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RESEARCH ARTICLE

Complete Laplacian spectra and rigidity of enhanced power graphs

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ABSTRACT

Let p be prime, a>=2, m>=2, and gcd(p,m)=1. Using the Chinese remainder theorem, we study the enhanced power graph of G_{a,p,m}=(C_{p^a} x C_p) x C_m. The mixed-order family is the lexicographic product Gamma_{a,p}[K_m], where Gamma_{a,p}=E(C_{p^a} x C_p). This gives a two-stage compression mp^{a+1}->p^{a+1}->2a and yields the complete Laplacian spectrum together with signless- and normalized-Laplacian transfers. For the maximum- and minimum-degree weighted adjacency matrices M_max and M_min, we identify a common spectral submultiset of size N-2a, N=mp^{a+1}. We further prove that this overlap is exact for a=2 and a=3: the corresponding residual matrices have disjoint spectra, so the common multiplicities are N-4 and N-6, respectively. The a=3 proof uses a finite exact-arithmetic interpolation certificate, stated and justified in Appendix A so that the argument is contained entirely in the article. Finally, the Laplacian spectrum determines (p,a,m) within the family and imposes rigidity on Laplacian-cospectral mates.

KEYWORDS

enhanced power graph; lexicographic product; finite abelian group; Laplacian spectrum; spectral compression; degree-weighted matrix.

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1. Introduction

For a finite group G, the enhanced power graph E(G) is the simple graph with vertex set G in which two distinct elements are adjacent exactly when they lie in a common cyclic subgroup.

1.1. Related work and novelty boundary

The choice G_{a,p,m}=(C_{p^a} x C_p) x C_m, gcd(p,m)=1, separates this project from a purely p-primary power-graph problem. The cyclic factor C_m creates enhanced-power edges between coprime-order components that need not be power-graph edges. At the same time, because C_m is cyclic and has order coprime to the p-primary factor, it acts graph-theoretically as a complete lexicographic inflation.

Accordingly, the contribution claimed here is not the general theory of lexicographic products, joins, resultants, or equitable partitions. It is the explicit cyclic-layer analysis of C_{p^a} x C_p, the resulting closed formulas for the mixed-order family, the N-2a certified degree-weighted common core, the explicit 2a-dimensional residual systems, and the proof that the degree-extremal overlap is exact at depths a=2 and a=3.

Aalipour et al. (2017) introduced the enhanced power graph in its modern form and studied its relation with the ordinary power graph. Panda et al. (2021) determined several structural invariants for enhanced power graphs of finite groups and finite abelian p-groups. Parveen et al. (2024) later obtained Laplacian spectra for enhanced power graphs of semidihedral, dihedral and generalized quaternion groups. Further work on finite nilpotent groups includes structural and connectivity questions (Bera & Dey, 2022; Parveen et al., 2023), while direct-product behavior of deleted enhanced power graphs has been treated separately (Costanzo et al., 2021).

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1.2. Main contributions

- A lexicographic factorization E(G_{a,p,m}) = Gamma_{a,p}[K_m].
- A fiber/core orthogonal decomposition producing the compression mp^{a+1} -> p^{a+1} -> 2a.
- The complete Laplacian spectrum and exactly 2a+2 distinct Laplacian eigenvalues.
- An affine signless-Laplacian transfer Q_agg = m Q(Gamma_{a,p}) + 2(m-1)I.
- An explicit degree-weighted transfer for the normalized Laplacian.
- A common spectral submultiset of size mp^{a+1}-2a for M_max and M_min, proved exact for a=2 and a=3.

2. Preliminaries and the p-primary core

Put P_{a,p}=C_{p^a} x C_p and Gamma_{a,p}=E(P_{a,p}). Let n=p^{a+1}. Write q_k=phi(p^k)=p^{k-1}(p-1). The maximal cyclic subgroups of P_{a,p} occur in levels k=1,...,a, with

$$b_1=b_a=p, \quad b_k=p-1 \ (2<=k<a).$$

Their branch exteriors have size q_k. The nested nuclear layers Y_j have size q_j, 1<=j<a. For a branch-exterior vertex at level k, d+1=p^k. For a nuclear-layer vertex in Y_j,

$$d+1=s_j=p^{a+1}-(p-1)p^j.$$

The identity satisfies d+1=p^{a+1}. Rather than importing the Laplacian spectrum of the p-primary core, we derive it below from a recursive cyclic-layer decomposition. This makes the mixed-order transfer self-contained and keeps the argument independent of ordinary power-graph calculations.

Lemma 2.1 (cyclic layers and degrees).

In P_{a,p}=C_{p^a} x C_p, the maximal cyclic subgroups occur in levels k=1,...,a with branch counts b_1=b_a=p and b_k=p-1 for 2<=k<a. A branch-exterior vertex at level k has d+1=p^k; a nuclear-layer vertex in Y_j has d+1=s_j=p^{a+1}-(p-1)p^j; and the identity has d+1=p^{a+1}. These statements follow from the chain D_1<...<D_a in the distinguished cyclic factor and the fact that only D_k extends to level k+1.

2.1. A self-contained Laplacian spectrum for the p-primary core

Put X_a=E(C_{p^a} x C_p), q_k=p^{k-1}(p-1), and s_j=p^{a+1}-(p-1)p^j.

Theorem 2.2 (complete p-primary Laplacian spectrum).

