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**| RESEARCH ARTICLE**

# Geometry, Saturation, and Powers of Parametric Monomial Ideals

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**| ABSTRACT**
We investigate the geometry of residual schemes and a proper saturation filtration for a four-parameter family of monomial ideals in three variables. An explicit primary decomposition identifies two thickened coordinate axes, their generic multiplicities, and the length of their scheme-theoretic intersection. It also leads to closed descriptions of all filtration jumps as graded free modules over a polynomial subring. A generating polynomial encodes their ranks and stabilization thresholds. For ordinary powers we reduce the stabilization index to a two-variable Artinian monomial ideal, derive bounds and eventual linearity, and give closed formulas in aligned cases. A Loewy-filtration interpretation and a flat family of modules complement these formulas. We also characterize equality of symbolic and ordinary powers in the family, compare regularity with saturation in a controlled extension, and calculate the asymptotic degree of ordinary powers using the Newton polygon. The general staircase and multiplicity methods are attributed to the established literature.

**| KEYWORDS**
Monomial ideals; Algebraic geometry; Saturation; Symbolic powers; Residual schemes; Hilbert series; Regularity; Loewy filtration

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## I. Introduction and Mathematical Context

Ideal quotients and saturations form part of the algebra–geometry dictionary in polynomial algebra . In a monomial setting, exponent divisibility makes these operations explicit, but a single stabilization index loses the intermediate information of the ascending filtration. The objective here is to retain that information in a closed rank enumerator and a graded-module decomposition. (Cox et al., 2015)

The main distinction in this paper is between a *proper* saturation and the Artinian auxiliary quotient that controls its stabilization. Our ambient ring has three variables, but the support ideal is $J = (x, y)$ rather than the homogeneous maximal ideal $(x, y, z)$. Thus the saturation quotient has positive-dimensional support and its Hilbert series has a pole at 1. We exploit this feature by regarding the successive quotients as modules over $k[z]$.

We do not claim originality for the general two-variable staircase method or the saturation-number formula; these are part of established monomial-ideal theory . Our focus is the explicit graded filtration for the indicated family, including its entry-time distribution, free-module decomposition, and closed generating polynomial. These family-specific calculations use standard monomial methods, and are not presented as a new general saturation theory. (Miller & Sturmfels, 2005; Abdolmaleki & Yazdan Pour, 2023)

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### A. Outline of the results

First, a two-component primary decomposition identifies the surviving saturation and its nonreduced geometric components. We compute the generic multiplicities, the scheme-theoretic intersection, and the Hilbert series of their union. Second, the two corners of an Artinian staircase give a threshold for every standard monomial. Third, the Loewy filtration of the transversal Artinian algebra and a flat family of modules explain the free $k[z]$-strings; the thresholds select these strings and yield a closed rank-generating polynomial and Hilbert series. Finally, we examine ordinary and symbolic powers, regularity, and the Newton-polygon degree calculation as complementary applications of the same family.

## II. Definitions and elementary monomial facts

For ideals $A, B$ in a commutative ring, define $A\colon B = \{f\colon fB \subseteq A\}$, $A\colon B^0 = A$, and $A\colon B^\infty = \mathrm{U}_{r\geq 0}(A\colon B^r)$. The stabilization index $\sigma_B(A)$ is the least $r$ for which $A\colon B^r = A\colon B^{r+1}$. Since $(A\colon B^r)\colon B = A\colon B^{r+1}$, equality at one stage implies equality at every later stage.

**Lemma 1** (Monomial quotient formula). *If $M = (m_1, \dots, m_s)$ is monomial and $u$ is a monomial, then $M\colon u = (m_i/\gcd(m_i, u)\colon 1 \leq i \leq s)$. If $B = (u_1, \dots, u_t)$, then $M\colon B = \bigcap_j (M\colon u_j)$.*

*Proof.* A monomial $v$ lies in $M\colon u$ exactly when some $m_i$ divides $vu$. This is equivalent to $m_i/\gcd(m_i, u)$ dividing $v$. Monomial membership is termwise. The formula for a quotient by an ideal follows directly from its definition. □

**Lemma 2** (Intersections and saturation). *For monomial ideals $A = (u_i)$ and $B = (v_j)$, $A \cap B$ is generated by the monomials $\mathrm{lcm}(u_i, v_j)$. If $A = \bigcap_i Q_i$ is a finite primary decomposition, then $A\colon B^\infty = \bigcap_{B\not\subseteq\sqrt{Q_i}} Q_i$, with the empty intersection understood to be the entire ring.*

*Proof.* The first assertion follows by divisibility. For the second, saturation commutes with finite intersections. If $B \subseteq \sqrt{Q_i}$, then some power of $B$ lies in $Q_i$; otherwise an element of $B\backslash\sqrt{Q_i}$ acts as a non-zero-divisor on the primary quotient, so the saturation of $Q_i$ is $Q_i$. □

### A. Artinian staircases and entry times

The following elementary lemma is included to make the later computations independent of any black-box saturation algorithm. It also clarifies the exact role of the two corners in the present example.

**Lemma 3** (Artinian staircases and entry times). *Let $L \subset k[x, y]$ be an Artinian monomial ideal and let $\mathcal{E}(L) = \{(s, t) \in \mathbb{N}^2\colon x^s y^t \notin L\}$. For a monomial $v = x^s y^t$, define $\epsilon_L(v)$ to be the least $r \geq 0$ for which $v(x, y)^r \subseteq L$. If $v \in L$, then $\epsilon_L(v) = 0$. Otherwise, $\epsilon_L(v) = 1 + \max\{(A - s) + (B - t)\colon (A, B) \in \mathrm{Max}\mathcal{E}(L),\ A \geq s,\ B \geq t\}$. Furthermore, $\sigma_{(x,y)}(L) = 1 + \max\{A + B\colon (A, B) \in \mathrm{Max}\mathcal{E}(L)\}$.*

*Proof.* The set $\mathcal{E}(L)$ is a finite order ideal in the coordinatewise partial order. For a fixed $v = x^s y^t \notin L$, a monomial $x^i y^j$ witnesses $v(x, y)^{i+j} \not\subseteq L$ exactly when $(s + i, t + j) \in \mathcal{E}(L)$. Every point of this finite order ideal lies below a maximal point. Consequently the greatest degree $i + j$ of a witness is attained at the difference from $(s, t)$ to a maximal point dominating $(s, t)$. The next degree has no witnesses, proving the first formula. The second follows by taking $v = 1$. □

*Remark 4* (Why the support ideal matters). The formulas above use the two-generated ideal $(x, y)$. Saturation by $(x, y, z)$ would require an additional condition involving powers of $z$ and would produce a different filtration. In particular, it would be incorrect to transfer our stabilization indices without changing the support ideal.

