Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Double Elzaki Transform Technique for Linear Fractional Partial Differential Equations

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| ABSTRACT
This study presents a novel approach for obtaining analytical solutions to certain classes of linear partial differential equations (PDEs) of fractional order regarding Mittag-Leffler function according to initial and boundary conditions by employing a newly developed double integral transformation technique named Double Elzaki Transformation (DET). Fractional-order PDEs have gained significant attention due to their ability to accurately describe various complex phenomena in physics, engineering, and other applied sciences. The suggested transformation method broadens traditional integral transform techniques by introducing a double integral framework, improving its ability to handle fractional derivatives efficiently and systematically. The transformation converts the original fractional PDEs into algebraic equations within the transform domain, which are subsequently solved and inverted to yield series-form or exact solutions in the original domain. Several illustrative examples are provided to demonstrate the effectiveness and applicability of the method. The results indicate that the new DET is a powerful and reliable tool for solving linear fractional partial differential equations (LFPDEs), providing more accuracy and simplicity compared to existing methods.

| KEYWORDS
Double Elzaki Transformation, Inverse Double Elzaki Transformation, Fractional Partial Differential Equations, Caputo-Derivative.

| ARTICLE INFORMATION

ACCEPTED: 08 September 2026 **PUBLISHED:** 07 October 2026 **DOI:** 10.32996/jmss.2026.7.6.8

1.Introduction

Fractional calculus is the extension of classical differentiation and integration to any arbitrary non-integer orders. Numerous Fractional partial differential equations in mathematical physics have significant real-world applications in the field of science, engineering, control theory, anomalous diffusion, biological and physical systems. However, solving such equations analytically remains a challenging task due to the non-local nature of fractional operators. Integral transforms such as Laplace, Fourier, Shehu [27], Sawi [21-22], Upadhyaya integral transforms [23-25], have proven effective in analyzing and solving system of ordinary and partial differential equations, various types of integral equations and linear fractional partial differential equations (LFPDEs). Last few years, several researchers have been utilized and established various analytical and numerical techniques to gain the solution of LFPDEs such as Double Laplace Transform [7-9], Elzaki transform [19,28], Elzaki Projected Differential Transform Method [15], Homotopy Analysis Method (HAM) [6], Adomian's Decomposition Method (ADM) [10], Sumudu Transform Method [16-17,20], Double Sumudu-Elzaki Transform Method [13], Double Laplace-Sumudu Transform Method [11].

In [31], a significant contribution in this field was made by Rashid et al. (2021), who proposed a novel analytical approach for solving time-fractional Korteweg-De Vries (KdV) equations using a new integral transform.This study advances the analytical treatment of fractional nonlinear partial differential equations by leveraging symmetry properties and offering computationally efficient solutions.

In 2019, Chand et al. [30] made significant strides by investigating fractional integrals and exact solutions of fractional kinetic equations involving products of S-functions. Their work provides analytical tools to address complex dynamical systems with non-local operators.In [29], a notable contribution by Chu et al. (2023) addressed the multi-dimensional, time-fractional Navier–

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Stokes equations using a novel integral transformation. This study combines analytical rigor with computational efficiency, providing exact solutions and numerical schemes for fractional fluid flow problems.

The study of fractional partial differential equations plays a vital role in modeling numerous physical and engineering processes, including diffusion, wave propagation, and viscoelastic behavior, where memory and hereditary effects cannot be captured by classical integer-order models. Although several analytical and numerical techniques have been used to solve LFPDEs, these methods often require complicated inversion procedures and become less effective for multi-dimensional fractional problems due to the nonlocal nature of fractional derivatives. Motivated by these limitations, the present study introduces a new double integral transform, called the Double Elzaki Transform (DET), as a direct, efficient, and systematic analytical tool for solving linear LFPDEs in both space and time, particularly when initial and boundary conditions are provided. Notably, DET [1-4] is a newly developed and novel study, whereas previous research focused on definitions, elementary theories and solution of partial differential equations (PDEs) and integral equations. For instances, in paper [34], Elzaki and Hilal (2012) proposed a modified Double Sumudu (Elzaki) Transform to obtain exact solutions for the telegraph equation, demonstrating the transform's capability in effectively handling time-dependent physical models. In 2018, Elzaki and Alderremy[33] developed a new double integral transform that effectively addressed singular hyperbolic systems and expanded the scope of transform-based solution methods.In [32], Hassan and Elzaki (2021) developed the Double Elzaki Transform Decomposition Method (DETDM) to solve third-order Korteweg–De Vries equations, illustrating the capability of the Double Elzaki Transform in efficiently addressing nonlinear partial differential equations.

In this study, we elaborated the application of DET method to a board range of LFPDEs where DET kernel is easier to handle fractional operators. The proposed method systematically reduces fractional PDEs into simpler algebraic forms without cumbersome intermediate steps. The main contributions of this work can be summarized as follows. First, the proposed approach yields closed-form analytical solutions expressed in terms of the Mittag–Leffler function, demonstrating the effectiveness and accuracy of the DET. Second, comprehensive numerical and graphical analyses confirm that the obtained fractional-order solutions closely match the corresponding classical (integer-order) solutions, thereby validating the reliability and convergence of the DET method. Finally, the developed DET framework offers a computationally efficient and systematically generalizable technique, which can be readily extended to nonlinear fractional systems and other complex models arising in applied mathematics and physics.