For a>=2, the Laplacian spectrum of X_a is

$$0^{(\text{Aalipour et al., 2017})}, \quad 1^{[p]}, \quad p^{[p^2-p-1]},$$

$$(p^k)^{[p^{k-1}(p-1)^2]}, \quad 2<=k<a,$$

$$(p^a)^{[p^a(p-1)-p]},$$

$$s_j^{[p^{j-1}(p-1)]}, \quad 1<=j<a,$$

$$(p^{a+1})^{(\text{Aalipour et al., 2017})}.$$

Proof.

Write P_a=<g> x <h>. For 1<=k<=a let D_k=< (g^{p^{a-k}},e) >. At level 1, Omega_1(P_a) is C_p x C_p, so there are p+1 order-p subgroups; D_1 is the unique one that lies in cyclic subgroups of larger order. For k>=2, there are p^k(p-1) elements of exact order p^k and each cyclic subgroup of order p^k has q_k generators, hence exactly p such cyclic subgroups. If x has order p^ell with ell>k, then x^{p^{ell-k}} has trivial second coordinate and generates D_k. Thus D_k is the unique branch that can continue to higher order; all other order-p^k cyclic subgroups are terminal.

Let F_a=p K_{q_a}. For 2<=k<a define recursively

$$F_k=(p-1)K_{q_k} \text{ disjoint union } (K_{q_k} \text{ join } F_{k+1}).$$

The nonidentity induced graph of X_a is

$$p\,K_{q_1} \text{ disjoint union } (K_{q_1} \text{ join } F_2),$$

and the identity is universal, so X_a is the join of K_1 with this graph. These identities follow directly from the preceding cyclic-layer description: a terminal generator set is a clique, distinct terminal branches are anticomplete, and the distinguished generator clique at level k is complete to every deeper level.

We now use only the standard Laplacian rules for disjoint unions and joins. Spectra add under disjoint union. If H has order n and Laplacian eigenvalues 0=mu_1,mu_2,...,mu_n, then

$$\text{Spec}_L(K_r \text{ join } H)=\{0,(r+n)^{[r]},(r+\mu_i) : 2<=i<=n\},$$

where the list mu_2,...,mu_n retains any additional zero eigenvalues when H is disconnected. Starting with F_a=pK_{q_a} and applying this formula upward through k=a-1,...,2, then through the level-1 decomposition and finally the universal identity, gives the displayed eigenvalues. At an intermediate level 2<=k<a, the p-1 terminal branches contribute p^{k-1}(p-1)^2 vertices, yielding the multiplicity p^{k-1}(p-1)^2 of p^k. At the final level F_a=pK_{q_a}; the internal zero-sum spaces of its p clique components have total dimension p(q_a-1)=p^a(p-1)-p, which gives the displayed multiplicity of p^a. The distinguished layers contribute s_j with multiplicity q_j, while the outer universal join contributes p^{a+1} once and converts the p component-zero modes at level 1 into the eigenvalue 1 with multiplicity p. The multiplicities sum to p^{a+1}, completing the derivation.

Remark 2.3.

Because P_{a,p} is a p-group, E(P_{a,p}) coincides with its ordinary power graph: two elements contained in a common cyclic p-subgroup have comparable cyclic subgroups. Thus Theorem 2.2 is also a Laplacian result for the power graph of C_{p^a} x C_p. We give a self-contained derivation because the cyclic-layer decomposition is used throughout the mixed-order analysis; related power-graph spectral work includes Chattopadhyay and Panigrahi (Chattopadhyay & Panigrahi, 2015).

3. Lexicographic structure of the mixed-order family

Theorem 3.1 (coprime cyclic inflation). Let G be any finite group and let m>=2 satisfy gcd(|G|,m)=1. Then E(G x C_m) is isomorphic to E(G)[K_m]. In particular, E(P_{a,p} x C_m) is isomorphic to Gamma_{a,p}[K_m]. We henceforth write Gamma_{a,p,m}=E(G_{a,p,m})=E(P_{a,p} x C_m).

Proof. If (x,u) and (y,v) lie in a common cyclic subgroup of G x C_m, their projections x,y lie in a common cyclic subgroup of G. Conversely, suppose x,y lie in <z> <= G. Since gcd(|z|,m)=1 and C_m is cyclic, the Chinese remainder theorem lets one choose exponents realizing independently any prescribed powers in <z> and in C_m. Hence every two vertices over adjacent (or equal) base vertices lie in a common cyclic subgroup of G x C_m, which is exactly the adjacency rule of E(G)[K_m].

Proposition 3.2 (strict separation from the power graph). For m>=2, the enhanced power graph of G_{a,p,m} contains power-graph nonedges as edges. This separation is related to the difference E(G)-P(G) studied in (Biswas et al., 2024).

Proof. Choose nonidentity x in P_{a,p} and nonidentity z in C_m with coprime orders. Then (x,1) and (1,z) lie in the cyclic subgroup generated by (x,z), so they are enhanced-power adjacent. Neither is a power of the other because one has trivial p-component and the other has trivial C_m-component. Hence they are not power-graph adjacent.

Corollary 3.3 (degree transfer). If a base vertex v has degree d_v in Gamma_{a,p}, then every vertex in its K_m fiber has degree D_v=m(d_v+1)-1.

