Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Arithmetic Enumeration of Klein-Graded Orders in a Biquadratic Field

Luis Alfredo Dupont García[1] ✉, Juan Rafael Acosta Portilla[2], Manuel Alejandro Torres Sotelo[1]
[1] *Facultad de Matemáticas, Universidad Veracruzana, Xalapa, Veracruz, Mexico*
[2] *Instituto de Investigaciones y Estudios Superiores Económicos y Sociales, Universidad Veracruzana, Xalapa, Veracruz, Mexico*
Corresponding Author: *Luis Alfredo Dupont García;* **E-mail**: *ldupont@uv.mx*

| ABSTRACT

Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{7})$ and $\mathcal{O}_K$ be its ring of integers. We classify the conductor ideals realized by the embedded Klein-graded orders $R_{a,b,c} = \mathbb{Z} \oplus a\mathbb{Z}\sqrt{3} \oplus b\mathbb{Z}\sqrt{7} \oplus c\mathbb{Z}\sqrt{21}$, and determine the cardinality of every conductor fiber within this family. Every such conductor has the form $2t(\sqrt{3})^{-\varepsilon_3}(\sqrt{7})^{-\varepsilon_7}\mathcal{O}_K$, with $\varepsilon_i \in \{0,1\}$ where $\varepsilon_3 = 1$ requires $3 \mid t$ and $\varepsilon_7 = 1$ requires $7 \mid t$. Its fiber cardinality is a product of explicit local factors $A(k) = 1 + 3k(k+1)/2$, $B_0(k) = (k+1)^2$, and $B_1(k) = k(k+1)/2$. We prove the classification by local conductor calculations, including a perfect-pairing argument at 2. Scalar conductor fibers, index spectra, and Dirichlet series by conductor and by index follow as consequences. The enumeration concerns embedded graded orders, not all orders or isomorphism classes of orders in K.

| KEYWORDS
biquadratic number field; Klein grading; conductor ideals; conductor fibers; local enumeration; Dirichlet series.

| ARTICLE INFORMATION
ACCEPTED: 08 September 2026 **PUBLISHED:** 23 October 2026 **DOI:** 10.32996/jmss.2026.7.6.12

1. Introduction

A conductor records how a nonmaximal order sits inside the maximal order of a number field, but it need not determine the order uniquely. Here we solve a concrete inverse problem: for a naturally graded family of orders in a fixed biquadratic field, which conductor ideals occur, and how many embedded orders realize each ideal? The central result is a complete conductor-and-fiber classification within that family, including nonscalar conductors at the exceptional primes 3 and 7. The enumeration of orders with conductor $2t\mathcal{O}_K$ is recovered as a specialization.

The restriction to graded orders is essential and mathematically natural: the four rational radical lines are the homogeneous components of the Klein grading. At odd primes, grading agrees with Galois stability; at 2 these conditions differ. We do not enumerate all quartic orders or count abstract isomorphism classes.

Position within the literature.

The integral-closure background follows Alaca and Williams (2004); related explicit biquadratic-order constructions appear in Krásenský et al. (2022). Nakagawa (1996) studies quartic orders counted by index, while Lettl and Prabpayak (2014) investigate conductor ideals and Lettl and Prabpayak (2016) study orders in cubic number fields. Prabpayak (2017) examines conductors in ramified quartic fields. Kaplan et al. (2015) study order distribution through zeta functions. Kopp and Lagarias (2025) develop general order and ray-class theory, including conductor nonuniqueness.

Our classification concerns embedded orders graded by the Klein four-group in a fixed biquadratic field, and includes both scalar and nonscalar conductors. It is not a count of all quartic orders or of abstract isomorphism classes. In the scalar case the conductor-fiber Dirichlet series is indexed by t, whereas the index zeta function is indexed by $[\mathcal{O}_K : R]$. We have not established that the explicit restricted formula is absent from all prior classifications; no bibliographic priority is asserted.

Copyright: © 2026 the Author(s). This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) 4.0 license (https://creativecommons.org/licenses/by/4.0/). Published by Al-Kindi Centre for Research and Development, London, United Kingdom.

Main classification and organization.

Define the local counting functions

$$A(k) = 1 + \frac{3k(k+1)}{2}, \qquad B_0(k) = (k+1)^2, \qquad B_1(k) = \frac{k(k+1)}{2}.$$

Write $t = 2^{\alpha}3^{\beta}7^{\gamma}\prod_{p\nmid 42} p^{k_p}$. The main classification (Theorem 9) states that the conductor of every order in this family can be written uniquely as

$$2t\left(\sqrt{3}\right)^{-\varepsilon_3}\left(\sqrt{7}\right)^{-\varepsilon_7}\mathcal{O}_K, \qquad \varepsilon_3 = 1 \Rightarrow 3 \mid t, \quad \varepsilon_7 = 1 \Rightarrow 7 \mid t,$$

and the number $N(t;\varepsilon_3,\varepsilon_7)$ of realizing orders is

$$A(\alpha)B_{\varepsilon_3}(\beta)B_{\varepsilon_7}(\gamma)\prod_{p\nmid 42} A\left(k_p\right).$$

The proof proceeds from the integral basis and the closure divisibilities to odd-prime and 2-adic conductors, and then to a local-to-global fiber count. Subsequent sections treat the scalar specialization, indices, two Dirichlet series, and analytic consequences. In particular, the analytic formulas are deductions from the classification, not independent claims of a new counting method.