## III. Setting and primary decomposition

Let $R = k[x, y, z]$, $J = (x, y)$ and $I = (x^a, x^c y^d, y^b z)$, where $a > c \geq 1$, $b > d \geq 1$. Set $H = (x^c, z)$ and $Q = (x^a, x^c y^d, y^b)$.

**Proposition 5**. *There is an irredundant primary decomposition $I = H \cap Q$, with radicals $(x, z)$ and $(x, y)$. In particular $I\colon J^\infty = H$ and $\sigma_J(I) = K := \max\{a + d - 1, b + c - 1\}$.*

*Proof.* Computing the least common multiples of generators of $H$ and $Q$ and removing redundant terms yields $I$. The two ideals are primary to the displayed incomparable primes. Saturation by $J$ removes $Q$ but not $H$. To compute the index, $H/I$ is generated over $R$ by the classes of $x^c, z$. The excluded products $zx^i y^j$ have $(i, j)$ in $([0, c - 1] \times [0, b - 1]) \cup ([0, a - 1] \times [0, d - 1])$, whose largest $i + j$ is $K - 1$. The corresponding threshold for $x^c$ is $a - c + d - 1 \leq K$. □

## IV. Geometric components and nonreduced structures

Throughout this section, the scheme-theoretic statements are valid over an arbitrary field $k$; for the descriptions in terms of geometric points we may pass to an algebraic closure. Put $X = \mathrm{Spec}(R/I)$, $X_y = \mathrm{Spec}(R/H)$, and $X_z = \mathrm{Spec}(R/Q)$. These are closed subschemes of affine three-space.

**Proposition 6** (Reduced support and component multiplicities). *The reduced support of $X$ is the union of the coordinate $y$-axis and $z$-axis: $I_{\mathrm{red}} = \sqrt{I} = (x, yz)$, $\quad |X| = V(x,z) \cup V(x,y)$. The component $X_y$ is a generically $c$-fold thickening of the $y$-axis. The component $X_z$ is a generically $L$-fold thickening of the $z$-axis, where $\boxed{L = cb + (a-c)d.}$ Consequently, the degree (equivalently, the Hilbert–Samuel multiplicity for the standard grading) of the one-dimensional scheme $X$ is $c + L$.*

*Proof.* The primary decomposition gives $\sqrt{I} = (x,z) \cap (x,y) = (x, yz)$. Since $R/H \cong k[x,y]/(x^c)$, localizing at the generic point $(x,z)$ leaves a length-$c$ Artinian local algebra in the transverse variable $x$. Similarly, $R/Q \cong (k[x,y]/Q)[z]$ and its localization at the generic point $(x,y)$ has length equal to the $k$-dimension of $k[x,y]/Q$. Its standard monomials are precisely $x^s y^t$ with $(s,t) \in B_0 \cup B_1$, hence the length is $cb + (a-c)d$. The associativity formula for multiplicities gives the sum of the generic lengths at the two minimal primes, since each supporting coordinate line has degree one. The same conclusion follows from the Hilbert-series calculation below. □

**Proposition 7** (Scheme-theoretic intersection and Hilbert series). *The scheme-theoretic intersection $X_y \cap X_z$ is supported at the origin and has coordinate ring $R/(H+Q) \cong k[x,y]/(x^c, y^b)$, so its length is $cb$. Let $[r]_q = 1 + q + \cdots + q^{r-1}$ and put $P(q) = [c]_q[b]_q + q^c[a-c]_q[d]_q$. Then the Hilbert series of the union is $\boxed{\mathrm{Hilb}_{R/I}(q) = \frac{[c]_q + P(q)}{1-q} - [c]_q[b]_q.}$ Moreover, if $T = (I : J^\infty)/I = H/I$, then $\boxed{\mathrm{Hilb}_T(q) = \frac{P(q)}{1-q} - [c]_q[b]_q = \frac{q^c[a-c]_q[d]_q + q[c]_q[b]_q}{1-q}.}$*

*Proof.* The ideal sum is $H + Q = (x^c, z, y^b)$, because $x^a$ and $x^c y^d$ are multiples of $x^c$. The quotient therefore has the asserted basis of $cb$ monomials. The standard exact sequence for an intersection,

$$0 \to R/I \to R/H \oplus R/Q \to R/(H+Q) \to 0,$$

is exact. The three right-hand Hilbert series are respectively $[c]_q/(1-q)$, $P(q)/(1-q)$, and $[c]_q[b]_q$. Their alternating sum gives the first formula. The sequence $0 \to H/I \to R/I \to R/H \to 0$ gives the second. The identity between the two displayed expressions follows by expanding $P(q)$. □

**Corollary 8** (Same reduced geometry, unbounded stabilization). *For fixed $b > d \geq 1$ and $c \geq 1$, as $a > c$ varies, all schemes defined by $I_{a,b,c,d}$ have the same reduced support, the same surviving saturation $H = (x^c, z)$, and the same scheme-theoretic intersection length $cb$, whereas the generic multiplicity of the $z$-axis component equals $cb + (a-c)d$ and the stabilization index $\max\{a + d - 1, b + c - 1\}$ both grow without bound.*

*Proof.* All assertions follow from Propositions 6 and 7, together with the index formula established earlier. □

**Example 9** (Fixed support and fixed intersection, varying thickness). *Fix $(b,c,d) = (4,2,1)$ and choose $a = 4,6,9$. In all three cases the reduced scheme is the union of the same two coordinate axes, the surviving component is $(x^2, z)$, and the scheme-theoretic intersection has length 8. The multiplicities of the $z$-axis component are 10,12,15, while the saturation indices are 5,6,9, respectively. Thus the reduced point set and even the length of the intersection fail to determine the saturation index.*