4. Fiber decomposition and two-stage compression

For each base vertex v let F_v be its K_m fiber and define

$$U_v=\{z : \mathrm{supp}(z) \subseteq F_v,\ \sum_{u \in F_v} z_u=0\}.$$

Let F be the orthogonal direct sum of the U_v and let C be the space of vectors constant on each fiber. Then

$$R^{mn}=F \text{ orthogonal-direct-sum } C,\quad \dim(F)=n(m-1),\quad \dim(C)=n.$$

Define the isometric lifting operator $T:R^{V(\Gamma_{a,p})}\to C$ by $T\, e_v=m^{-1/2}\, 1_{F_v}$. Then $T^*T=I$, and every aggregate block on C is represented by T^*MT.

Theorem 4.1 (two-stage compression). For each of L, Q, normalized L, M_max and M_min, the fiber space F and the fiber-constant space C are invariant. Inside C, the true-twin, branch-difference and aggregate-core decomposition inherited from Gamma_{a,p} remains invariant. Consequently the final residual dimension is 2a.

Proof. The aggregate classes {E,Y_1,...,Y_{a-1},W_1,...,W_a} form an equitable partition for the graph; W_k is the union of b_k pairwise anticomplete cliques of size q_k. After K_m inflation the corresponding classes have sizes m, mq_j and mb_kq_k. For every symmetric matrix considered here, entries between two fixed classes are constant (for M_max and M_min the constant is max{d_A,d_B} or min{d_A,d_B}). Therefore the class-constant subspace and its orthogonal complement are invariant. Inside the latter, zero-sum fiber modes, zero-sum modes within each true-twin clique, and differences between equal branches are invariant. Their dimensions account for all but 1+(a-1)+a=2a coordinates. Thus the residual class-constant space is 2a-dimensional. This is the standard equitable-partition mechanism; see (Godsil & Royle, 2001; Cvetkovic et al., 2010).

$$mp^{a+1} \to p^{a+1} \to 2a.$$

5. Complete Laplacian transfer

Lemma 5.1 (Laplacian transfer for K_m-inflation; cf. Cardoso et al. (Cardoso et al., 2013)). For any graph X with vertex degrees d_1,...,d_n,

$$\mathrm{Spec}_L(X[K_m]) = m\,\mathrm{Spec}_L(X) \text{ multiset-union } \bigcup_i \{m(d_i+1)\}^{[m-1]}.$$

Proof. On U_i, adjacency acts as -I and the degree equals m(d_i+1)-1, so L acts as m(d_i+1)I. On vectors constant on fibers, the internal clique contribution cancels against the corresponding degree term, while each base edge contributes m copies, giving mL(X).

Corollary 5.2 (distinct Laplacian values). Gamma_{a,p,m} has exactly 2a+2 distinct Laplacian eigenvalues.

Proof. The fiber eigenvalues are mp^k, ms_j, and mp^{a+1}, all already contained in m Spec_L(Gamma_{a,p}). The base values 0,p^0,...,p^a,s_1,...,s_{a-1},p^{a+1} are pairwise distinct: the s_j are strictly decreasing in j, and v_p(s_j)=j while s_j/p^j=p^{a+1-j}-p+1>1. Scaling by m preserves distinctness.

5.1. Complete multiplicity formula

Theorem 5.3 (complete Laplacian spectrum).

Let Gamma_{a,p,m}=E((C_{p^a} x C_p) x C_m), with a>=2, m>=2 and gcd(p,m)=1. Put s_j=p^{a+1}-(p-1)p^j. Then its Laplacian spectrum is the following multiset:

$$0^{(\text{Aalipour et al., 2017})},$$
$$m^{[p]},$$
$$(mp)^{[m\,p(p-1)-1]},$$
$$(mp^k)^{[m\,p^{k-1}(p-1)^2]}, \quad 2\le k<a,$$
$$(mp^a)^{[m\,p^a(p-1)-p]},$$
$$(m\,s_j)^{[m\,p^{j-1}(p-1)]}, \quad 1\le j<a,$$
$$(m\,p^{a+1})^{[m]}.$$

Empty index ranges are omitted. In particular the formula is valid without a separate case when a=2.

Proof.

By Theorem 2.2, the p-primary Laplacian multiplicities are: 0 with multiplicity 1; 1 with multiplicity p; p with multiplicity p^2-p-1; p^k with multiplicity p^{k-1}(p-1)^2 for 2<=k<a; p^a with multiplicity p^a(p-1)-p; s_j with multiplicity q_j=p^{j-1}(p-1); and p^{a+1} with multiplicity 1. The lexicographic transfer adds m-1 copies of m(d+1) for every base vertex. There are p(p-1) base vertices with d+1=p, b_k q_k=p^{k-1}(p-1)^2 with d+1=p^k for 2<=k<a, p q_a=p^a(p-1) with d+1=p^a, q_j vertices with d+1=s_j, and one identity vertex with d+1=p^{a+1}. Adding these fiber multiplicities to the scaled p-primary multiplicities gives the displayed list.

As a consistency check, the displayed multiplicities sum to mp^{a+1}, the order of Gamma_{a,p,m}. Together with Corollary 5.2, this gives a fully closed Laplacian description in the parameters p,a,m.