2. The Klein grading and intrinsic scope

Notation.

Write $x = \sqrt{3}$, $y = \sqrt{7}$, $z = xy = \sqrt{21}$ and $S = \mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z = \mathbb{Z}[x,y]$. We call these orders Klein-graded (or diagonal). The decomposition

$$K = \mathbb{Q} \oplus \mathbb{Q}x \oplus \mathbb{Q}y \oplus \mathbb{Q}z$$

is a grading by the Klein four-group: products of homogeneous components lie in the component indexed by the group product. This makes the diagonal restriction natural.

Proposition 1. *For an order $R \subseteq S$, the following are equivalent: (i) $R = R_{a,b,c}$ for positive integers a,b,c; (ii) $R = \oplus_{w\in\{1,x,y,z\}}(R \cap \mathbb{Q}w)$ as additive groups. At any odd prime p, a $\mathbb{Z}_p$-order in $\mathcal{O}_{K,p} = S_p$ is graded if and only if it is stable under* $\mathrm{Gal}(K/\mathbb{Q})$.

Proof. The first equivalence follows since $R \cap \mathbb{Q} = \mathbb{Z}$ and each remaining intersection is a rank-one integral lattice. For odd p, the character projectors $\frac{1}{4}\sum_{\sigma\in\mathrm{Gal}(K/\mathbb{Q})}\chi(\sigma)\sigma$ preserve every Galois-stable $\mathbb{Z}_p$-lattice because 4 is invertible in $\mathbb{Z}_p$. Their images are precisely the four homogeneous components. Conversely, each Galois automorphism acts on each component by a sign, so every graded order is stable. □

At $p = 2$ Galois-stability need not imply grading. For example, $\mathbb{Z} + \mathbb{Z}(x+y) + 2\mathbb{Z}x + \mathbb{Z}z$ is a Galois-stable order contained in S, but is not diagonal: it contains $x+y$ without containing x and y separately. Indeed $(x+y)^2 = 10 + 2z$, $(x+y)z = 3(x+y) + 4x$, and $(2x)z = 6(x+y) - 3(2x)$.

3. Integral basis

The ring of integers has integral basis

$$\mathcal{O}_K = \mathbb{Z}\left\langle 1, x, \omega = \frac{1+z}{2}, \eta = \frac{x+y}{2}\right\rangle.$$

For positive integers a,b,c, define

$$R_{a,b,c} = \mathbb{Z} \oplus a\mathbb{Z}x \oplus b\mathbb{Z}y \oplus c\mathbb{Z}z.$$

The conductor of R is $(R:\mathcal{O}_K) = \{\alpha \in \mathcal{O}_K : \alpha\mathcal{O}_K \subseteq R\}$. The three quadratic subfields of K are $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(\sqrt{21})$. Their fundamental discriminants are 12, 28 and 21, respectively. The biquadratic discriminant formula gives

$$\mathrm{disc}(K) = 12 \cdot 28 \cdot 21 = 7056.$$

The trace pairing in the radical basis $(1,x,y,z)$ is diagonal, with entries $(4,12,28,84)$; hence its discriminant is $112896 = 16 \cdot 7056$. The displayed lattice in the integral basis above is closed under multiplication and has index four over S, and therefore has discriminant 7056. This verifies that it is the full ring of integers; compare the integral-basis classification in Williams (1970).

4. Diagonal orders and indices

Proposition 2. *The lattice $R_{a,b,c}$ is an order if and only if $c \mid ab$, $a \mid 7bc$, $b \mid 3ac$. If these conditions hold, then $[\mathcal{O}_K : R_{a,b,c}] = 4abc$.*

Proof. The squares of ax, by, cz are rational integers, while their pairwise products are $abz, 3acy, 7bcx$. Closure under multiplication requires the three divisibilities stated in the proposition. The additive index in S is abc and $[\mathcal{O}_K : S] = 4$. Consequently, the discriminant of the order is $7056(4abc)^2 = 112896a^2b^2c^2$. □

5. An elementary conductor bound

Lemma 3. *For every admissible triple (a, b, c), let $d = \mathrm{lcm}(a, b, c)$. Then $2d\mathcal{O}_K \subseteq (R_{a,b,c} : \mathcal{O}_K) \subseteq R_{a,b,c} \subseteq S$. Moreover, for each positive integer t, $2t\mathcal{O}_K \subseteq R_{a,b,c} \iff a \mid t,\ b \mid t,\ c \mid t$.*

Proof. The integral basis $1, x, (1+z)/2, (x+y)/2$ shows that $2d\mathcal{O}_K$ is additively generated by $2d$, $2dx$, $d(1+z)$, and $d(x+y)$. Each generator belongs to $R_{a,b,c}$ because a, b, c divide d. As $2d\mathcal{O}_K$ is an ideal of $\mathcal{O}_K$, its inclusion in $R_{a,b,c}$ implies its inclusion in the conductor. The conductor is contained in $R_{a,b,c}$ because $1 \in \mathcal{O}_K$. The same four-generator calculation with d replaced by t shows that $2t\mathcal{O}_K \subseteq R_{a,b,c}$ precisely when $a \mid t$, $b \mid t$, and $c \mid t$. □

6. Conductors at odd primes

For a prime p write $r = v_p(a)$, $s = v_p(b)$, $u = v_p(c)$, $\delta_3 = v_p(3)$ and $\delta_7 = v_p(7)$. Localization at p is denoted by a subscript p.