This article is organized as follows: Section two introduces some mathematical preliminaries and important characteristics on Caputo's fractional derivatives. Also, some definitions and important properties with convolution theorem of DEL are covered in the same section. In third section provides a description of the DET method to solve general FLPDEs. Some illustrations including the fractional heat equation, fractional wave equation, fractional Klein-Gordon equation, fractional Fokker-Planck equation, fractional telegraph equation in section four are presented to clarify the efficiency and applicability of the suggested method. To demonstrate the accuracy and convergence of the proposed method, we include numerical tables for several fractional-order values and present a detailed analysis of the corresponding absolute errors. These results support the graphical illustrations and offer a clearer understanding of the numerical behavior of the solutions. In section five, we explain the accuracy of the DET technique and analyze the numerical result. Finally, conclusion and future research directions given in the last section.

2. Preliminaries

2.1. Double Elzaki Transformation

Suppose $\varphi(y,t)$defines a function of the variables y and t where $y,t \in R^+$, then the DET is symbolized by $E_yE_t[\varphi(y,t)]$ and expressed by the integral equation

$$E_yE_t[\varphi(y,t)] = T(u,v) = uv\int_0^\infty\int_0^\infty \varphi(y,t)e^{-(\frac{y}{u}+\frac{t}{v})}dydt,\ y,t>0 \quad (1)$$

Here u,v are transformation variables of $y$and t respectively, whenever the improper integral is convergent.

2.2. Inverse Double Elzaki Transformation

If $T(u,v)$ be the DET of $\varphi(y,t)$, then $\varphi(y,t)$ is called the inverse DET of $T(u,v)$.The inverse DET symbolized by the operator$E_y^{-1}E_t^{-1}(\cdot)$and is defined by the complex integral formula

$$\varphi(y,t) = E_y^{-1}E_t^{-1}[T(u,v)] = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{uv}e^{\frac{y}{u}+\frac{t}{v}}T(u,v)\,dudv \quad (2)$$

2.3. Linearity Property of DET

Let $\varphi(y,t)$and $\psi(y,t)$be two functions of the variables y and t such that $E_yE_t[\varphi(y,t)] = T(u,v)$and $E_yE_t[\psi(y,t)] = M(u,v)$ then $E_yE_t[p\varphi(y,t)+q\psi(y,t)] = pE_yE_t[\varphi(y,t)] + qE_yE_t[\psi(y,t)]$

Or $E_y E_t\,[p\varphi(y,t)+q\psi(y,t)] = pT(u,v)+qM(u,v)$

Where $p$and $q$are two constants.

Proof: By definition, $E_yE_t[p\varphi(y,t)+q\psi(y,t)] = uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}[p\varphi(y,t)+q\psi(y,t)]dydt$

$$= p\left[uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\varphi(y,t)\,dydt\right] + q\left[uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\psi(y,t)\,dydt\right]$$
$$= pE_yE_t[\varphi(y,t)] + qE_yE_t[\psi(y,t)]$$
$$= pT(u,v)+qM(u,v)$$

$\therefore\quad E_yE_t[p\varphi(y,t)+q\psi(y,t)] = pT(u,v)+qM(u,v)$ (3)

Also, the inverse double Elzaki transform $E_y^{-1}E_t^{-1}[T(u,v)]$satisfies the linearity property.

$E_y^{-1}E_t^{-1}[pT(u,v)+qM(u,v)] = pE_y^{-1}E_t^{-1}[T(u,v)] + qE_y^{-1}E_t^{-1}[M(u,v)]$ (4)

2.4. Fundamental Properties of DET

Table 1

Serial. No	$\varphi(y,t)$	$E_yE_t[\varphi(y,t)] = T(u,v)$
1	1	u^2v^2
2	yt	u^3v^3
3	$y^mt^n,\ y>0, t>0$, where $m>-1$ and $n>-1$ are real numbers	$m!\,u^{m+2}n!\,v^{n+2}$
4	$e^{ay+bt},\ \forall y,t$	$\frac{u^2v^2}{(1-au)(1-bv)}$
5	$cos(ay+bt)$	$\frac{(u^2v^2-abu^3v^3)}{(1+a^2u^2)(1+b^2v^2)}$
6	$sin(ay+bt)$	$\frac{(au^3v^2+bu^2v^3)}{(1+a^2u^2)(1+b^2v^2)}$
7	$e^{ay}\,cos\,b\,t$	$\frac{u^2v^2}{(1-au)(1+b^2v^2)}$
8	$e^{ay}\,sin\,b\,t$	$\frac{au^3v^2}{(1-au)(1+b^2v^2)}$
9	$J_0(a\sqrt{yt})$, where $J_0$is the zero order Bessel function.	$\frac{4u^2v^2}{4+a^2uv}$
10	$g(y)h(t)$	$T(u)T(v)$
11	$\varphi(ay,bt)$	$\frac{1}{a^2b^2}T(au,bv)$
12	$e^{cy+dt}(y,t)$	$(1-cu)(1-dv)T\left(\frac{u}{1-cu},\frac{v}{1-dv}\right)$