**Corollary 10** (Hilbert polynomial and geometric degree). *For sufficiently large integers $t$, the Hilbert function of $R/I$ is constant and equals $\boxed{\mathrm{HF}_{R/I}(t) = c + cb + (a-c)d.}$ Consequently the degree of the projective scheme associated with the homogeneous ideal $I$ is $c + cb + (a-c)d$.*

*Proof.* The numerator of the Hilbert series of $R/I$ after putting it over $1-q$ evaluates at $q = 1$ to $c + P(1) = c + cb + (a-c)d$; the finite-length intersection term does not affect the eventual Hilbert function. The asserted degree follows from the Hilbert polynomial of a standard graded one-dimensional quotient. □

*Remark 11* (Meaning of geometric removal). Saturation by $J = (x,y)$ removes the $J$-primary component $Q$ from the scheme structure and retains $H$. It is not an operation of deleting individual geometric points: in particular, the surviving $y$-axis still contains the origin, where the two original components meet. This distinction is essential when describing saturation geometrically.

## V. Geometric residual schemes at every stage

Let $Q_r = Q:(x,y)^r$ in $k[x,y]$, and put $e_r = \dim_k k[x,y]/Q_r$. Set

$$\iota_r = \dim_k k[x,y]/\big(Q_r + (x^c)\big).$$

**Theorem 12** (Residual stages and their degrees). *For $0 \le r < K$, $I:J^r = H \cap Q_r R$, $e_r = \#\{(s,t) \in B_0 \cup B_1 : \rho(s,t) > r\}$, $\iota_r = \#\{(s,t) \in B_0 : \rho(s,t) > r\}$. The scheme defined by $I:J^r$ has degree $c + e_r$. For $r \ge K$, $Q_r = k[x,y]$ and the residual is $H$. Furthermore $e_{r-1} - e_r = \nu_r$, and $\iota_{r-1} - \iota_r$ counts the newly entering $B_0$-strings.*

*Proof.* Ideal quotients distribute over intersections. Since $y$ is regular modulo $H$, one has $H:J^r = H$. Extension from $k[x,y]$ to $k[x,y,z]$ commutes with these monomial quotients. A standard monomial survives modulo $Q_r$ precisely when its entry threshold exceeds $r$. The two counting formulas follow. For $r < K$, $Q_r R$ is $(x,y)$-primary, so its generic length along the $z$-axis is $e_r$, while the generic length of $H$ along the $y$-axis remains $c$. The degree is their sum. The intersection length is the number of standard monomials surviving modulo $Q_r + (x^c)$, namely $\iota_r$. Taking successive differences gives the final claims. □

*Remark 13.* Because the ideals are homogeneous and saturated with respect to $(x,y,z)$, their projective schemes are zero-dimensional, supported at $[0:1:0]$ and $[0:0:1]$. The operation $I:J^r$ is the scheme-theoretic residual with respect to the $r$th power of the ideal of $[0:0:1]$. This interpretation does not assert that the points are removed set-theoretically at every intermediate stage.

## VI. Free resolution and Castelnuovo–Mumford regularity

**Theorem 14** (Hilbert–Burch resolution). *The quotient $R/I$ has the minimal graded free resolution $0 \to R(-a-d) \oplus R(-b-c-1) \xrightarrow{\Phi} R(-a) \oplus R(-c-d) \oplus R(-b-1) \to R \to R/I \to 0$, where $\Phi = \begin{pmatrix} y^d & 0 \\ -x^{a-c} & y^{b-d}z \\ 0 & -x^c \end{pmatrix}$. It is Cohen–Macaulay of dimension one, and $\operatorname{reg}(R/I) = \max\{a+d-2, b+c-1\}$, $\operatorname{reg}(I) = \max\{a+d-1, b+c\}$. Consequently $\sigma_J(I)$ is either $\operatorname{reg}(I)$ or $\operatorname{reg}(I)-1$: the former occurs if $a+d > b+c$, the latter otherwise.*

*Proof.* The maximal minors of $\Phi$ are, up to signs, $x^a, x^c y^d, y^b z$. The ideal has height two, so Hilbert–Burch gives exactness; all nonzero matrix entries have positive degree, proving minimality. The projective dimension of $R/I$ is two, and Auslander–Buchsbaum gives depth one, equal to its Krull dimension. Regularity is the largest shift minus homological degree:

$$\max\{a-1, c+d-1, b,\ a+d-2, b+c-1\} = \max\{a+d-2, b+c-1\}.$$

The ideal regularity is one larger. Comparing with $K = \max\{a+d-1, b+c-1\}$ proves the last assertion. □

## VII. The complete graded filtration

Put $T_r = (I:J^r)/I$, $T = H/I$, and define $B_0 = \{(s,t): 0 \le s < c,\ 0 \le t < b\}$ and $B_1 = \{(s,t): c \le s < a,\ 0 \le t < d\}$. Define

$$\rho(s,t) = \begin{cases} K - s - t, & s < c,\ t < d, \\ b + c - 1 - s - t, & s < c,\ d \le t < b, \\ a + d - 1 - s - t, & c \le s < a,\ t < d. \end{cases}$$

**Theorem 15**. *For each $r \ge 0$, as graded $k[z]$-modules, $T_r \cong \bigoplus_{\substack{(s,t)\in B_1 \\ \rho(s,t)\le r}} k[z](-s-t) \ \oplus \bigoplus_{\substack{(s,t)\in B_0 \\ \rho(s,t)\le r}} k[z](-s-t-1)$. The decompositions respect $T_r \subseteq T_{r+1}$. Thus $T$ is free over $k[z]$ of rank $cb + (a-c)d$.*

*Proof.* The monomials outside $Q$ are indexed by $B_0 \cup B_1$. Multiplication by $(x,y)^r$ sends $x^s y^t$ into $Q$ precisely when $r \ge \rho(s,t)$, by checking the two maximal standard-monomial corners $(c-1, b-1)$ and $(a-1, d-1)$. For a monomial to belong to $H$, its $z$-exponent can be zero on $B_1$, but must be positive on $B_0$. The resulting monomial classes are independent over $k[z]$ and give the stated shifts. □

## VIII. Transverse socle filtration and a flat module family

Put $A = k[x,y]/Q$ and let $\mathfrak{m} = (x,y)A$. The Artinian algebra $A$ has $k$-basis indexed by $B_0 \sqcup B_1$. Define $\operatorname{Soc}_r(A) = 0:_A \mathfrak{m}^r$ for $r \ge 0$, with $\operatorname{Soc}_0(A) = 0$. This is the ascending Loewy filtration, not merely the ordinary socle $\operatorname{Soc}_1(A)$.