Theorem 4 (Odd-prime conductors). *For every odd prime p the conductor of $R_{a,b,c}$ localized at p is the following diagonal $\mathbb{Z}_p$-lattice:*

$$
\begin{aligned}
(R_{a,b,c} : \mathcal{O}_K)_p &= p^{e_0}\mathbb{Z}_p \oplus p^{e_x}\mathbb{Z}_p x \oplus p^{e_y}\mathbb{Z}_p y \oplus p^{e_z}\mathbb{Z}_p z,\\
e_0 &= \max(r, s, u),\\
e_x &= \max(r, u, s - \delta_3),\\
e_y &= \max(s, u, r - \delta_7),\\
e_z &= \max(u, s - \delta_3, r - \delta_7).
\end{aligned}
$$

Proof. Since $[\mathcal{O}_K : S] = 4$, we have $\mathcal{O}_{K,p} = S_p$ for odd p. An element $A + Bx + Cy + Dz$ is in the conductor exactly when its products by $1, x, y, z$ are in $R_{a,b,c,p}$. Their coordinates in the basis $1, x, y, z$ are respectively

$$(A, B, C, D), \quad (3B, A, 3D, C), \quad (7C, 7D, A, B), \quad (21D, 7C, 3B, A).$$

The divisibility requirements on the x-, y- and z-coordinates of these four products give the four stated bounds. They are necessary and sufficient. □

Corollary 5. *If $p \nmid 42$, the conductor at p of R_{p^r,p^s,p^u} is $p^M\mathcal{O}_{K,p}$, where $M = \max(r, s, u)$. At $p = 3$ the four exponents reduce to (M, ℓ, M, ℓ), with $\ell = \max(r, u, s-1)$; at $p = 7$ they reduce to (M, M, ℓ, ℓ) with $\ell = \max(s, u, r-1)$.*

7. An explicit conductor-membership test

The local results can also be checked directly over the integers. Write $\alpha = A + Bx + Cy + Dz \in S$. Since $\mathcal{O}_K$ has basis $(1, x, \omega, \eta)$, α belongs to $(R_{a,b,c} : \mathcal{O}_K)$ precisely when all four products α, αx, $\alpha\omega$, $\alpha\eta$ belong to $R_{a,b,c}$. The last two products have radical-coordinate vectors

$$\frac{1}{2}(A + 21D,\ B + 7C,\ C + 3B,\ A + D), \qquad \frac{1}{2}(3B + 7C,\ A + 7D,\ A + 3D,\ B + C).$$

Consequently conductor membership is a finite system of integer congruences, with the nonconstant coordinates divisible by a, b, c. This provides a direct check independent of the local counting argument.

For completeness, the conditions can be displayed without matrix notation. The first two products require

$$a \mid B,\ b \mid C,\ c \mid D, \qquad a \mid A,\ b \mid 3D,\ c \mid C.$$

The product by ω requires

$$2 \mid A + 21D, \quad 2a \mid B + 7C, \quad 2b \mid C + 3B, \quad 2c \mid A + D,$$

whereas the product by η requires

$$2 \mid 3B + 7C, \quad 2a \mid A + 7D, \quad 2b \mid A + 3D, \quad 2c \mid B + C.$$

The constant-coordinate congruences ensure integrality of the products, and the other three congruences in each row ensure membership in the diagonal lattice. These conditions are useful for checking an individual order, including cases where its conductor is not a scalar ideal.

8. The prime two

For $j \in \{x, y, z\}$ let $[v]_j$ denote the coefficient of j in the radical basis $1, x, y, z$. Define on $V = \mathcal{O}_K/2\mathcal{O}_K$ the $\mathbb{F}_2$-bilinear pairing $\lambda_j(v, w) = 2[vw]_j \pmod 2$. This is well defined because all radical coordinates of algebraic integers are half-integral, and replacing either argument by an element of $2\mathcal{O}_K$ changes $2[vw]_j$ by an even integer. The radical-coordinate multiplication rule is

$$\begin{aligned}(A + Bx + Cy + Dz)(E + Fx + Gy + Hz) = \quad & (AE + 3BF + 7CG + 21DH) \\ & +(AF + BE + 7CH + 7DG)x \\ & +(AG + CE + 3BH + 3DF)y \\ & +(AH + DE + BG + CF)z.\end{aligned}$$

Lemma 6. *Each λ_j is nondegenerate.*

Proof. In the integral basis $(1, x, \omega, \eta)$ the matrices of λ_x and λ_y are both