2.5. DET of Partial Derivatives

If $E_yE_t[\varphi(y,t)] = T(u,v)$, then

$I.\ E_yE_t\left[\frac{\partial\varphi(y,t)}{\partial y}\right]=\frac{1}{u}T(u,v)-uE[\varphi(0,t)]$

$II.\ E_yE_t\left[\frac{\partial\varphi(y,t)}{\partial t}\right]=\frac{1}{v}T(u,v)-vE[\varphi(y,0)]$

$III.\ E_yE_t\left[\frac{\partial^2\varphi(y,t)}{\partial y^2}\right]=\frac{1}{u^2}T(u,v)-E[\varphi(0,t)]-uE[\varphi_y(0,t)]$

$IV.\ E_yE_t\left[\frac{\partial^2\varphi(y,t)}{\partial t^2}\right]=\frac{1}{v^2}T(u,v)-E[\varphi(y,0)-vE[\varphi_t(y,0)]$

$V.\ E_yE_t\left[\frac{\partial^2\varphi(y,t)}{\partial y\partial t}\right]=\frac{1}{uv}T(u,v)-\frac{v}{u}E[\varphi(y,0)]-\frac{u}{v}E[\varphi(0,t)]+uvE[\varphi(0,0)]$

2.6. Convolution and convolution theorem of DET

Let $\varphi(y,t)$and $\psi(y,t)$be two functions of variables y and t. The convolution of two functions $\varphi(y,t)$and$\psi(y,t)$is symbolized by $(\varphi**\psi)(y,t)$ and expressed by the double integral,

$$(\varphi**\psi)(y,t)=\int_0^y\int_0^t\varphi(y-\lambda,t-\eta)\,\psi(\lambda,\eta)d\lambda d\eta \quad (5)$$

Theorem 1: If $E_yE_t[\varphi(y,t)]=T(u,v)$ then $E_yE_t[\varphi(y-\lambda,t-\eta)H(y-\lambda,t-\eta)]=e^{-\left(\frac{\lambda}{u}+\frac{\eta}{v}\right)}T(u,v)$

Here, $H(y,t)$be Heaviside unit step function expressed by

$$H(y-\lambda,t-\eta)=\begin{Bmatrix}1,\ y>\lambda,\ t>\eta\\0,\ otherwise\end{Bmatrix}$$

Proof: By the definition of DET

$$E_yE_t[\varphi(y-\lambda,t-\eta)H(y-\lambda,t-\eta)]=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\varphi(y-\lambda,t-\eta)H(y-\lambda,t-\eta)dydt$$

$$=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\varphi(y-\lambda,t-\eta)dydt$$

Putting $y-\lambda=p, t-\eta=q$, the above integral is reduced to-

$$=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{p+\lambda}{u}+\frac{q+\eta}{v}\right)}\varphi(p,q)dpdq$$

$$=e^{-\frac{\lambda}{u}}e^{-\frac{\eta}{v}}uv\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}+\frac{q}{v}\right)}\varphi(p,q)dpdq$$

$$=e^{-\frac{\lambda}{u}}e^{-\frac{\eta}{v}}T(u,v)$$

Therefore,$E_yE_t[\varphi(y-\lambda,t-\eta)H(y-\lambda,t-\eta)]=e^{-\left(\frac{\lambda}{u}+\frac{\eta}{v}\right)}T(u,v)$ (6)

Theorem 2: Convolution Theorem of DET

Let $\varphi(y,t)$and $\psi(y,t)$be two functions of variables y and t such that $E_yE_t[\varphi(y,t)]=T(u,v)$ and $E_yE_t[\psi(y,t)]=M(u,v)$, then

$$E_yE_t[(\varphi**\psi)(y,t)]=\frac{1}{uv}E_yE_t[\varphi(y,t)]E_yE_t[\psi(y,t)]=\frac{1}{uv}T(u,v)M(u,v)$$

Or equivalently, $E_y^{-1}E_t^{-1}\left[\frac{1}{uv}T(u,v)M(u,v)\right]=(\varphi**\psi)(y,t)$ (7)

Where, $(\varphi**\psi)(y,t)$is expressed by the double integral formula (5).

Proof: Using the definition of DET

$$E_yE_t[(\varphi**\psi)(y,t)]=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}(\varphi**\psi)(y,t)dydt$$

$$=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\left[\int_0^y\int_0^t\varphi(y-\lambda,t-\eta)\,\psi(\lambda,\eta)d\lambda d\eta\right]dydt$$

Using the Heaviside step function,

$$=uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\left[\int_0^\infty\int_0^\infty\varphi(y-\lambda,t-\eta)\,H(y-\lambda,t-\eta)\psi(\lambda,\eta)d\lambda d\eta\right]dydt$$

$$E_yE_t[(\varphi**\psi)(y,t)]=\int_0^\infty\int_0^\infty\psi(\lambda,\eta)\ d\lambda d\eta\ \left[uv\int_0^\infty\int_0^\infty e^{-\left(\frac{y}{u}+\frac{t}{v}\right)}\varphi(y-\lambda,t-\eta)\,H(y-\lambda,t-\eta)dydt\right] \quad (8)$$