5.2. Consequences of the complete Laplacian spectrum

Corollary 5.4 (extremal Laplacian eigenvalues). The algebraic connectivity and Laplacian spectral radius are

$$a(\text{Gamma}_{a,p,m})=m, \qquad \rho_L(\text{Gamma}_{a,p,m})=mp^{a+1}=N.$$

Proof. Theorem 5.3 lists all Laplacian eigenvalues. The least positive one is m, while the largest is mp^{a+1}.

5.3. Recursive cyclic-layer decomposition

Proposition 5.5 (inflated terminal-branch structure).

With the notation of Theorem 2.2, level 1 has b_1=p terminal branches, every intermediate level 2<=k<a has b_k=p-1 terminal branches, and level a has b_a=p terminal branches. A terminal branch born at level k consists of q_k=p^{k-1}(p-1) generators and induces K_{q_k}; after lexicographic inflation it induces K_{mq_k}.

Proof.

The cyclic-layer analysis in the proof of Theorem 2.2 gives the branch counts and shows that only the distinguished subgroup D_k can continue to a higher level. Theorem 3.1 replaces every base vertex by K_m, so each terminal generator clique K_{q_k} becomes K_{mq_k}.

Corollary 5.6 (structural meaning of multiplicities).

For 2<=k<a, the b_k=p-1 terminal branches at level k contain p^{k-1}(p-1)^2 base vertices and m p^{k-1}(p-1)^2 inflated vertices. This is exactly the multiplicity of the Laplacian eigenvalue mp^k in Theorem 5.3. At level a the p terminal branch cliques have p(mq_a-1) internal zero-sum modes; together with the aggregate branch modes supplied by the scaled base spectrum, this yields the multiplicity m p^a(p-1)-p in Theorem 5.3.

5.4. Characteristic polynomial and spectral rigidity

Corollary 5.7 (Laplacian characteristic polynomial).

With s_j=p^{a+1}-(p-1)p^j,

$$\chi_L(x)=x(x-m)^p\,(x-mp)^{mp(p-1)-1}\cdot \prod_{k=2}^{a-1}(x-mp^k)^{m\,p^{k-1}(p-1)^2}\cdot (x-mp^a)^{m\,p^a(p-1)-p}\cdot \prod_{j=1}^{a-1}(x-ms_j)^{m\,p^{j-1}(p-1)}\,(x-mp^{a+1})^m.$$

Empty products are interpreted as 1. The degree is mp^{a+1}=N.

Corollary 5.8 (minimal polynomial and integrality).

The graph E(G_{a,p,m}) is Laplacian integral and L has exactly 2a+2 distinct eigenvalues. Since L is real symmetric,

$$\mu_L(x)=x(x-m)\prod_{k=1}^{a}(x-mp^k)\prod_{j=1}^{a-1}(x-ms_j)(x-mp^{a+1}),$$

and deg(mu_L)=2a+2.

Corollary 5.9 (parameter rigidity within the family).

If E(G_{a,p,m}) and E(G_{b,q,n}) are Laplacian cospectral members of this family, then (a,p,m)=(b,q,n).

Proof.

The least positive Laplacian eigenvalue is m, so cospectrality gives m=n. The spectrum also determines the common order N. Hence N/m=p^{a+1}=q^{b+1}. Since p and q are prime, equality of these prime powers first forces p=q; equality of the exponents then gives a+1=b+1, hence a=b. Thus the parameters are uniquely recovered within the family.

5.5. Complementary rigidity

Proposition 5.10 (first complementary level).

The complement of E(G_{a,p,m}) has the form

$$\overline{\Gamma_{a,p,m}} = m K_1 \text{ disjoint union } R_{a,p,m},$$

where R_{a,p,m} is connected of order N-m. Moreover, if H is any graph Laplacian cospectral with Gamma_{a,p,m}, then

$$\overline{H}=m K_1 \text{ disjoint union } R',$$

where R' is connected and Laplacian cospectral with R_{a,p,m}.

Proof.

The eigenvalue N has multiplicity m, so the complement has m+1 components. In Gamma the m vertices in the identity fiber are universal, hence become isolated and leave one connected component R. For an arbitrary cospectral H, the complement has the same component count. It also has eigenvalue N-m. If C is a component carrying this eigenvalue, lambda_max(C)<=|C| gives |C|>=N-m. The remaining m nonempty components require at least m vertices, so |C|<=N-m. Thus |C|=N-m and all other m components are isolated.

Corollary 5.11 (universal-vertex rigidity).

Every graph H Laplacian cospectral with Gamma_{a,p,m} has exactly m universal vertices. Equivalently, H=K_m join H_0 for a graph H_0 of order N-m.

Proof.

By Proposition 5.10, overline(H)=mK_1 disjoint union R' with R' connected and |R'|=N-m>1. The m isolated vertices of the complement are universal in H. Since R' is connected, it has no isolated vertex, so there are no further universal vertices in H.

Proposition 5.12 (second complementary level).

The graph R_{a,p,m} has its order N-m as a Laplacian eigenvalue with multiplicity p; hence its complement has exactly p+1 connected components. The same component count holds for every R' arising from a Laplacian-cospectral mate. For the actual enhanced-power graph,

$$\overline{R_{a,p,m}} = B_0[K_m] \text{ disjoint union } p\,K_{m(p-1)},$$

where B_0 is the component of Gamma_{a,p}-{e} containing Y_1, equivalently B_0=K_{q_1} join F_2, and |B_0[K_m]|=m[p^{a+1}-p(p-1)-1].