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

whereas the matrix of λ_z is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

All three have determinant 1 over $\mathbb{F}_2$. The matrices of λ_x and λ_y coincide modulo 2, while that of λ_z differs; all three are nonsingular. □

Theorem 7 (2-adic conductors). *If $r, s, u \geq 0$ and $R_{2^r,2^s,2^u}$ is an order, then* $\left(R_{2^r,2^s,2^u} : \mathcal{O}_K\right) = 2^{M+1}\mathcal{O}_K, \qquad M = \max(r, s, u).$

Proof. The integral basis displayed above shows that $2^{M+1}\mathcal{O}_K \subseteq R_{2^r,2^s,2^u}$; hence it is contained in the conductor I. Suppose I strictly contains $2^{M+1}\mathcal{O}_K$. The finite nonzero 2-group $I/2^{M+1}\mathcal{O}_K$ has an element of order 2, represented by $2^M\beta$ with $\beta \in \mathcal{O}_K \backslash 2\mathcal{O}_K$. Choose $j \in \{x, y, z\}$ such that the corresponding exponent (r, s or u) equals M. For every $\gamma \in \mathcal{O}_K$, membership of $2^M\beta\gamma$ in the order forces $[\beta\gamma]_j \in \mathbb{Z}$. Thus $\lambda_j(\beta, \gamma) = 0$ for all γ. Lemma 6 gives $\beta \in 2\mathcal{O}_K$, a contradiction. □

9. Local conductor fibers: detailed counting arguments

The scalar-conductor problem differs from merely computing the conductor of a single order: we must enumerate all exponent triples producing the same *ideal*, rather than merely the same index. This section records the necessary restrictions and their exact counts.

9.1 Unramified odd primes

Fix an odd prime $p \nmid 42$, and put $a = p^r$, $b = p^s$ and $c = p^u$. The three closure conditions are equivalent to

$$u \leq r + s, \qquad r \leq s + u, \qquad s \leq r + u.$$

By Theorem 4, the conductor localized at p is $p^M(\mathcal{O}_K)_p$, where $M = \max(r, s, u)$. Thus the local fiber of $p^k(\mathcal{O}_K)_p$ is the set of nonnegative integer triangles with largest side k. To count these triangles, let E_r denote the subset with $r = k$. Then $0 \leq s, u \leq k$, and the only nonautomatic inequality is $s + u \geq k$. There are $(k + 1)^2 - k(k + 1)/2 = (k + 1)(k + 2)/2$ solutions. The same count applies to E_s and E_u. Each pairwise intersection has $k + 1$ elements, and the triple intersection is the single point (k, k, k). Inclusion–exclusion gives

$$|E_r \cup E_s \cup E_u| = 3\frac{(k + 1)(k + 2)}{2} - 3(k + 1) + 1 = 1 + \frac{3k(k + 1)}{2}.$$

This argument also works at $k = 0$, where the only triple is $(0,0,0)$.

9.2 The exceptional prime three

At $p = 3$, the closure conditions become

$$u \le r + s, \qquad r \le s + u, \qquad s \le r + u + 1.$$

Theorem 4 yields the local conductor exponents

$$(e_0, e_x, e_y, e_z) = (M, \ell, M, \ell), \qquad M = \max(r, s, u), \quad \ell = \max(r, u, s - 1).$$

For $k \ge 1$, equality with the scalar ideal $3^k(\mathcal{O}_K)_3$ is equivalent to $M = \ell = k$. To see this without suppressing a boundary case, first $M = k$ forces $r, s, u \le k$. Since then $s - 1 < k$, the equality $\ell = k$ forces $r = k$ or $u = k$. Conversely, if $\max(r, u) = k$ and $s \le k$, then $M = \ell = k$.

If $r = k$, the inequalities $r \le s + u$ and $u \le r + s$ reduce to $s + u \ge k$ and an automatic inequality, while $s \le r + u + 1$ is automatic. Hence there are $(k + 1)(k + 2)/2$ possibilities. The same count holds for $u = k$. The intersection $r = u = k$ has $k + 1$ possibilities for s, so

$$\#\{(r, s, u) : \text{conductor } 3^k(\mathcal{O}_K)_3\} = 2\frac{(k + 1)(k + 2)}{2} - (k + 1) = (k + 1)^2.$$

For $k = 0$, the scalar conductor is the unit ideal and only the zero exponent triple occurs.

9.3 The exceptional prime seven

At $p = 7$ the inequalities are

$$u \le r + s, \qquad r \le s + u + 1, \qquad s \le r + u.$$

The conductor exponents are (M, M, ℓ, ℓ) with $M = \max(r, s, u)$ and $\ell = \max(s, u, r - 1)$. For $k \ge 1$, the scalar condition is therefore $\max(s, u) = k$ with $r \le k$. Fixing $s = k$ or $u = k$ gives $(k + 1)(k + 2)/2$ choices each, with $k + 1$ choices in the overlap. Thus the count is again $(k + 1)^2$. The exceptional formulas are symmetric as counts, even though the relevant radical coordinates and closure inequalities are different.