Applying theorem 2, the above equation is converted to the new one

$$E_yE_t[(\varphi**\psi)(y,t)]=\int_0^\infty\int_0^\infty\psi(\lambda,\mu)e^{-\left(\frac{\lambda}{u}+\frac{\eta}{v}\right)}T(u,v)d\lambda d\eta=\frac{1}{uv}T(u,v)\left[uv\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda}{u}+\frac{\eta}{v}\right)}\psi(\lambda,\eta)d\lambda d\eta\right]$$

$$=\frac{1}{uv}T(u,v)\left[uv\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda}{u}+\frac{\eta}{v}\right)}\psi(\lambda,\eta)d\lambda d\eta\right]$$

Hence $E_yE_t[(\varphi**\psi)(y,t)]=\frac{1}{uv}T(u,v)M(u,v)$ (9)

2.7. Existence Condition for DET

If $\varphi(y,t)$ be a function of exponential order, then for $a>0, b>0, y\in[0,\infty]$ and $t\in[0,\infty]$, there exists a constant $K>0$such that $\forall\ y>Y$and $t>T$:

$$|\varphi(y,t)| \le Ke^{ay+bt}$$

And we can write $\varphi(y,t) = O(e^{ay+bt})$ as $y \to \infty, t \to \infty$

Or, Equivalently, $\lim_{y\to\infty,y\to\infty} e^{-\alpha y-\beta t}|\varphi(y,t)| == K \lim_{y\to\infty,t\to\infty} e^{-(\alpha-a)y-(\beta-b)t} = 0;\ \alpha > a, \beta > b$

Consequently, if $\varphi(y,t)$ is a continuous function in$(0,Y)$ and$(0,T)$. Also for $a = \frac{1}{c}(c > 0)$ and $b = \frac{1}{d}(d > 0)$ a function is of exponential order $e^{\frac{y}{c}+\frac{t}{d}}$: $|\varphi(y,t)| \le Ke^{\frac{y}{c}+\frac{t}{d}}$, then the DET of $\varphi(y,t)$ exists for all$u$and v provided $Re\left(\frac{1}{u}\right) > \frac{1}{c}$ and$Re\left(\frac{1}{v}\right) > \frac{1}{d}$.

2.8. Caputo Fractional Derivatives

The $\alpha > 0$ order Caputo- time fractional derivative of a function $\varphi(y,t)$ is expressed by

$$\frac{\partial^\alpha \varphi(y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(r-\alpha)}\int_0^t (t-\xi)^{r-\alpha-1}\frac{\partial^r \varphi(y,t)}{\partial \xi^r}d\xi, \quad r-1 < \alpha \le r, \quad r \in N, \tag{10}$$

and the $\delta > 0$order Caputo-space fractional derivative of a function $\varphi(y,t)$ is expressed by

$$\frac{\partial^\delta \varphi(y,t)}{\partial y^\delta} = \frac{1}{\Gamma(s-\delta)}\int_0^y (y-\tau)^{s-\delta-1}\frac{\partial^s \varphi(y,t)}{\partial \tau^s}d\tau, \quad s-1 < \delta \le s, \quad s \in N. \tag{11}$$

Caputo fractional derivatives of DET are

$$E_y E_t\left[\frac{\partial^\alpha \varphi(y,t)}{\partial t^\alpha}\right] = \frac{1}{v^\alpha}T(u,v) - \sum_{j=0}^{m-1} v^{-\alpha+2+j}E\left[\frac{\partial^j \varphi(y,0)}{\partial t^j}\right] \tag{12}$$

$$E_y E_t\left[\frac{\partial^\delta \varphi(y,t)}{\partial y^\delta}\right] = \frac{1}{u^\delta}T(u,v) - \sum_{k=0}^{n-1} u^{-\delta+2+k}E\left[\frac{\partial^k \varphi(0,t)}{\partial y^k}\right] \tag{13}$$

2.9. Mittag-Leffler Function

Mittage-Leffler function $E_\alpha(z)$ in one parameter $\alpha > 0$ be expressed by

$$E_\alpha(z) = \sum_{p=0}^{\infty}\frac{z^p}{\Gamma(\alpha p+1)}, \qquad z, \alpha \in c, \qquad R_e(\alpha) > 0 \tag{14}$$

Mittage-Leffler function$E_{\alpha,\delta}(z)$ in two parameters α & δ be expressed by the formula

$$E_{\alpha,\delta}(z) = \sum_{p=0}^{\infty}\frac{z^p}{\Gamma(\alpha p+\delta)}, \qquad z, \alpha, \beta \in c, \qquad R_e(\alpha) > 0, R_e(\delta) > 0 \tag{15}$$

Taking single Elzaki transformation of the function$t^{\delta-1}E_{\alpha,\delta}(\lambda t^\alpha)$, then we have $E_t[t^{\delta-1}E_{\alpha,\delta}(\lambda t^\alpha)] = \frac{v^{\delta+1}}{1-\lambda v^\alpha}, for|\lambda| < |v^\alpha|$. (16)