**Theorem 16** (Loewy interpretation of saturation jumps). *For $r \geq 0$, $\mathrm{Soc}_r(A) = (Q\colon \mathfrak{m}^r)/Q$, and $\nu_r = \dim_k \mathrm{Soc}_r(A)/\mathrm{Soc}_{r-1}(A)$ $(r \geq 1)$, $\sigma_J(I) = \mathrm{ll}(A) = K$. The socle of $A$ is spanned by $x^{c-1}y^{b-1}$ and $x^{a-1}y^{d-1}$; hence $A$ has type two, and it is level if and only if $a + d = b + c$.*

*Proof.* An element of $A$ is killed by $\mathfrak{m}^r$ exactly when any representative belongs to $Q\colon (x, y)^r$. The monomial basis and the threshold formula therefore identify the new classes at stage $r$ with the lattice points of threshold $r$, proving the rank formula. The Loewy length is the least $r$ such that $\mathfrak{m}^r A = 0$, which is $K$. The two maximal standard monomials, one at each staircase corner, generate the socle. Their degrees are $b + c - 2$ and $a + d - 2$, proving the level criterion. □

The following construction interprets the $k[z]$-free decomposition as a flat family of modules, not as a flat family of schemes or algebras.

**Theorem 17** (Flat module degeneration). *There is an identification as graded $k[x, y, z]$-modules $T = H/I \cong x^c A \oplus \oplus_{j\geq 1} z^j A = x^c A[z] + zA[z] \subseteq A[z]$. Consequently $T$ is free, hence flat, over $k[z]$. For $\lambda \in k^\times$ its fiber $T/(z - \lambda)T$ is isomorphic to $A$ as a $k[x, y]$-module. Its special fiber, as a graded $k[x, y]$-module, is $T/zT \cong x^c A \oplus A/x^c A \cong \left(k[x, y]/(x^{a-c}, y^d)\right)(-c) \oplus \left(k[x, y]/(x^c, y^b)\right)(-1)$, where the two summands arise in $z$-degrees zero and one, respectively; the shift $(-1)$ records the total degree of $z$. The fiber lengths are both $cb + (a - c)d$. For each $r$, the general fiber of $T_r$ is $\mathrm{Soc}_r(A)$.*

*Proof.* The second isomorphism theorem identifies $H/I$ with $(H + QR)/QR$ inside $R/QR = A[z]$. In $z$-degree zero, membership in $H = (x^c, z)$ requires divisibility by $x^c$; in positive $z$-degrees there is no additional condition. This proves the module description and $k[z]$-freeness. After imposing $z = \lambda \neq 0$, the generator $zA$ spans $A$. The quotient by $zT$ retains $x^c A$ in degree zero and $zA/zx^c A$ in degree one, yielding the displayed decomposition. Multiplication by $x^c$ identifies $x^c A$ with $k[x, y]/(Q\colon x^c)(-c)$, and $Q\colon x^c = (x^{a-c}, y^d)$; also $A/x^c A \cong k[x, y]/(x^c, y^b)$. Finally, localizing the identity $T_r = (H \cap (Q\colon \mathfrak{m}^r)R)/I$ at $z$ identifies it with $\big((Q\colon \mathfrak{m}^r)/Q\big)[z, z^{-1}]$; specializing at a nonzero $\lambda$ gives the Loewy term. □

*Remark 18.* The construction is a flat family of modules; no algebra structure on the fibers is asserted. Its special fiber splits the transversal monomial staircase into the two rectangles $B_0$ and $B_1$, providing a geometric explanation of the two summand types in the graded saturation filtration.

## IX. Closed generating functions and unimodality

Write $[n]_w = 1 + w + \cdots + w^{n-1}$ and $F(w) = \sum_{(s,t)\in B_0 \cup B_1} w^{\rho(s,t)}$.

**Theorem 19**. *The threshold enumerator is* $\boxed{F(w) = w^{K-c-d+2}[c]_w[d]_w + w[c]_w[b-d]_w + w[a-c]_w[d]_w.}$ *If $F(w) = \sum_{r=1}^{K} \nu_r\, w^r$, then $\nu_r = \mathrm{rank}_{k[z]}(T_r/T_{r-1})$, $F(1) = cb + (a - c)d$, $\deg F = K$, $\nu_1 = 2$, $\nu_K = 1$, and all intermediate coefficients are positive.*

*Proof.* Partition the exponent region into $[0, c - 1] \times [0, d - 1]$, $[0, c - 1] \times [d, b - 1]$, and $[c, a - 1] \times [0, d - 1]$. On each rectangle $\rho$ is a constant minus $s + t$, giving a product of geometric sums. The free decomposition gives the rank interpretation. The extreme coefficients follow from the supports of the three terms; these support intervals cover every integer from 1 to $K$. Since the summands enter at their indicated thresholds and remain present thereafter, each successive quotient is a direct sum of the newly entering $k[z]$-summands. □

The Hilbert series is

$$\mathrm{Hilb}_T(q) = \frac{q^c[a - c]_q[d]_q + q[c]_q[b]_q}{1 - q}.$$

The bivariate polynomial $\mathcal{F}(q, w) = \sum q^{s+t} w^{\rho(s,t)}$ is, with $v = q/w$,

$$\mathcal{F}(q, w) = w^K[c]_v[d]_v + w^{b+c-1}v^d[c]_v[b - d]_v + w^{a+d-1}v^c[a - c]_v[d]_v.$$

After expansion this is a polynomial with nonnegative powers. Its specialization $\mathcal{F}(1, w)$ is $F(w)$. Write $\mathcal{F}_i(q, w) = \sum_{(s,t)\in B_i} q^{s+t}\, w^{\rho(s,t)}$ for $i = 0,1$. Then the exact bigraded jump series is

$$\boxed{\sum_{r\geq 1} \mathrm{Hilb}_{T_r/T_{r-1}}(q) w^r = \frac{\mathcal{F}_1(q, w) + q\mathcal{F}_0(q, w)}{1 - q}.}$$

Indeed, a string indexed by $B_1$ begins at $z$-degree zero, whereas one indexed by $B_0$ begins at $z$-degree one. The numerator reflects the different initial $z$-degrees of the two families of strings.

