{h_2(z)}\, dz \int_H^{\square} \overline{q_1(t)}\, q_2(t) dt = \langle h_1, h_2 \rangle \overline{\langle q_1, q_2 \rangle}.
\end{aligned}$$

□

Corollary 3.2.
The theorem above is also valid for the Fourier-Migner transform $\mathcal{N}$.

Proposition 3.3.
For all $h, q \in C_C(H)$, we have
(i) $\mathcal{M}\big(\tilde{h}, \tilde{q}\big)(t,\gamma) = \mathcal{M}(h,q)(t^{-1}, \gamma^{-1}), (t,\gamma) \in H \times \hat{H}$ where $\tilde{h}(t) = h(t^{-1})$ for all $t \in H$.
(ii) $\overline{\mathcal{M}(h,q)(t,\gamma)} = \mathcal{M}(q,h)(t^{-1}, \gamma^{-1})$ for all $(t,\gamma) \in H \times \hat{H}$.

Proof:
(i) By the definition of the Migner transformation and replacing z^{-1} by z, we obtain

$$\mathcal{M}(\tilde{h},\tilde{q})(t,\gamma)=\int_H^{\square}\overline{\gamma(z)}\,\tilde{h}(t\varphi(z))\overline{\tilde{q}(t^{-1}\varphi(z))}dz=\int_H^{\square}\overline{\gamma(z)}\,h(t^{-1}\varphi(z^{-1}))\overline{q(t\varphi(z^{-1}))}dz=\int_H^{\square}\overline{\gamma(z^{-1})}\,h(t^{-1}\varphi(z))\overline{q(t\varphi(z))}dz$$
$$=\int_H^{\square}\overline{\gamma^{-1}(z)}\,h(t^{-1}\varphi(z))\overline{q(t\varphi(z))}dz=\mathcal{M}(h,q)(t^{-1},\gamma^{-1})$$

(ii) By the definition of the Migner transformation, the following holds

$$\overline{\mathcal{M}(h,q)(t,\gamma)}=\overline{\int_H^{\square}\overline{\gamma(z)}\,h(t\varphi(z))\overline{q(t^{-1}\varphi(z))}dz}=\int_H^{\square}\gamma(z)\,\overline{h(t\varphi(z))}q(t^{-1}\varphi(z))dz=\int_H^{\square}\overline{\gamma^{-1}(z)}\,q(t^{-1}\varphi(z))\overline{h(t\varphi(z))}dz$$
$$=\mathcal{M}(q,h)(t^{-1},\gamma^{-1})$$

Proposition 3.4.
For any $h,q\in C_C(H)$ and $(t,\omega)\in \mathrm{H}\times\widehat{H}$, the following holds
(i) for all $(\gamma,z)\in\widehat{H}\times H$

$$\mathcal{M}\big(M_\gamma L_z h,M_\gamma L_y q\big)(t,\omega)=\gamma(t^2)\overline{\xi(z^2)}\mathcal{M}(h,q)(t,\omega)=\mathcal{M}\big(L_z M_\gamma h,L_z M_\gamma q\big)(t,\omega).$$

(ii) For all $(\gamma,z),(\eta,r)\in\widehat{H}\times H$,

$$\mathcal{M}(\psi(\gamma,z)h,\psi(\eta,r)q)(t,\omega)=\overline{\omega(zr)\gamma\big(t\varphi(r)\big)\eta\big(t\varphi(z^{-1})\big)}\mathcal{M}(h,q)\big(t^{-1}\varphi(zr^{-1}),\omega\hat{\varphi}(\gamma\eta^{-1})\big).$$

where $\hat{\varphi}$ is an automorphism on $\widehat{H}$ such that $\varphi(t)(\omega)=\hat{\varphi}(\omega)(t)$ for any $(\omega,t)\in\widehat{H}\times H$.
More specifically,

$$\mathcal{M}(\psi(\gamma,z)h,\psi(\gamma^{-1},z^{-1})h)(t,\omega)=\mathcal{M}(h,h)(t^{-1}z,\omega\gamma).$$