2.10. Properties of Mittag-Leffler function:

- $E_0(z) = E_{0,0}(z)\frac{1}{1-z},\ |z| < 1$
- $E_{1,1}(z) = E_1(z) = exp\, z$
- $E_{1,2}(z) = \frac{e^z-1}{z}$
- $E_{2,1}(z^2) = E_2(z^2) = cosh\, z$
- $E_{2,1}(-z^2) = E_2(-z^2) = cos\, z$
- $E_{2,2}(z^2) = \frac{sinh\, z}{z}$
- $E_{2,2}(-z^2) = \frac{sin\, z}{z}$

3. Description of Double Elzaki Transform (DET) Method

A general form of non-homogeneous FLPDE is considered of the following:

$$b\frac{\partial^\alpha \varphi(y,t)}{\partial t^\alpha} + c\frac{\partial^\delta \varphi(y,t)}{\partial y^\delta} + d\ L\varphi(y,t) = g(y,t) \qquad y,t \ge 0 \tag{17}$$

with $r-1 < \alpha \le r, \qquad s-1 < \delta \le s, \quad r,s \in N$ and $g(y,t)$ is a source term.

Under initial conditions (ICs):

$$\frac{\partial^i \varphi(y,t)}{\partial t^i} = f_i(y) \qquad i = 0,1,2,\cdots\ \cdots\ \cdots, r-1 \tag{18}$$

And boundary conditions (BCs):

$\frac{\partial^j \varphi(y,t)}{\partial y^j} = h_j(t) \qquad j = 0,1,2,\cdots \quad \cdots \quad \cdots, s-1$ (19)

In the part, DET method is applied to solve the equation (17). Initially, using DET on both sides of (17),

$$b\left[\frac{1}{v^\alpha}T(u,v) - \sum_{i=0}^{r-1} v^{-\alpha+2+i} E\left[\frac{\partial^i \varphi(y,0)}{\partial t^i}\right]\right] + c\left[\frac{1}{u^\delta}T(u,v) - \sum_{j=0}^{s-1} u^{-\delta+2+j} E\left[\frac{\partial^j \varphi(0,t)}{\partial y^j}\right]\right]$$
$$+ dE_y E_t[L\varphi(y,t)] = G(u,v) \quad (20)$$

Furthermore, applying single Elzaki transform to the ICs and BCs, we get

$E\left[\frac{\partial^i \varphi(y,0)}{\partial t^i}\right] = \frac{\partial^i}{\partial t^i}T(u,0) = F_i(u), i = 0,1,\cdots \quad \cdots \quad \cdots, r-1$ (21)

And $E\left[\frac{\partial^j \varphi(0,t)}{\partial y^j}\right] = \frac{\partial^j}{\partial y^j}T(0,v) = H_j(v), j = 0,1,\cdots \quad \cdots \quad \cdots, s-1$ (22)

By replacing (21) and (22) in (20), the equation becomes

$T(u,v) = \frac{v^\alpha u^\delta}{au^\delta + bv^\alpha}\left[G(u,v) + b\sum_{i=0}^{m-1} v^{-\alpha+2+i} F_i(u) + c\sum_{j=0}^{n-1} u^{-\delta+2+j} H_j(v) - dE_t[L\varphi(y,t)]\right]$ (23)

Taking inverse DET ${E_y}^{-1}{E_t}^{-1}[T(u,v)]$ of (23), we obtain the required solution of (17)

$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{v^\alpha u^\delta}{au^\delta + bv^\alpha}\left[G(u,v) + b\sum_{i=0}^{m-1} v^{-\alpha+2+i} F_i(u) + c\sum_{j=0}^{n-1} u^{-\delta+2+j} H_j(v) - d[L\varphi(y,t)]\right]\right]$ (24)

4. Illustrative Examples

This section contains some different examples of linear fractional differential equations that show the efficiency and effectiveness of the DET method.

4.1. Linear Fractional Heat (Diffusion) Equation

Putting $b = 1,\ c = -p,\ d = 0,\ r = 1,\ s = 2$ and $g(y,t) = 0$ in (17) to get

$\frac{\partial^\alpha \varphi(y,t)}{\partial t^\alpha} = p\frac{\partial^\delta \varphi(y,t)}{\partial y^\delta}, \quad y,t \geq 0,\ 0 < \alpha \leq 1,\ 1 < \delta \leq 2$ (25)

With ICs $\varphi(y,0) = f_0(y)$

And BCs $\varphi(0,t) = h_0(t), \quad \frac{\partial \varphi(0,t)}{\partial y} = \varphi_y(0,t) = h_1(t)$

Therefore, (24) provides the required solution to (25),

$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{v^\alpha u^\delta}{u^\delta - pv^\alpha}\left[v^{-\alpha+2}F_0(u) - pu^{-\delta+2}H_0(v) - pu^{-\delta+3}H_1(v)\right]\right]$ (26)

Example 1: Substituting $\delta = 2$and $p = 1$ in (25), we get

$$\frac{\partial^\alpha \phi(y,t)}{\partial t^\alpha} = \frac{\partial^2 \phi(y,t)}{\partial y^2}, \quad 0 < \alpha \leq 1 \quad (27)$$

With $\varphi(y,0) = f_0(y) = \sin y, \quad \varphi(0,t) = h_0(t) = 0, \quad \varphi_y(0,t) = h_1(y) = E_\alpha(-t^\alpha)$ (28)