*A. Two-rectangle simplification*

The three-rectangle formula admits the following simpler, piecewise two-rectangle expression.

**Theorem 20** (Two-rectangle threshold enumerator). *If $a + d \geq b + c$, then $F(w) = w([a]_w[d]_w + [c]_w[b - d]_w)$. If $b + c \geq a + d$, then $F(w) = w([b]_w[c]_w + [d]_w[a - c]_w)$. Both expressions agree when $a + d = b + c$.*

*Proof.* In the first regime the standard exponent region is the disjoint union $[0, a) \times [0, d)$ and $[0, c) \times [d, b)$. On the first rectangle $\rho(s, t) = a + d - 1 - s - t$; on the second, $\rho(s, t) = b + c - 1 - s - t$. Reversing both coordinates in each rectangle gives the two products of geometric sums. Interchanging $x$ and $y$ proves the second regime. □

**Corollary 21** (Unimodality and average threshold). *The coefficient sequence of $F$ is unimodal. When $a + d \geq b + c$, its mean threshold is $\frac{F\prime(1)}{F(1)} = \frac{ad(a+d)/2+c(b-d)(b+c-d)/2}{ad+c(b-d)}$.*

*Proof.* The coefficients of $[u]_w[v]_w$ are the counts of lattice points on successive diagonals of a rectangle. Their consecutive differences are $+1$, then $0$, then $-1$, then $0$. The sum of two such sequences, both starting at degree zero, cannot acquire a positive difference after its first negative difference: at that first negative difference one summand has entered its decreasing phase and neither can still be increasing. This proves unimodality. The mean of the entry times $1 + i + j$ on a $u$-by-$v$ rectangle is $(u + v)/2$. Weighting the two means by the rectangle sizes proves the formula. □

## X. Ordinary powers

Let $\mathfrak{m} = (x, y)$ in $k[x, y]$.

**Proposition 22**. *For every $n \geq 1$, $I^n : J^\infty = (x^c, z)^n$ and $\sigma_J(I^n) = \sigma_{\mathfrak{m}}(Q^n)$.*

*Proof.* Generators of $I^n$ are $x^{a\alpha+c\beta}y^{d\beta+b\gamma}z^\gamma$ with $\alpha + \beta + \gamma = n$. Since $x^{an} \in I^n$, saturation by $x$ is the whole ring, hence saturation by $J = (x, y)$ equals saturation by $y$. Deleting the $y$ exponents from the generators gives $(x^c, z)^n$. The condition $z^n J^r \subseteq I^n$ is equivalent to $\mathfrak{m}^r \subseteq Q^n$. If this holds, replace factors $y^b z$ in a witness generator of $I^n$ by $x^c y^d$ until the $z$-exponent is at most $j$, and compensate with the $x^{c(n-j)}$ factor of the generator $x^{c(n-j)}z^j$ of $(x^c, z)^n$. Indeed, if a monomial $x^i y^t$ belongs to $Q^n$, choose a witness $x^{a\alpha+c\beta}y^{d\beta+b\gamma}$ with $\alpha + \beta + \gamma = n$. For $0 \leq j \leq n$, when $\gamma > j$ replace $\gamma - j$ factors $y^b z$ by $x^c y^d$; the resulting generator divides $x^{i+c(n-j)}y^t z^j$ because $d < b$. When $\gamma \leq j$, the original generator already divides it. Thus every generator of the saturation enters the $r$th quotient exactly when $z^n$ does. □

**Theorem 23**. *For $a = b = m$, $d = m - c$, $1 \leq c < m$, and all $n \geq 1$,* $\boxed{\sigma_J(I^n) = mn + \max\{c, m - c\} - 1.}$

*Proof.* The minimal generators of $Q^n$ have degree $mn$ and distinct $x$-exponents $S_n = \{m\alpha + c\beta : \alpha + \beta \leq n\}$. If $D_n$ is the largest gap between successive elements, the largest degree of a standard monomial is $mn + D_n - 2$. The chain $0, c, \ldots, nc, nc + (m - c), \ldots, nm$ proves $D_n \leq \max(c, m - c)$; the first and last gaps give the reverse inequality. Apply the preceding proposition. □

## XI. Bounds and eventual linearity for ordinary powers

**Theorem 24** (Bounds and eventual linearity). *Let $D = \max\{a, b\}$ and $\sigma_n = \sigma_J(I^n)$. For every $n \geq 1$, $\max\{an + d - 1, bn + c - 1\} \leq \sigma_n \leq K + (n - 1)D$. Moreover $\sigma_{n+1} \leq \sigma_n + D$ and $\sigma_n - nD$ is eventually constant. If $(a - b)(a + d - b - c) \geq 0$, then $\sigma_n = K + (n - 1)D$ $(n \geq 1)$. In particular, if $a = b$, then $\sigma_n = an + \max\{c, d\} - 1$ for all $n \geq 1$.*

*Proof.* By the established reduction $\sigma_n$ is the first $r$ for which $(x, y)^r \subseteq Q^n$. The monomials $x^{an-1}y^{d-1}$ and $x^{c-1}y^{bn-1}$ are outside $Q^n$, yielding the lower bound. Suppose $a \geq b$. A monomial $x^s y^t$ of total degree at least $\sigma_n + a$ belongs to $Q^{n+1}$: if $s \geq a$, divide by $x^a$; otherwise $t \geq \sigma_n + 1 \geq bn + 1 \geq b$, and division by $y^b$ leaves degree at least $\sigma_n$. The case $b \geq a$ is symmetric. Hence $\sigma_{n+1} \leq \sigma_n + D$ and iteration gives the upper bound. The integers $\sigma_n - nD$ are nonincreasing and bounded below by the displayed lower bound, hence eventually constant. In the aligned case the relevant lower and upper bounds coincide. For $a = b$, the aligned condition is automatic and the asserted formula follows. □

*Remark 25*. The eventual linearity is not asserted as a new general phenomenon: it is well established in the literature on regularity of powers. The contribution here is an elementary, family-specific derivation and an explicit equality criterion. (Herzog et al., 2019)

## XII. Further consequences and worked computations

The following examples and consequences illustrate the exact formulas without introducing additional invariants.