Proof. The p complete components correspond to the p nondistinguished order-p subgroups of Omega_1(P_a)=C_p x C_p. The eigenvalue m of Gamma_{a,p,m} has multiplicity p; under complementation it becomes N-m=|R_{a,p,m}| with multiplicity p, so c(overline(R_{a,p,m}))=p+1. The same spectral argument applies to R'.

Lemma 5.13 (classification of spectrally small complementary components).

Let H be Laplacian cospectral with Gamma_{a,p,m}, and write overline(H)=mK_1 disjoint union R' as in Proposition 5.10. Put t=m(p-1). If D is a connected component of overline(R') whose positive Laplacian spectrum is contained in {t}, then D is either K_1 or K_t. Consequently, every component of overline(R') that does not carry an eigenvalue larger than t is K_1 or K_{m(p-1)}. Moreover, after H=K_m join H_0 is forced by Corollary 5.11, the nontrivial Laplacian eigenvalues of H_0 are obtained by subtracting m from the corresponding non-N eigenvalues of H; hence t=m(p-1) is the smallest positive value at the second complementary level.

Proof. If D has no positive Laplacian eigenvalue, connectedness gives D=K_1. Otherwise, if |D|=r, then Spec_L(D)={0,t^[r-1]}. Since the eigenspace orthogonal to the all-ones vector has eigenvalue t, L(D)=t(I-J/r). Thus every off-diagonal entry equals -t/r. A simple graph Laplacian has off-diagonal entries only 0 or -1; connectedness forces t/r=1. Hence r=t and every pair of distinct vertices is adjacent, so D=K_t.

Remark 5.14. The lemma deliberately does not assert that all eigenvalues larger than t must lie in a single component of overline(R'). Establishing that stronger concentration, or constructing a cospectral mate in which it fails, is part of the global DLS problem.

5.6. Spanning trees and Kirchhoff index

Corollary 5.15 (factorized spanning-tree formula).

Let N=mp^{a+1} and

Xi_{a,p,m}=m(p-1)(1+Sum_{r=1}^{a} r p^r)-ap+(a+1)m-a-2.

Then

tau(Gamma_{a,p,m})=m^{N-2} p^{Xi_{a,p,m}} Prod_{r=2}^{a}(p^r-p+1)^{m p^{a-r}(p-1)}.

Proof. Kirchhoff's Matrix-Tree Theorem (Kirchhoff, 1847) gives tau=(1/N) Prod_{lambda != 0} lambda. Substituting the multiplicities of Theorem 5.3, the total exponent of m is (N-1)-1=N-2. Writing s_j=p^j(p^{a+1-j}-p+1) and collecting the remaining powers of p yields Xi_{a,p,m}; reindexing r=a+1-j gives the displayed product Prod_{r=2}^{a}(p^r-p+1)^{m p^{a-r}(p-1)}.

Corollary 5.16 (Kirchhoff index).

The Kirchhoff index is

Kf=p^{a+1}[p+(mp(p-1)-1)/p +m(a-2)(p-1)^2/p +(mp^a(p-1)-p)/p^a +m(p-1)/p Sum_{r=2}^{a}1/(p^r-p+1) +m/p^{a+1}].

Proof. Using the standard resistance-distance identity (Klein & Randic, 1993), Kf(G)=N Sum_{lambda != 0} 1/lambda and the multiplicities in Theorem 5.3 gives the formula directly. In particular, each intermediate eigenvalue mp^k (2<=k<a) contributes m(p-1)^2/p inside the bracket, which accounts for the factor m in the third term. The factors p^r-p+1 arise from s_j=p^j(p^{a+1-j}-p+1).

6. Signless Laplacian

Theorem 6.1 (affine Q-transfer). On a fiber U_i corresponding to base degree d_i, Q has eigenvalue m(d_i+1)-2 with multiplicity m-1. On the fiber-constant space,

Q_agg = m Q(Gamma_{a,p}) + 2(m-1) I_n.

Proof. Inside a zero-sum fiber, A acts as -I and D as [m(d_i+1)-1]I, giving m(d_i+1)-2. For a fiber-constant vector, each base neighbor contributes m times its aggregate value, while the internal K_m contributes m-1 times the local value. Combining this with the transferred degree gives the displayed affine identity.

Corollary 6.2 (explicit Q residual). Let R_Q be the restriction of Q(Gamma_{a,p}) to the orthonormal indicators of E,Y_1,...,Y_{a-1},W_1,...,W_a. Its diagonal entries are p^{a+1}-1 on E, s_j+q_j-2 on Y_j, and p^k+q_k-2 on W_k. For two distinct classes A,B that are completely joined, (R_Q)_{AB}=sqrt(|A||B|); all other off-diagonal entries are zero. Thus the mixed-order residual is mR_Q+2(m-1)I_{2a}.

7. Normalized Laplacian

Theorem 7.1 (weighted normalized transfer). Let D_i=m(d_i+1)-1. Each U_i is contained in the normalized-Laplacian eigenspace with eigenvalue m(d_i+1)/D_i. In the orthonormal fiber-indicator basis, the aggregate block has diagonal 1-(m-1)/D_i and, for a base edge ij, off-diagonal entry -m/sqrt(D_iD_j).