Taking single Elzaki transform in (28),

$F_0(u) = \frac{u^3}{1+u^2}, \quad H_0(v) = 0, \quad H_1(v) = \frac{v^2}{1+v^\alpha}$ (29)

Substituting (29) in (26) and simplifying, the solution to the equation (27) is provided by

$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{u^3v^2}{(1+u^2)\ (1+v^\alpha)}\right] = \sin y\, E_\alpha(-t^\alpha)$ (30)

If we take $\alpha = 1$, the closed form solution is

$$\varphi(y,t) = e^{-t}siny \quad (31)$$

Table 1: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $t = 1.5$ for Eq. (27).

y	$\alpha = .75$	$\alpha = .85$	$\alpha = .95$	$\alpha = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .95$
1	0.254185	0.228355	0.201489	0.187758	0.066427	0.040597	0.013732
2	0.274673	0.246761	0.217730	0.202892	0.071782	0.043869	0.014839
3	0.042628	0.038297	0.033791	0.031488	0.011140	0.006808	0.002303
4	−0.228609	−0.205378	−0.181215	−0.168865	0.059743	0.036512	0.012350
5	−0.289664	−0.260229	−0.229613	−0.213965	0.075699	0.046264	0.015648
6	−0.084404	−0.075827	−0.066906	−0.062346	0.022058	0.013481	0.004560
7	0.198457	0.178290	0.157315	0.146594	0.051864	0.031697	0.010721

Figure 1 shows the approximate 2D and 3D solution of equation (30) for various fractional order values.

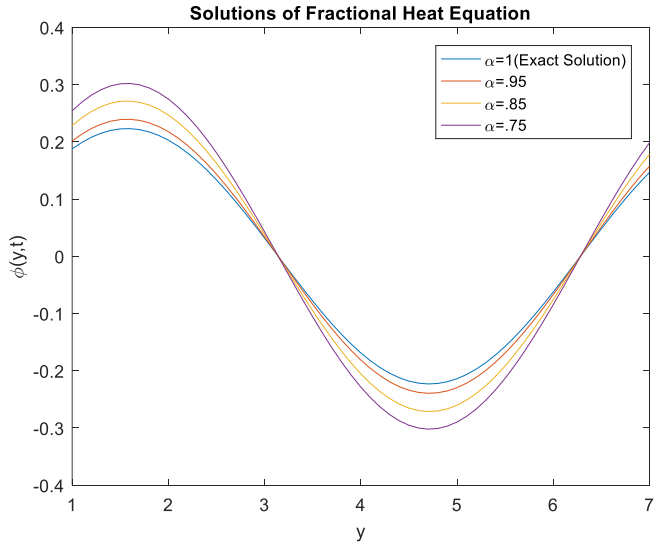


Figure 1- 2D solutions of linear fractional heat (diffusion) equation

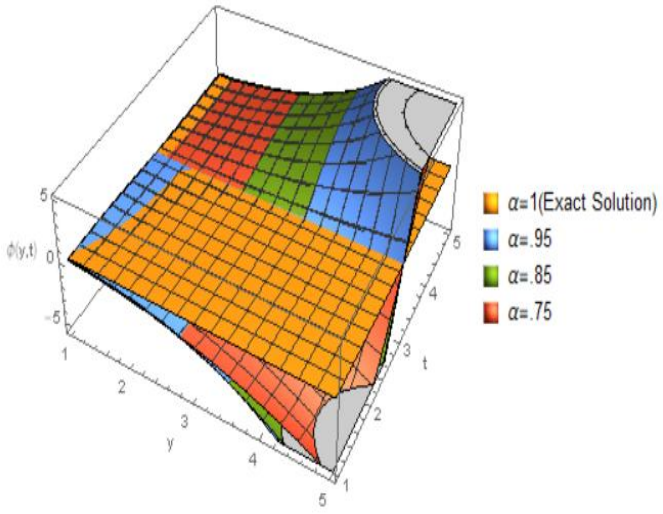


Figure 1- 3D solutions of linear fractional heat (diffusion) equation

4.2. Linear Space-Fractional Wave Equation

Putting $b = 1,\ d = 0,$ and $r = s = 1$in (17) to have inhomogeneous linear wave equation

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} + c\frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} = g(y,t)\ ,\ 0 < \alpha,\delta \leq 1,\ y,t \geq 0 \quad (32)$$

with ICs $\varphi(y,0) = f_0(y)$ and BCs $\varphi(0,t) = h_0(t)$

Thus, the solution to (32) is provided by (24) in the following form

$$\varphi(y,t) = E_y{}^{-1}E_t{}^{-1}\left[\frac{v^{\alpha}u^{\delta}}{u^{\delta}+cv^{\alpha}}\left[G(u,v) + v^{-\alpha+2}F_0(u) + cu^{-\delta+2}H_0(v)\right]\right] \quad (33)$$

Example 2: Taking $\alpha = 1,\ c = 1$ and $g(y,t) = sin\, y + t\, cos\, y$ in (32) to obtain

$$\frac{\partial\varphi(y,t)}{\partial t} + \frac{\partial^{\delta}\varphi(y.t)}{\partial y^{\delta}} = siny + t\ cosy, \qquad 0 < \delta \leq 1 \quad (34)$$