9.4 The prime two

Theorem 7 gives the local conductor $2^{M+1}(\mathcal{O}_K)_2$, with $M = \max(r, s, u)$. The closure conditions are the ordinary triangle inequalities because 3 and 7 are odd. Hence the same counting argument as in the local counting formula gives $A(k)$ with $k = M = v_2(t)$. Notice that the extra factor 2 in the conductor is caused by the index $[\mathcal{O}_K : S] = 4$; it must not be dropped when comparing global conductor ideals.

9.5 Small exponent examples

For $k = 1$, the triangle triples of maximum one are exactly $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$. For $k = 2$, the ten ordinary triangle triples are

$$(2,2,0), (2,0,2), (0,2,2),$$
$$(2,2,1), (2,1,2), (1,2,2),$$
$$(2,2,2), (2,1,1), (1,2,1), (1,1,2).$$

For $k = 2$, the exceptional fibers at 3 and 7 contain nine triples rather than ten; the preceding inequalities specify exactly which local triples occur, without relying on a computer search.

10. Summary of local scalar conductor conditions

For a rational prime p, put $(r, s, u) = \left(v_p(a), v_p(b), v_p(c)\right)$. The following table makes the precise local conditions transparent. In the first row $k \ge 0$ is the valuation of t at 2; at odd primes it is likewise the valuation of t. All triples are nonnegative integers.

Table 1. Local scalar conductor conditions

Prime	Multiplicative closure	Scalar conductor condition
$p = 2$	$r \le s + u,\ s \le r + u,\ u \le r + s$	$\max(r, s, u) = k$; conductor $2^{k+1}\mathcal{O}_{K,2}$
$p = 3$	$u \le r + s,\ r \le s + u,\ s \le r + u + 1$	$\max(r, u) = k,\ 0 \le s \le k$; conductor $3^k\mathcal{O}_{K,3}$

$p = 7$	$u \le r + s,\ r \le s + u + 1,\ s \le r + u$	$\max(s,u) = k,\ 0 \le r \le k$; conductor $7^k\mathcal{O}_{K,7}$
$p \nmid 42$	$r \le s + u,\ s \le r + u,\ u \le r + s$	$\max(r,s,u) = k$; conductor $p^k\mathcal{O}_{K,p}$

For $k = 0$ the only admissible triple in each row is $(0,0,0)$. The table is a summary of the proved local conductor formulas, not an independent assumption. It also shows why the 3- and 7-fibers have identical cardinalities despite their different coordinate constraints.

11. Local-to-global principle

Lemma 8. *Let $R \subseteq \mathcal{O}_K$ be an order and $I = (R:\mathcal{O}_K)$. For every rational prime p, localization gives $I \otimes_{\mathbb{Z}} \mathbb{Z}_p = \left(R \otimes \mathbb{Z}_p : \mathcal{O}_K \otimes \mathbb{Z}_p\right)$. Moreover, two finite-index ideals of $\mathcal{O}_K$ are equal if and only if their localizations at every rational prime are equal.*

Proof. Choose a finite integral basis $e_1, \dots, e_4$ of $\mathcal{O}_K$. Then I is the intersection of the four additive sublattices $\{\alpha \in \mathcal{O}_K : \alpha e_i \in R\}$. Localization is exact and commutes with finite intersections of submodules, so it commutes with the conductor construction. If two full-rank $\mathbb{Z}$-lattices have the same localization at every prime, their quotient lattices have no nonzero finite primary component; hence they are equal. Apply this to the two conductor ideals. □

12. Main theorem: complete conductor classification and fiber enumeration

We now state the central result of the paper. It classifies all conductor ideals realized within the diagonal family, not only the scalar ones. The functions A, B_0 and B_1 were defined in the Introduction.

Theorem 9 (Complete conductor classification and fiber enumeration). *The conductor of every diagonal order can be written uniquely as $\mathfrak{f}(R) = 2t\left(\sqrt{3}\right)^{-\varepsilon_3}\left(\sqrt{7}\right)^{-\varepsilon_7}\mathcal{O}_K$, $\varepsilon_3, \varepsilon_7 \in \{0,1\}$, where t is a positive integer, $\varepsilon_3 = 1$ implies $3 \mid t$, and $\varepsilon_7 = 1$ implies $7 \mid t$. These expressions denote integral ideals under the stated divisibility restrictions. Writing $t = 2^\alpha 3^\beta 7^\gamma \prod_{p \nmid 42} p^{k_p}$, the number of diagonal orders with this prescribed conductor is $N(t;\varepsilon_3,\varepsilon_7) = A(\alpha)B_{\varepsilon_3}(\beta)B_{\varepsilon_7}(\gamma)\prod_{p\nmid 42} A\left(k_p\right)$. In particular, the scalar-conductor enumeration is the case $\varepsilon_3 = \varepsilon_7 = 0$.*