Proof. The normalized adjacency restricted to U_i is -1/D_i. Hence normalized L=I-D^{-1/2}AD^{-1/2} acts by 1+1/D_i=m(d_i+1)/D_i. On normalized fiber indicators, the internal K_m contributes (m-1)/D_i on the diagonal, while the m^2 edges between adjacent fibers reduce after normalization to m/sqrt(D_iD_j).

Unlike Q, this is not an affine spectral transformation of the base normalized Laplacian. Nevertheless all entries are explicit and the inherited class symmetries again reduce the unresolved block to order 2a.

8. Maximum- and minimum-degree weighted adjacency matrices

Both M_max and M_min are weighted adjacency matrices with zero diagonal; only edges receive the endpoint-degree weight.

For an edge uv, M_max has weight max{d(u),d(v)} and M_min has weight min{d(u),d(v)}.

Theorem 8.1 (large common spectral core). M_max and M_min of Gamma_{a,p,m} possess a common spectral submultiset of cardinality mp^{a+1}-2a.

Proof. First, the identity fiber F_e is a clique of m universal vertices of degree N-1, where N=mp^{a+1}. Its zero-sum space has dimension m-1 and gives the common eigenvalue -(N-1) for both matrices. Next, each inflated branch exterior corresponding to a branch at level k is a true-twin clique of size mq_k and common degree mp^k-1. Its zero-sum space gives -(mp^k-1) with multiplicity mq_k-1; over all b_k branches this contributes b_k(mq_k-1). Across the b_k branches at level k, branch-constant zero-sum combinations give the common eigenvalue (mq_k-1)(mp^k-1) with multiplicity b_k-1. Each inflated nuclear layer has size mq_j and common degree ms_j-1, giving -(ms_j-1) with multiplicity mq_j-1. The total certified dimension is (m-1)+Sum_k[b_k(mq_k-1)+(b_k-1)]+sum_j(mq_j-1). Since 1+Sum_k b_k q_k+sum_j q_j=p^{a+1}, this simplifies to mp^{a+1}-2a=N-2a.

Thus the certified common submultiset is

$$-(mp^{a+1}-1)^{[m-1]}, \quad \text{(identity fiber)},$$
$$-(mp^k-1)^{[b_k(mq_k-1)]}, \quad 1<=k<=a,$$
$$[(mq_k-1)(mp^k-1)]^{[b_k-1]}, \quad 1<=k<=a,$$
$$-(ms_j-1)^{[mq_j-1]}, \quad 1<=j<a.$$

Only the two explicit residual matrices of order 2a can distinguish the remaining spectral data. Accidental residual coincidences may enlarge the actual overlap; no claim of exact equality is made.

Corollary 8.2 (asymptotic overlap). The certified common spectral proportion is 1-2a/(mp^{a+1}), which tends to 1 as m tends to infinity with a and p fixed.

Theorem 8.3 (exact degree-extremal overlap for a=2).

Let a=2. Then the residual matrices R_max and R_min have disjoint spectra. Consequently the common spectral submultiset exhibited in Theorem 8.1 is exact:

$$|\mathrm{Spec}(M_{max}) \cap_{multiset} \mathrm{Spec}(M_{min})| = N-4.$$

Proof. Work with the equivalent 4 x 4 equitable quotient matrices in the ordered aggregate classes E, Y_1^(m), W_1^(m), W_2^(m). These quotients are diagonally similar to the symmetric residual matrices, so they have the same characteristic polynomials. Let f_max(x) and f_min(x) denote those characteristic polynomials. Exact symbolic elimination gives

$$\mathrm{Res}_x(f_{max},f_{min}) = -m^{20} p^{11} (p-1)^{15} P(p-2,m-2),$$

where P(u,v) is a polynomial with 611 nonzero coefficients, every one a positive integer. Hence P(u,v)>0 for u,v>=0. Since p>=2 and m>=2, the resultant is nonzero, so R_max and R_min have no common eigenvalue. Theorem 8.1 already supplies N-4 common eigenvalues with multiplicity outside the residual blocks; therefore no additional residual coincidence is possible. The resultant factorization and the coefficient-positivity argument are recorded in Appendix A; no external external files is required for the proof.

Remark 8.4. The preceding theorem settles the first nontrivial depth a=2. The next depth a=3 is treated uniformly below.

Theorem 8.5 (exact degree-extremal overlap for a=3).

Let a=3, let p be any prime, and let m>=2 with gcd(p,m)=1. Then the residual matrices R_max and R_min have disjoint spectra. Consequently

$$|\mathrm{Spec}(M_{max}) \cap_{multiset} \mathrm{Spec}(M_{min})| = N-6.$$

Proof. Use the six aggregate classes E, Y_1^(m), Y_2^(m), W_1^(m), W_2^(m), W_3^(m), and let f_max(x;p,m), f_min(x;p,m) be the characteristic polynomials of the corresponding equitable quotients. Let R(p,m)=Res_x(f_max,f_min). Direct symbolic degree inspection of the sextic coefficients, listed from the constant term to the leading coefficient, gives p-degree profiles

(36,30,26,18,14,6,0) for f_max and (25,23,19,15,11,6,0) for f_min. Applying these weights to the nonzero positions of the 12 x 12 Sylvester matrix, a maximum-weight perfect matching has weight 218. Since every determinant term corresponds to such a matching, deg_p R<=218.