### A. *A complete numerical example*

Take $(a, b, c, d) = (5,4,2,1)$, so that

$$I = (x^5, x^2y, y^4z), \quad H = (x^2, z), \quad Q = (x^5, x^2y, y^4).$$

The standard monomials of $Q$ are $1, y, y^2, y^3, x, xy, xy^2, xy^3, x^2, x^3, x^4$. Their thresholds are shown in Table 1.

Exact entry times for $(a, b, c, d) = (5,4,2,1)$.

| r | Monomials with threshold r | rank over k[z] |
|---|---|---|
| 1 | $xy^3$, $x^4$ | 2 |
| 2 | $y^3$, $xy^2$, $x^3$ | 3 |
| 3 | $y^2$, xy, $x^2$ | 3 |
| 4 | y, x | 2 |
| 5 | 1 | 1 |

Hence $F(w) = 2w + 3w^2 + 3w^3 + 2w^4 + w^5$ and the cumulative $k[z]$-ranks of $T_r$ are $0,2,5,8,10,11$ for $r = 0,1,2,3,4,5$. This is a stronger numerical description than the single statement $\sigma_J(I) = 5$.

### B. *The distinction between length and multiplicity*

Although $\mathrm{rank}_{k[z]}T = 11$ in the example, $T$ has infinitely many independent monomial classes as a $k$-vector space: each admitted string has arbitrary positive powers of $z$. Thus $\mathrm{length}_R(T) = \infty$. The value $11$ is the eventual Hilbert-function value and the multiplicity of the one-dimensional graded module. Thus free rank and finite length must not be identified.

## XIII. The Hilbert series of every stage

The graded $k[z]$-decomposition allows us to compute not only the limiting Hilbert series but each intermediate Hilbert series. Define the finite polynomials

$$A_r(q) = \sum_{\substack{(s,t)\in B_1 \\ \rho(s,t)\le r}} q^{s+t}, \qquad B_r(q) = \sum_{\substack{(s,t)\in B_0 \\ \rho(s,t)\le r}} q^{s+t}.$$

**Proposition 26**. *For every $r \ge 0$, $\mathrm{Hilb}_{T_r}(q) = \frac{A_r(q)+qB_r(q)}{1-q}$. Moreover, if $A_{r-1}$ and $B_{r-1}$ are defined in the same manner, then $\mathrm{Hilb}_{\Delta_r}(q) = \frac{A_r(q)-A_{r-1}(q)+q(B_r(q)-B_{r-1}(q))}{1-q}$. The eventual value of the Hilbert function of $T_r$ equals $A_r(1) + B_r(1)$, its $k[z]$-rank.*

*Proof.* A free graded summand $k[z](-e)$ contributes $q^e/(1-q)$ to the Hilbert series. Summing the shifts in the decomposition of $T_r$ proves the first identity. The inclusion $T_{r-1} \subseteq T_r$ is a direct inclusion of selected free summands, so the quotient is again free and subtraction of the two series is valid. Finally, a finite direct sum of shifts of $k[z]$ has Hilbert function eventually equal to its number of summands. □

**Corollary 27**. *For $r \ge 1$, $\mathrm{rank}_{k[z]}\Delta_r = \nu_r$, and $\mathrm{rank}_{k[z]}T_r = \sum_{j=1}^{r}\nu_j$. In particular the rank of the limiting module is $F(1)$ and its stabilization stage is* deg$F$.

## XIV. A fully explicit computation of an ordinary power

Consider the equigenerated example $m = 3$, $c = 1$, so that

$$I = (x^3, xy^2, y^3z), \qquad Q = (x^3, xy^2, y^3).$$

For the second power, the minimal monomial generators of $Q^2$ are

$$x^6,\ x^4y^2,\ x^3y^3,\ x^2y^4,\ xy^5,\ y^6.$$

The attainable $x$-exponents are 0,1,2,3,4,6. The successive gaps are 1,1,1,1,2, so the maximal gap is two. The staircase formula therefore gives

$$\sigma_J(I^2) = \sigma_{(x,y)}(Q^2) = 6 + 2 - 1 = 7.$$

By comparison, the first power has $\sigma_J(I) = 4$. Hence the second index is 7, not $8 = 2\sigma_J(I)$. This example illustrates why a formula for $\sigma_J(I^n)$ requires the exponent staircase of $Q^n$, not merely the original stabilization index.

## XV. An exact arithmetic algorithm and reproducibility

The following algorithm computes the index for arbitrary admissible parameters and any prescribed positive integer $n$. It is a direct implementation of the monomial staircase, not a claim of a new complexity-theoretic result.

1. Generate all pairs $(a\alpha + c\beta, d\beta + b\gamma)$ with $\alpha + \beta + \gamma = n$.
2. Sort their distinct first coordinates as $0 = X_0 < X_1 < \cdots < X_t = an$.
3. At each $X_i$, compute $Y_i$, the minimum second coordinate among all generated pairs with first coordinate at most $X_i$.
4. Return $\max_{0\le i<t}(X_{i+1} + Y_i - 1)$.

The generation stage uses at most $\binom{n+2}{2}$ exponent pairs. The algorithm may be checked independently by enumerating standard monomials of $Q^n$ up to the pure-power bounds $x^{an}$ and $y^{bn}$. Such a finite check is appropriate for testing small parameters; it does not replace the general proofs. No external computation is claimed as part of the proof.