Proof:
(i) By the definitions of the Migner transformation and the modulation and translation operators, and replacing s by z^2s, it follows that

$$\mathcal{M}\big(M_\gamma L_z h,M_\gamma L_z q\big)(t,\omega)=\int_H^{\square}\overline{\omega(s)}\,\big(M_\gamma L_z h\big)\big(t\varphi(s)\big)\overline{\big(M_\gamma L_z q\big)\big(t^{-1}\varphi(s)\big)}ds$$
$$=\int_H^{\square}\overline{\omega(s)}\,\gamma\big(t\varphi(s)\big)h\big(tz^{-1}\varphi(s)\big)\overline{\gamma\big(t^{-1}\varphi(s)\big)q\big(t^{-1}z^{-1}\varphi(s)\big)}ds$$
$$=\gamma(t^2)\int_H^{\square}\overline{\omega(z^2s)}\,h\big(tz^{-1}\varphi(z^2s)\big)\overline{q\big(t^{-1}z^{-1}\varphi(z^2s)\big)}ds=\gamma(t^2)\overline{\omega(z^2)}\int_H^{\square}\overline{\omega(s)}\,h\big(t\varphi(s)\big)\overline{q\big(t^{-1}\varphi(s)\big)}ds$$
$$=\gamma(t^2)\overline{\omega(z^2)}\mathcal{M}(h,q)(t,\omega).$$

Proceeding as above, one obtains that

$$\mathcal{M}\big(L_z M_\gamma h,L_z M_\gamma q\big)(t,\omega)=\gamma(t^2)\overline{\xi(z^2)}\mathcal{M}(h,q)(t,\omega).$$

This completes the proof of (i).

For (ii), from (3,1), (2,1) and substituting zrs instead of s,

$$\mathcal{M}(\psi(\gamma,z)h,\psi(\eta,r)q)(t,\omega)=\int_H^{\square}\overline{\omega(s)}\,(\psi(\gamma,z)h)\big(t\varphi(s)\big)\overline{(\psi(\eta,r)q)\big(t^{-1}\varphi(s)\big)}ds$$
$$=\int_H^{\square}\overline{\omega(s)}\,\gamma\big(t^{-1}\varphi(zs^{-1})\big)h\big(zt^{-1}\varphi(s^{-1})\big)\overline{\eta\big(t\varphi(rs^{-1})\big)q\big(rt\varphi(s^{-1})\big)}ds$$
$$=\int_H^{\square}\overline{\omega(zrs)}\,\gamma\big(t^{-1}\varphi(r^{-1}s^{-1})\big)h\big(t^{-1}\varphi(zr^{-1}s^{-1})\big)\overline{\eta\big(t\varphi(z^{-1}s^{-1})\big)q\big(t\varphi(z^{-1}rs^{-1})\big)}ds$$
$$=\overline{\omega(zr)\gamma\big(t\varphi(r)\big)\eta\big(t\varphi(z^{-1})\big)}\int_H^{\square}\overline{\omega(s)}\,(\gamma^{-1}\eta)\big(\varphi(s)\big)h\big(t^{-1}\varphi(zr^{-1}s^{-1})\big)\overline{q\big(t\varphi(z^{-1}rs^{-1})\big)}ds$$
$$=\overline{\omega(zr)\gamma\big(t\varphi(r)\big)\eta\big(t\varphi(z^{-1})\big)}\int_H^{\square}\overline{\omega(p)}\,\hat{\varphi}(\gamma^{-1}\eta)(s)h\big(t^{-1}\varphi(zr^{-1}s^{-1})\big)\overline{q\big(t\varphi(z^{-1}rs^{-1})\big)}ds$$
$$=\overline{\omega(zr)\gamma\big(t\varphi(r)\big)\eta\big(t\varphi(z^{-1})\big)}\int_H^{\square}\overline{\big(\omega\hat{\varphi}(\gamma\eta^{-1})\big)(s)}\,h\big(t^{-1}\varphi(zr^{-1}s^{-1})\big)\overline{q\big(t\varphi(z^{-1}rs^{-1})\big)}ds$$
$$=\overline{\omega(zr)\gamma\big(t\varphi(r)\big)\eta\big(t\varphi(z^{-1})\big)}\mathcal{M}(h,q)\big(t^{-1}\varphi(zr^{-1}),\omega\hat{\varphi}(\gamma\eta^{-1})\big).$$

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ORCID iD: https://orcid.org/0000-0002-0565-8763, https://orcid.org/0000-0002-9625-5803, https://orcid.org/0000-0003-4879-1190
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