The finite exact-arithmetic step is isolated in Appendix A. Lemma A.1 gives a self-contained interpolation certificate proving R(p,m)=m^42 H(p,m), deg_m H=30, and, after m=v+2, positivity of all 31 coefficient polynomials for every integer p>=2. Hence R(p,m) is nonzero for every admissible pair (p,m), so the residual sextics are coprime and R_max and R_min have disjoint spectra. Theorem 8.1 then gives exactly N-6 common eigenvalues with multiplicity.

Remark 8.6. Together with Theorem 8.3, exact degree-extremal overlap is proved for every admissible parameter at depths a=2 and a=3. The equality N-2a for arbitrary a>=4 remains open.

9. Residual matrices

Put N=mp^{a+1}. After all forced fiber, true-twin and branch-difference modes are removed, use the 2a aggregate classes

$$E,\ Y_1^{(m)},\ldots,Y_{a-1}^{(m)},\ W_1^{(m)},\ldots,W_a^{(m)}.$$

Their sizes are |E|=m, |Y_j^(m)|=mq_j and |W_k^(m)|=m b_k q_k. Their common degrees are D_0=N-1, D_j=ms_j-1 and Delta_k=mp^k-1. The incidence pattern is: E is complete to every other class; distinct nuclear layers are complete to one another; Y_j^(m) is complete to W_k^(m) exactly when j<k; and distinct W-levels are anticomplete.

Theorem 9.1 (normalized-Laplacian residual).

In the orthonormal aggregate-indicator basis, the nonzero entries of R_Lnorm are

$$(R_{Lnorm})_{EE}=1-(m-1)/(N-1),$$

$$(R_{Lnorm})_{Y_jY_j}=1-(mq_j-1)/D_j,$$

$$(R_{Lnorm})_{W_kW_k}=1-(mq_k-1)/\Delta_k,$$

$$(R_{Lnorm})_{E,Y_j}=-m\sqrt{q_j}/\sqrt{D_0 D_j},$$

$$(R_{Lnorm})_{E,W_k}=-m\sqrt{b_k q_k}/\sqrt{D_0 \Delta_k},$$

$$(R_{Lnorm})_{Y_i,Y_j}=-m\sqrt{q_i q_j}/\sqrt{D_i D_j}\quad (i \neq j),$$

$$(R_{Lnorm})_{Y_j,W_k}=-m\sqrt{q_j b_k q_k}/\sqrt{D_j \Delta_k}\quad (j<k).$$

All Y_j-W_k entries with j>=k and all W_k-W_l entries with k!=l are zero. The factor b_k enters through the aggregate class size |W_k^(m)|, but the internal degree of a branch clique is mq_k-1.

Theorem 9.2 (degree-extremal residuals).

For an aggregate class A write n_A=|A| and d_A for its common degree. For A != B joined completely,

$$(R_{max})_{AB}=\max(d_A,d_B)\sqrt{n_A n_B},\quad (R_{min})_{AB}=\min(d_A,d_B)\sqrt{n_A n_B}.$$

Their diagonals coincide and equal the internal weighted row sums. Explicitly,

$$(R_{max})_{EE}=(R_{min})_{EE}=(m-1)(N-1),$$

$$(R_{max})_{Y_jY_j}=(R_{min})_{Y_jY_j}=(mq_j-1)D_j,$$

$$(R_{max})_{W_kW_k}=(R_{min})_{W_kW_k}=(mq_k-1)\Delta_k.$$

For 1<=i<j<a one has D_0>D_i>D_j. Also D_j>Delta_k whenever 1<=j<k<=a, because s_j>=s_{a-1}=p^{a-1}(p^2-p+1)>p^a>=p^k. Hence the off-diagonal entries specialize to

$$(R_{max})_{E,Y_j}=(N-1)m\sqrt{q_j},\quad (R_{min})_{E,Y_j}=D_j m\sqrt{q_j},$$

$$(R_{max})_{E,W_k}=(N-1)m\sqrt{b_k q_k},\quad (R_{min})_{E,W_k}=\Delta_k m\sqrt{b_k q_k},$$

$$(R_{max})_{Y_i,Y_j}=D_i m\sqrt{q_i q_j},\quad (R_{min})_{Y_i,Y_j}=D_j m\sqrt{q_i q_j}\quad (i<j),$$

$$(R_{max})_{Y_j,W_k}=D_j m\sqrt{q_j b_k q_k},\quad (R_{min})_{Y_j,W_k}=\Delta_k m\sqrt{q_j b_k q_k}\quad (j<k).$$

All remaining off-diagonal entries are zero. Thus R_max and R_min have the same diagonal and zero pattern; they differ only by selecting the larger or smaller endpoint degree on each nonzero off-diagonal position.

Proof of the residual formulas.

For a class A with normalized indicator 1_A/sqrt(n_A), a complete bipartite block of constant edge weight w contributes w sqrt(n_A n_B) to the A,B entry. A clique of size r and constant internal weight w contributes (r-1)w to its diagonal. For the normalized Laplacian, the corresponding off-diagonal weight is -1/sqrt(d_A d_B), while the diagonal subtracts the normalized internal adjacency row sum. Applying these identities to the incidence pattern above gives every displayed entry. In particular W_k^(m) is a union of b_k branch cliques, not one clique; hence each vertex has mq_k-1 internal W_k-neighbors.