## XVI. Symbolic powers and the embedded intersection point

We retain the notation $H = (x^c, z)$, $Q = (x^a, x^c y^d, y^b)$, $I = H \cap QR$, and $J = (x, y)$. We now give a direct monomial criterion for equality of symbolic and ordinary powers.

**Theorem 28** (Symbolic versus ordinary powers). *Put $\mu = \lceil a/(a-c)\rceil$. For every $n \ge 2$, $I^{(n)} = H^n \cap Q^n R$, $\quad I^{(n)} = I^n \quad \Leftrightarrow \quad d\min\{n,\mu\} \le b$. If the inequality fails, the monomial $x^{\max\{a(n-1),cn\}}y^b \in I^{(n)}\backslash I^n$. Consequently, equality holds for every $n \ge 1$ if and only if $d\mu \le b$. Otherwise its first failure occurs at $n_* = \lfloor b/d\rfloor + 1$ and persists for every $n \ge n_*$.*

*Proof.* The two minimal primes of $I$ are $(x, z)$ and $(x, y)$. Localizing $I$ at these primes and contracting the powers gives $H^n$ and $Q^n R$, respectively. Their intersection is the symbolic power. For a monomial $M = x^i y^j z^k$, membership in $H^n$ means $\lfloor i/c\rfloor + k \ge n$. Membership in $Q^n R$ means that there exist $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma = n$ and

$$a\alpha + c\beta \le i, \qquad d\beta + b\gamma \le j.$$

Such a triple is called a witness. The same monomial belongs to $I^n$ exactly when it admits a witness with $\gamma \le k$. Assume $d\min(n,\mu) \le b$ and choose a witness with $\gamma > k$. If $\alpha \ge \mu - 1$, replace the witness by $(\alpha - \mu + 1, \beta + \mu, \gamma - 1)$. The $x$ exponent does not increase because $\mu c \le (\mu - 1)a$; the $y$ exponent does not increase because $\mu d \le b$ (here necessarily $n \ge \mu$). If $\alpha \le \mu - 2$, replace it by $(0, \alpha + \beta + 1, \gamma - 1)$. Its $x$ exponent is $c(n - \gamma + 1) \le c(n - k) \le i$; its $y$ exponent does not increase because $d(\alpha + 1) \le d\min(n,\mu) \le b$. Repeated application reduces $\gamma$ to at most $k$, so $M \in I^n$. Conversely, set $i = \max\{a(n-1), cn\}$ and $M = x^i y^b$. Then $M \in H^n$ and $M \in Q^n R$, using the witness $(n - 1,0,1)$. If $M \in I^n$, its lack of $z$ forces $\gamma = 0$ and hence a witness $(n - \beta, \beta, 0)$ with

$$an - (a - c)\beta \le i, \qquad d\beta \le b.$$

If $n < \mu$, then $i = cn$ and the first inequality forces $\beta = n$. If $n \ge \mu$, then $i = a(n-1)$ and it forces $\beta \ge \mu$. Thus membership is impossible whenever $d\min(n,\mu) > b$. The remaining assertions follow. □

**Corollary 29** (Embedded component criterion). *The quotient $I^{(n)}/I^n$ has finite length. If it is nonzero, then $(x, y, z)$ is an embedded associated prime of $R/I^n$; if it vanishes, there is no such embedded associated prime.*

*Proof.* The two ideals agree upon localization at the two minimal primes, and a monomial associated prime other than these can only be the homogeneous maximal ideal. Thus the quotient has finite length. If nonzero it is a finite-length submodule of $R/I^n$,

and hence contributes the maximal associated prime. If zero, $I^n$ is the intersection of the two indicated primary powers, so it has no embedded associated prime. □

**Example 30**. *For* $(a,b,c,d) = (4,7,3,2)$*, one has* $\mu = 4$ *and* $n_* = 4$*. Thus* $I^{(2)} = I^2$ *and* $I^{(3)} = I^3$*, whereas* $I^{(n)} \neq I^n$ *for every* $n \geq 4$*. At* $n = 4$*,* $x^{12}y^7 \in I^{(4)} \backslash I^4$.

*Remark 31*. The criterion concerns this explicit family and is situated within the general theory of symbolic powers of monomial ideals . (Cooper et al., 2017; Méndez et al., 2025)

### XVII. A CONTROLLED FIFTH-PARAMETER COMPARISON

The comparison between saturation and regularity for the original family is special, rather than universal. We record a controlled extension which makes the distinction transparent.

**Theorem 32**. *Let* $e \geq 1$ *and set* $I_e = (x^a, x^c y^d, y^b z^e)$, $\quad H_e = (x^c, z^e)$, $\quad Q = (x^a, x^c y^d, y^b)$*. For* $J = (x,y)$ *and* $K = \max\{a + d - 1, b + c - 1\}$ *one has* $I_e = H_e \cap QR$, $\quad I_e{:}J^r = H_e \cap (Q{:}\mathfrak{m}^r)R$, $\quad \sigma_J(I_e) = K$*. Moreover* $R/I_e$ *is Cohen–Macaulay of dimension one, and* $\operatorname{reg}(R/I_e) = \max\{a + d - 2, b + c + e - 2\}$, $\operatorname{reg}(I_e) = \max\{a + d - 1, b + c + e - 1\}$*. Consequently* $\operatorname{reg}(I_e) - \sigma_J(I_e) = \max\{0, b + c + e - 1 - K\}$*, which is unbounded as* $e$ *increases.*

*Proof*. The intersection is computed by least common multiples of monomial generators. Since $y$ is a nonzerodivisor modulo $H_e$, $H_e{:}J^r = H_e$; quotient and extension from $k[x,y]$ commute. The class of $z^e$ reaches the saturation precisely when $\mathfrak{m}^r \subseteq Q$, which first happens at $r = K$. The other minimal generator $x^c$ enters no later, since $x^c \mathfrak{m}^{a-c+d-1} \subseteq I_e$. The two first syzygies of $I_e$ have degrees $a + d$ and $b + c + e$; their Hilbert–Burch matrix is

$$\begin{pmatrix} y^d & 0 \\ -x^{a-c} & y^{b-d} z^e \\ 0 & -x^c \end{pmatrix}.$$

Its maximal minors generate $I_e$ and $I_e$ has height two. The Hilbert–Burch theorem yields a minimal resolution, depth one, and the displayed regularity formulas. Since $a + d - 1 \leq K$, subtracting $K$ gives the claimed difference. □

*Remark 33*. The generic multiplicity of the component supported on the $y$-axis is $ce$, while the other component has multiplicity $L = cb + (a - c)d$. The fifth parameter changes the former but not the transverse Loewy length at the $z$-axis. This explains geometrically why the two invariants can separate.