Proof. At every odd prime $p \nmid 42$, Theorem 4 gives the scalar ideal $p^M\mathcal{O}_{K,p}$, where $M = \max(r,s,u)$. At $p = 3$ its exponent vector is (M, ℓ, M, ℓ), with $\ell = \max(r,u,s-1)$. Since $\ell \le M$ and $\ell \ge M - 1$, either $\ell = M$, giving $3^M\mathcal{O}_{K,3}$, or $\ell = M - 1$, giving $3^{M-1}\sqrt{3}\,\mathcal{O}_{K,3} = 3^M\left(\sqrt{3}\right)^{-1}\mathcal{O}_{K,3}$. In the latter case necessarily $s = M$ and $r, u \le M - 1$. The closure conditions reduce to $r + u \ge M - 1$, yielding

$$\#\{(r,u): 0 \le r,u \le M-1,\ r+u \ge M-1\} = \frac{M(M+1)}{2}.$$

The remaining admissible triples with scalar conductor number $(M+1)^2$ are counted by the scalar local counting formula for $B_0(k)$. At $p = 7$ the argument is symmetric, with the roles of r and s exchanged and $\sqrt{7}$ in place of $\sqrt{3}$. At $p = 2$, Theorem 7 gives $2^{M+1}\mathcal{O}_{K,2}$, and its fiber has $A(M)$ members. The local conditions assemble independently by Lemma 8 and unique prime factorization of the three parameters. At primes $q \nmid 2t$, all parameter valuations vanish. Uniqueness of $t, \varepsilon_3, \varepsilon_7$ follows from the prime-ideal valuations of these local conductors. □

The nonscalar factors do not assert a classification of all orders of K: they classify precisely the ideals realized by this embedded diagonal family.

13. Scalar conductor fibers as a corollary

For $t \ge 1$ let $N(t)$ be the number of admissible triples (a,b,c) with conductor $2t\mathcal{O}_K$. Use the functions A and B_0 defined in the Introduction.

Corollary 10. *If $t = 2^\alpha 3^\beta 7^\gamma \prod_{p\nmid 42} p^{k_p}$, then $N(t) = A(\alpha)B_0(\beta)B_0(\gamma)\prod_{p\nmid 42} A\left(k_p\right)$. In particular N is multiplicative.*

Proof. This is the specialization $\varepsilon_3 = \varepsilon_7 = 0$ of Theorem 9. □

Corollary 11. *For fixed $t = \prod p^{k_p}$ the possible indices in the fiber $2t\mathcal{O}_K$ are exactly $\left\{4\prod_{p\mid t} p^{j_p} : 2k_p \le j_p \le 3k_p\right\}$. In particular, the minimum and maximum are $4t^2$ and $4t^3$.*

Proof. The sum $r_p + s_p + u_p$ ranges from $2k_p$ to $3k_p$ for each local fiber, and every intermediate value occurs at $\left(k_p, k_p, j\right)$, $0 \le j \le k_p$. Apply Proposition 2 and combine local choices. □

14. Order inclusions

For any two admissible triples,

$$R_{a,b,c} \subseteq R_{a',b',c'} \quad \Leftrightarrow \quad a' \mid a,\ b' \mid b,\ c' \mid c.$$

Moreover, their intersection is the diagonal order indexed by the componentwise least common multiples. The proof follows from uniqueness of coordinates in $(1, x, y, z)$; the intersection is automatically an order. Thus the diagonal-order family is closed under finite intersections. Inside a fixed scalar-conductor fiber, inclusions can be studied entirely by componentwise comparison of the local exponent triples. These structural observations require no claim that the family includes all orders of K.

15. Dirichlet series

Corollary 12. *For* $\Re(s) > 1$, $\sum_{t\geq 1} \frac{N(t)}{t^s} = \frac{\zeta(s)^4}{\zeta(3s)} \prod_{p\in\{3,7\}} \frac{1-p^{-2s}}{1-p^{-3s}}$.

Proof. The generating functions of the prime-power factors are

$$\sum_{k\geq 0} A(k) z^k = \frac{1+z+z^2}{(1-z)^3}, \qquad \sum_{k\geq 0} B_0(k) z^k = \frac{1+z}{(1-z)^3}.$$

Apply Corollary 10 and the Euler products for $\zeta(s)$ and $\zeta(3s)$. □

16. Counting diagonal orders by index

The conductor enumeration is not the same as the index enumeration studied in the general theory of quartic orders. For comparison, the latter has an explicit Euler product within our restricted diagonal family.

Proposition 13. *For* $\Re(s) > 1/2$, $\sum_{R \text{ diagonal}} [\mathcal{O}_K : R]^{-s} = 4^{-s} \frac{\zeta(2s)^3 \zeta(3s)}{\zeta(6s)} \prod_{p\in\{3,7\}} \frac{1+p^{-s}}{1+p^{-3s}}$. *Here the sum runs over distinct embedded diagonal orders, not over isomorphism classes.*

Proof. At a prime $p \nmid 21$, admissibility of the exponent triple (r, u, v) is equivalent to the three triangle inequalities. Set $d_1 = s + u - r$, $d_2 = r + u - s$, and $d_3 = r + s - u$. Then $(r, s, u) \mapsto (d_1, d_2, d_3)$ is a bijection onto nonnegative triples of equal parity, and $d_1 + d_2 + d_3 = r + s + u$. Consequently the local index-generating function is

$$\sum_{\substack{r,u,v\geq 0 \\ \text{triangle inequalities}}} q^{r+u+v} = \frac{1+q^3}{(1-q^2)^3}.$$