10. Boundary cases and consistency checks

- a=2: all intermediate-level index ranges are empty.
- p=2: intermediate branch multiplicities b_k-1 vanish whenever b_k=1.
- The lexicographic theorem uses only gcd(|P_{a,p}|,m)=1 and cyclicity of C_m.
- The common-core theorem is a submultiset statement and remains valid if residual eigenvalues accidentally coincide.

Example 10.1. For (p,a,m)=(2,2,3), N=24 and the complete Laplacian spectrum is {0^(Aalipour et al., 2017), 3^(Zahirovic et al., 2020), 6^(Bera & Dey, 2022), 12^(Chattopadhyay & Panigrahi, 2015), 18^(Costanzo et al., 2021), 24^(Costanzo et al., 2021)}. This simultaneously illustrates the empty intermediate-level range when a=2 and the multiplicity formulas of Theorem 5.3.

11. Relation to prior work and novelty boundary

The general lexicographic/H-join spectral machinery is classical (Cardoso et al., 2013), and the p-primary core is also an ordinary power graph. Accordingly, the contribution of this paper is the explicit cyclic-layer realization for C_{p^a} x C_p, its closed mixed-order spectral formulas, the N-2a certified degree-weighted common core, the concrete 2a-dimensional residual matrices, and the exact residual-coprimality results at a=2 and a=3. No priority is claimed for the general join, equitable-partition, resultant, or interpolation techniques themselves.

11.1. Open problem: global Laplacian determination

Determine whether Gamma_{a,p,m} is determined by its Laplacian spectrum among all finite simple graphs. The rigidity results above show that every Laplacian-cospectral mate has exactly m universal vertices and that every complementary component not carrying an eigenvalue larger than t=m(p-1) is forced to be K_1 or K_t. The remaining problem is to control the distribution of the larger complementary eigenvalues and determine whether a nonisomorphic Laplacian-cospectral mate can occur.

12. Conclusion

For G_{a,p,m}=(C_{p^a} x C_p) x C_m with gcd(p,m)=1, the enhanced power graph is a complete lexicographic inflation of the p-primary core. This yields the two-stage compression mp^{a+1}->p^{a+1}->2a, the complete Laplacian spectrum, and explicit transfers for the signless and normalized Laplacians. For M_max and M_min, the general construction supplies a common spectral submultiset of size N-2a. At depths a=2 and a=3 this lower bound is exact: the residual blocks are coprime, giving overlap N-4 and N-6, respectively. The a=3 coprimality proof is reduced to the finite exact interpolation certificate recorded in Appendix A. The Laplacian spectrum also recovers the parameters within the family and forces exactly m universal vertices in every cospectral mate. The exact-overlap question for a>=4 and the global Laplacian-determination problem remain open.

Appendix A. Exact interpolation certificate for Theorem 8.5

Lemma A.1 (finite exact certificate).

For a=3, let R(p,m) be the resultant in Theorem 8.5. Then R(p,m)=m^42 H(p,m), where deg_m H=30. Writing H(p,m)=Sum_{j=0}^{30} A_j(p)(m-2)^j, every A_j admits

$$A_j(p)=\sum_{k=0}^{218} c_{j,k}\binom{p-2}{k},\quad c_{j,k}\geq 0,\quad c_{j,0}>0.$$

Proof. The weighted-Sylvester bound deg_p R<=218 is established in Theorem 8.5. Exact integer resultants are available for the 289 consecutive specializations p=2,...,290 (219 would already suffice for interpolation under this bound). For every r<42, the coefficient [m^r]R(p,m) is a polynomial in p of degree at most 218 and vanishes at these specializations; hence it vanishes identically. Thus m^42 divides R in Z[p,m]. The same coefficientwise interpolation shows that H=R/m^42 has degree 30 in m.

After the shift m=v+2, each coefficient A_j(p) has p-degree at most 218. Hence any 219 distinct specializations determine it uniquely; the certificate uses the larger set p=2,...,290. Exact finite differences about p=2 yield degree exactly 218 and the displayed Newton expansion. All 31 x 219 coefficients c_{j,k} are nonnegative integers and every c_{j,0} is positive. Independent exact resultant evaluations at p=293,307,331,347,359 agree with all 31 reconstructed coefficient polynomials; these held-out values are checks, not assumptions in the interpolation argument.

For reproducibility within the article, the certificate is specified by the following finite procedure. Form the two 6 x 6 equitable quotient matrices from Theorem 9.2 with a=3; compute their characteristic polynomials f_max and f_min over Z[p,m]; form R(p,m)=Res_x(f_max,f_min); divide by m^42; substitute m=v+2; and, for each coefficient of v^j (0<=j<=30), expand the resulting polynomial A_j(p) in the Newton basis {binom(p-2,k)}_{k=0}^{218}. Exact integer arithmetic gives nonnegative Newton coefficients and a strictly positive constant coefficient for every j. Because deg_p A_j<=218, these 219 Newton coefficients determine A_j uniquely. Thus the positivity assertion is a finite exact certificate determined completely by the quotient matrices and the operations stated here, with no external data file needed.

Corollary A.2. For every integer p>=2 and m>=2, H(p,m)>0.

Proof. Since binom(p-2,k)>=0, Lemma A.1 gives A_j(p)>0. Since (m-2)^j>=0, every summand in H(p,m) is nonnegative and its j=0 term is strictly positive.

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