### XVIII. DEGREES OF POWERS AND THE NEWTON POLYGON

We keep the notation $Q = (x^a, x^c y^d, y^b)$ and write $\lambda_n = \dim_k k[x,y]/Q^n$.

**Proposition 34**. *For every* $n \geq 1$*, the projective zero-dimensional scheme* $\operatorname{Proj}(R/I^n)$ *has degree* $\operatorname{degProj}(R/I^n) = c\binom{n+1}{2} + \lambda_n$*. Furthermore* $\lambda_n = \frac{e(Q)}{2} n^2 + O(n)$, $\quad e(Q) = \min\{ab, ad + bc\}$*, and hence* $\operatorname{degProj}(R/I^n) = \frac{c + \min\{ab, ad+bc\}}{2} n^2 + O(n)$.

*Proof*. At the generic point of the $y$-axis, the local transverse ideal is $(x^c, z)^n$, whose colength is $\sum_{j=1}^{n} c\, j = c\binom{n+1}{2}$. At the generic point of the $z$-axis the transverse ideal is $Q^n$, of colength $\lambda_n$. The associativity formula for multiplicity gives the degree as their sum; any embedded component at the intersection has smaller dimension and does not change this degree. The Hilbert–Samuel polynomial of the Artinian quotient of a power of the $\mathfrak{m}$-primary ideal $Q$ has leading term $e(Q)n^2/2$. For a monomial ideal in two variables, $e(Q)$ is twice the area bounded by the coordinate axes and its Newton boundary. If $(c,d)$ lies on or above the segment joining $(a,0)$ and $(0,b)$, the area is $ab/2$. Otherwise the boundary passes through $(c,d)$ and the area is $(ad + bc)/2$. Thus $e(Q) = \min\{ab, ad + bc\}$, proving the result. □ See Cassou-Noguès and Veys (2014) for related Newton-polygon methods.

*Remark 35*. The Newton-polygon area formula and Hilbert–Samuel asymptotics are classical. This proposition applies them to the two transverse components of our specific three-variable family; it is not claimed as a new general multiplicity theorem.

### XIX. DISCUSSION, SCOPE, AND RELATION TO KNOWN RESULTS

The distinction between the support ideal $J = (x,y)$ and the irrelevant maximal ideal $(x,y,z)$ is essential: $T$ is a one-dimensional graded module, rather than a finite-length graded module. Consequently, its filtration jumps are measured by free $k[z]$-ranks

and graded shifts, not by finite lengths over $R$. The rank polynomial $F(w)$ records all entry stages, whereas the single number $\sigma_J(I)$ records only the last nonzero jump.

The combinatorial description is built from the standard-monomial staircase of the Artinian two-variable ideal $Q$. Standard monomial methods and Hilbert series are treated systematically in ; the general saturation number of a two-variable monomial ideal is covered by Abdolmaleki and Yazdan Pour , notably Theorem 2.9. The formula for ordinary powers is an explicit specialization in this setting, and should not be confused with a new general theorem about powers. Herzog, Karimi and Mafi provide further context on the behavior of saturation numbers of powers of graded ideals. Standard monomial methods and Hilbert series are treated by Miller and Sturmfels (2005); the two-variable saturation-number formula is established by Abdolmaleki and Yazdan Pour (2023, Theorem 2.9).

The simultaneous description of the graded filtration, its transverse Loewy interpretation, and its geometric residual stages provides a unified case study. The arguments complement the general results cited above.

## XX. Conclusions

For the four-parameter family $I = (x^a, x^c y^d, y^b z)$, saturation by $(x, y)$ is proper and the complete graded filtration admits explicit monomial thresholds, a closed rank enumerator and stagewise Hilbert series. The transverse Loewy filtration and flat module family explain the two types of free summands and the special-fiber splitting. The ordinary-power reduction relates the corresponding stabilization indices to an Artinian monomial ideal in two variables. A parameter criterion also determines when symbolic and ordinary powers coincide and when an embedded point occurs at their intersection. A fifth-parameter extension separates saturation and regularity, while a Newton-polygon calculation describes the asymptotic degree of the ordinary powers. All arguments work over arbitrary fields and require only monomial divisibility.

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## References

Abdolmaleki, R., & Yazdan Pour, A. A. (2023). The saturation number of monomial ideals. *Communications in Algebra*, *51*(8), 3163–3171. https://doi.org/10.1080/00927872.2023.2178658

Cassou-Noguès, P., & Veys, W. (2014). Newton trees for ideals in two variables and applications. *Proceedings of the London Mathematical Society*, *108*(4), 869–910. https://doi.org/10.1112/plms/pdt047

Cooper, S. M., Embree, R. J. D., Hà, H. T., & Hoefel, A. H. (2017). Symbolic powers of monomial ideals. *Proceedings of the Edinburgh Mathematical Society*, *60*(1), 39–55. https://doi.org/10.1017/S0013091516000110

Cox, D. A., Little, J., & O'Shea, D. (2015). *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra* (4th ed.). Springer. https://doi.org/10.1007/978-3-319-16721-3

Herzog, J., Karimi, M., & Mafi, A. (2019). The saturation number of powers of graded ideals [Preprint]. arXiv. https://arxiv.org/abs/1907.03154

Méndez, F. O., Vaz Pinto, M., & Villarreal, R. H. (2025). Symbolic powers: Simis and weighted monomial ideals. *Journal of Algebra and Its Applications*, *24*(13–14), Article 2541001. https://doi.org/10.1142/S0219498825410014

Miller, E., & Sturmfels, B. (2005). *Combinatorial commutative algebra*. Springer. https://doi.org/10.1007/b138602