For $p = 3$ or 7, precisely one triangle inequality is relaxed by one unit. The additional triples are those for which the corresponding opposite side exceeds the sum of the other two by exactly one. Their contribution is $q/(1-q^2)^2$, since the exceptional triples satisfy $s = r + u + 1$ (or its symmetric analogue), so their total exponent is $2(r + u) + 1$. The ordinary triangles contribute $(1 + q^3)/(1 - q^2)^3$; adding the two contributions gives

$$\frac{1+q}{(1-q^2)^3}.$$

The factor 4^{-s} accounts for $[\mathcal{O}_K : S] = 4$. Substituting $q = p^{-s}$ and using $\prod_p (1 + p^{-3s}) = \zeta(3s)/\zeta(6s)$ gives the stated identity. The Euler products converge absolutely for $\Re(s) > 1/2$, since the local factors differ from 1 by $O\left(p^{-2\Re(s)}\right)$ away from the two exceptional primes. □

Analytic scope.

The pole at $s = 1/2$ indicates a leading growth scale of $X^{1/2}(\log X)^2$ for the cumulative index count. We do not assert numerical lower-order constants or a power-saving remainder without a separate analytic proof.

17. Asymptotic counting

Corollary 14. *As* $X \to \infty$, $\sum_{t\leq X} N(t) \sim \frac{X(\log X)^3}{6\zeta(3)} \prod_{p\in\{3,7\}} \frac{1-p^{-2}}{1-p^{-3}}$.

Proof. By Corollary 12, the Dirichlet series is $\zeta(s)^4 H(s)$, where

$$H(s) = \zeta(3s)^{-1} \prod_{p\in\{3,7\}} \frac{1-p^{-2s}}{1-p^{-3s}}.$$

The Dirichlet coefficients $h(n)$ of H satisfy $\sum_{n\geq 1} |h(n)|\, n^{-\sigma} < \infty$ for every $\sigma > 1/3$: this follows by expanding the reciprocal zeta factor and the two finite Euler corrections. Thus $N = d_4 * h$, where d_4 is the fourfold divisor function. Using the classical summatory estimate for the fourfold divisor function (see, for example, the treatment of divisor sums in Tenenbaum (2015)), $\sum_{n\leq Y} d_4\,(n) = Y(\log Y)^3/6 + O(Y(\log Y)^2)$, write the summatory convolution as

$$\sum_{n\leq X} N\,(n) = \sum_{d\leq X} h\,(d) \sum_{m\leq X/d} d_4\,(m).$$

The leading term from the divisor summatory estimate is

$$\frac{X}{6}\sum_{d\leq X} \frac{h(d)}{d}(\log(X/d))^3.$$

Absolute convergence of $\sum|h(d)|d^{-\sigma}$ for some $\sigma < 1$ implies $\sum|h(d)|\log^j(d)/d < \infty$ for each fixed j, and the truncated sum tends to $H(1)X(\log X)^3/6$ with a lower-order remainder. The convolution of the error term is $O(X(\log X)^2 \sum_{d\geq 1} |h(d)|\,/d)$. Thus the leading coefficient is $H(1)/6$, as claimed. □

18. Further exact examples

For a prime p not dividing 42 and $k = 1$, the four orders of conductor $2p\mathcal{O}_K$ correspond to $(1,p,p)$, $(p,1,p)$, $(p,p,1)$, and (p,p,p). Their indices are $4p^2, 4p^2, 4p^2, 4p^3$. For $p = 3$ or 7 the same four triples occur when $k = 1$, but the counts differ for larger k.

The following examples follow from the multiplicative formula:

t	1	2	9	16	30	60
$N(t)$	1	4	9	31	64	160

These values concern diagonal orders with the prescribed *scalar* conductor. They are scalar counts $N(t) = N(t;0,0)$; nonscalar conductor fibers are classified in Theorem 9.

19. Worked conductor examples and verification

19.1 A nonscalar conductor at the ramified prime 3

Consider $R_{1,3,1} = \mathbb{Z} \oplus \mathbb{Z}x \oplus 3\mathbb{Z}y \oplus \mathbb{Z}z$. It is an order because $1 \mid 3$, $1 \mid 21$, and $3 \mid 3$. At $p = 3$ its valuation triple is $(r,s,u) = (0,1,0)$. The odd-prime conductor formula gives $(e_0, e_x, e_y, e_z) = (1,0,1,0)$, hence

$$\left(R_{1,3,1}\colon \mathcal{O}_K\right)_3 = 3\mathbb{Z}_3 \oplus \mathbb{Z}_3 x \oplus 3\mathbb{Z}_3 y \oplus \mathbb{Z}_3 z.$$

This is not the scalar ideal $3\mathcal{O}_{K,3}$: its x-coefficient can be a unit. This is the unique order in the conductor fiber $(t, \varepsilon_3, \varepsilon_7) = (3,1,0)$ of Theorem 9, with global conductor $2\sqrt{3}\,\mathcal{O}_K$, consistent with $B_1(1) = 1$. It also shows why $\mathrm{lcm}(a,b,c)$ alone loses information at ramified primes.