With $\varphi(y,0) = f_0(y) = y^{\delta+1}E_{2,\delta+2}(-y^2) - y^{2\delta}E_{2,2\delta+1}(-y^2), \varphi(0,t) = h_0(t) = 0$ (35)

Using single Elzaki transform in (35),

$$F_0(u) = \frac{u^{\delta+3}}{1+u^2} - \frac{u^{2\delta+2}}{1+u^2}, \quad H_0(v) = 0 \tag{36}$$

Again, taking DET of $g(y,t)$, we obtain $G(u,v) = \frac{u^3v^2}{1+u^2} + \frac{u^2v^3}{1+u^2}$ (37)

Substituting (36) and (37) in (33) and simplifying, we obtain the result of (34)

$$\varphi(y,t) = y^{\delta+1}E_{2,\delta+2}(-y^2) + ty^{\delta}E_{2,\delta+1}(-y^2) - y^{2\delta}E_{2,2\delta+1}(-y^2) \tag{38}$$

If we take $\delta = 1$, then the exact solution is -

$$\varphi(y,t) = t\,sin\,y \tag{39}$$

Table 2: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $y = 1.5$ for Eq. (34).

t	$\delta = .75$	$\delta = .85$	$\delta = .95$	$\delta = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\delta = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\delta = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\delta = .95$
1	0.820062	0.886598	0.960031	0.997495	0.177433	0.110897	0.037464
2	1.687538	1.814868	1.937334	1.994990	0.307452	0.180122	0.057656
3	2.555013	2.743137	2.914638	2.992485	0.437472	0.249348	0.077847
4	3.422489	3.671406	3.891941	3.989980	0.567491	0.318574	0.098039
5	4.289964	4.599675	4.869245	4.987475	0.697511	0.387800	0.118230
6	5.157439	5.527945	5.846548	5.984970	0.827531	0.457025	0.138422
7	6.024915	6.456214	6.823852	6.982465	0.957550	0.526251	0.158613

Figure 2 shows the approximate 2D and 3D solution of equation (38) for various fractional order values.

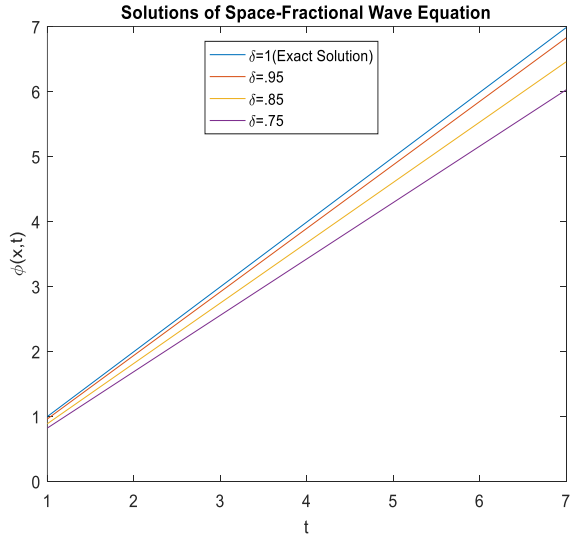


Figure 2- 2D solutions of linear space-fractional wave equation

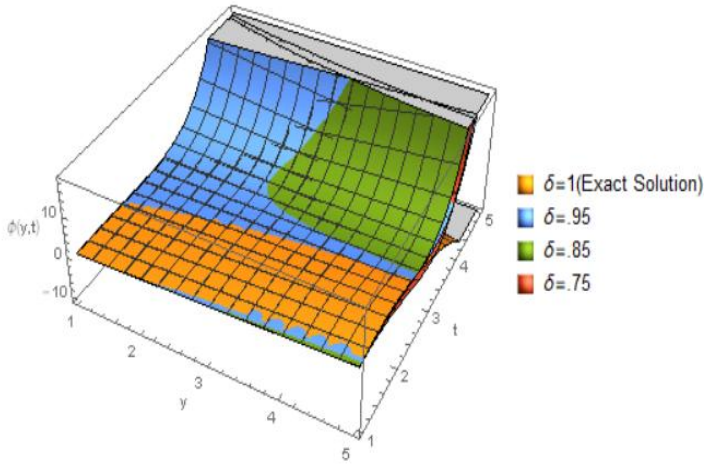


Figure 2- 3D solutions of linear space-fractional wave equation

4.3. Linear Fractional Klein-Gordon Equation

Putting $b = 1,\ c = -1,\ \ r = \ s = 2$and $L\varphi(y,t) = \varphi(y,t)$ in (17) to obtain

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - \frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} + d\varphi(y,t) \quad = g(y,t)\ ,\ 1 < \alpha, \delta \leq 2,\ \ y,t \geq 0 \tag{40}$$

with ICs $\varphi(y,0) = f_0(y),\quad \frac{\partial\varphi(y,0)}{\partial t} = \varphi_t(y,0) = f_1(y)$

and BCs $\varphi(0,t) = h_0(t),\quad \frac{\partial\varphi(0,t)}{\partial y} = \varphi_y(0,t) = h_1(t)$

Therefore, the solution to (40) is given in the following form by the equation (24)