19.2 A mixed-prime example with full conductor and index range

Take $t = 12 = 2^2 \cdot 3$. The local formulas give

$$N(12) = A(2)B_0(1) = 10 \cdot 4 = 40.$$

The scalar conductor is $24\mathcal{O}_K$. The possible indices are

$$4 \cdot 2^j 3^\ell, \qquad 4 \leq j \leq 6, \quad 2 \leq \ell \leq 3.$$

For instance, $R_{12,12,1}$ has conductor $24\mathcal{O}_K$ and index $4 \cdot 12^2 = 576$, while $R_{12,12,12}$ has the same conductor and index $4 \cdot 12^3 = 6912$. Two intermediate choices are $R_{12,6,2}$ and $R_{12,6,6}$: their local exponent triples are (2,1,1) and (2,1,1) at 2, and (1,1,0) and (1,1,1) at 3, respectively. Both satisfy the scalar-conductor criteria, and their indices are $4 \cdot 144 = 576$ and $4 \cdot 432 = 1728$. (The first shares the minimal index, illustrating nonuniqueness at a fixed index.) Their index ratio is 3. The extrema still have ratio 12. The two orders are distinct, and $R_{12,6,6} \subset R_{12,6,2}$ because $2 \mid 6$.

19.3 Finite verification protocol

The following procedure is a check of any fixed input triple; it is not used as a substitute for the proofs. First test the three divisibilities in the closure divisibilities. Then represent a candidate conductor element by its radical coordinates (A, B, C, D) and multiply it by the four integral basis elements in the integral basis. Membership in $R_{a,b,c}$ is equivalent to integrality of the constant coordinate and divisibility of the x, y, z coordinates by a, b, c, respectively. The resulting congruences are recorded in Section 7. A Hermite normal form calculation of the lattice of solutions yields a $\mathbb{Z}$-basis of the conductor; comparing two such bases gives a purely integral check of equality of conductor ideals. This procedure is especially useful when a, b, c have factors 3 or 7, where the conductor need not be a scalar ideal.

Statements and Declarations

Funding: This research received no external funding. The article processing charge (APC) will be paid personally by Juan Rafael Acosta Portilla.

Conflicts of Interest: The authors declare no conflict of interest.

Author Contributions: Luis Alfredo Dupont García, Juan Rafael Acosta Portilla, and Manuel Alejandro Torres Sotelo each contributed to all activities involved in the research and preparation of this article, including conceptualization, methodology, mathematical investigation, formal analysis, validation, writing of the original draft, and review and editing. All authors have read and approved the final manuscript.

ORCID iDs: Luis Alfredo Dupont García — 0000-0002-9044-3547; Juan Rafael Acosta Portilla — 0000-0002-8063-0752; Manuel Alejandro Torres Sotelo — 0009-0001-5646-7895.

Author emails: ldupont@uv.mx; juaacosta@uv.mx; mats.6528@gmail.com.

References

Alaca, Ş., & Williams, K. S. (2004). *Introductory algebraic number theory*. Cambridge University Press.

Kaplan, N., Marcinek, J., & Takloo-Bighash, R. (2015). Distribution of orders in number fields. *Research in the Mathematical Sciences*, 2, Article 6. https://doi.org/10.1186/s40687-015-0027-8

Kopp, G. S., & Lagarias, J. C. (2025). Ray class groups and ray class fields for orders of number fields. *Essential Number Theory*, 4(1), 1–65. https://doi.org/10.2140/ent.2025.4.1

Krásenský, J., Raška, M., & Sgallová, E. (2022). Pythagoras numbers of orders in biquadratic fields. *Expositiones Mathematicae*, 40(4), 1181–1228. https://doi.org/10.1016/j.exmath.2022.06.002

Lettl, G., & Prabpayak, C. (2014). Conductor ideals of orders in algebraic number fields. *Archiv der Mathematik*, 103, 133–138. https://doi.org/10.1007/s00013-014-0655-z

Lettl, G., & Prabpayak, C. (2016). Orders in cubic number fields. *Journal of Number Theory*, 166, 415–423. https://doi.org/10.1016/j.jnt.2016.02.006

Nakagawa, J. (1996). *Orders of a quartic field* (Memoirs of the American Mathematical Society, Vol. 122, No. 583). American Mathematical Society.

Prabpayak, C. (2017). Characterizations of conductors in quartic number fields with ramified prime. *Applied Mechanics and Materials*, 866, 402–405. https://doi.org/10.4028/www.scientific.net/AMM.866.402

Tenenbaum, G. (2015). *Introduction to analytic and probabilistic number theory* (3rd ed., Graduate Studies in Mathematics, Vol. 163). American Mathematical Society. https://doi.org/10.1090/gsm/163

Williams, K. S. (1970). Integers of biquadratic fields. *Canadian Mathematical Bulletin*, 13(4), 519–526. https://doi.org/10.4153/CMB-1970-094-8