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{v^{\alpha}u^{\delta}}{du^{\delta}v^{\alpha}+u^{\delta}-v^{\alpha}}\left[G(u,v) + v^{-\alpha+2}F_0(u) + v^{-\alpha+3}F_1(u) - u^{-\delta+2}H_0(v) - u^{-\delta+3}H_1(v)\right]\right] \tag{41}$$

Example 3: Taking $\delta = 2,\ d = -1$and $g(y,t) = 0$ in (40), we get

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - \frac{\partial^{2}\varphi(y,t)}{\partial y^{2}} + \varphi(y,t) = 0, \quad 0 < \alpha \leq 1 \tag{42}$$

With $\varphi(y,0) = f_0(y) = \sin y + 1,\quad \varphi_t(y,0) = f_1(y) = 0,$
$\varphi(0,t) = h_0(t) = E_{\alpha}(t^{\alpha}),\quad \varphi_y(0,t) = h_1(t) = 1$ (43)

Taking single Elzaki transform in (43),

$$F_0(u) = \frac{u^3}{1+u^2} + u^2,\quad F_1(u) = 0,\quad H_0(v) = \frac{v^2}{1-v^{\alpha}},\quad H_1(v) = v^2 \tag{44}$$

putting (44) in (41) and simplifying, the result of the equation (42) is obtained by

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{u^3v^2}{(1+u^2)} + \frac{u^2v^2}{1-v^{\alpha}}\right] = \sin y + E_{\alpha}(t^{\alpha}) \tag{45}$$

When $\alpha = 1$, we have the closed form solution

$$\varphi(y,t) = \sin y + e^{t} \tag{46}$$

Table 3: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $t = .9$ for Eq. (42).

y	$\alpha = .75$	$\alpha = .85$	$\alpha = .95$	$\alpha = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .95$
1	3.976684	3.659381	3.408120	3.301074	0.67561	0.358307	0.107046
2	4.044511	3.727207	3.475946	3.368901	0.67561	0.358307	0.107046
3	3.276333	2.959030	2.707769	2.600723	0.67561	0.358307	0.107046
4	2.378411	2.061107	1.809846	1.702801	0.67561	0.358307	0.107046
5	2.176289	1.858985	1.607725	1.500679	0.67561	0.358307	0.107046
6	2.855798	2.538494	2.287233	2.180188	0.67561	0.358307	0.107046
7	3.792199	3.474896	3.223635	3.116589	0.67561	0.358307	0.107046

Figure 3 shows the approximate 2D and 3D solution of equation (45) for various fractional order values.

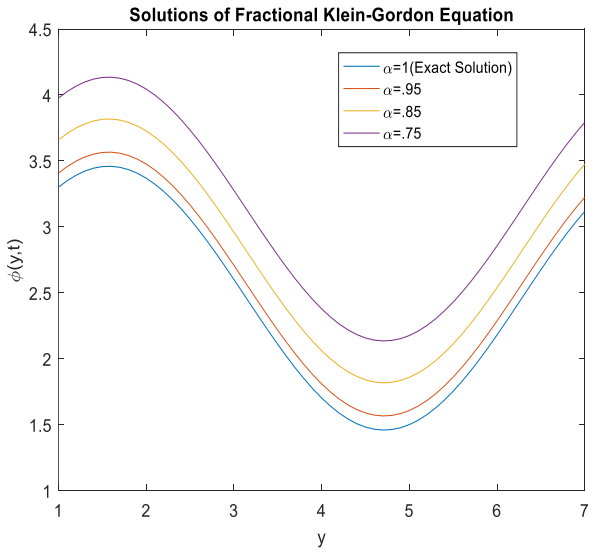


Figure 3- 2D solutions of linear fractional Klein-Gordon equation

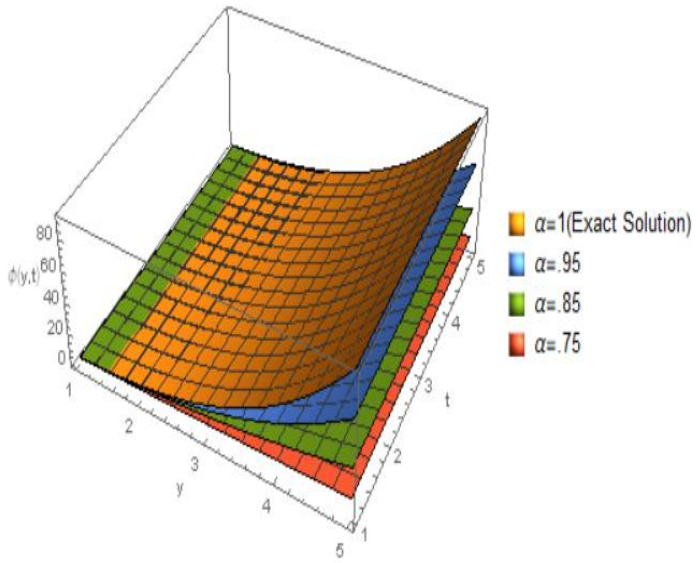


Figure 3- 3D solutions of linear fractional Klein-Gordon equation

4.4. Linear Fractional Fokker-Planck Equation

Taking $b = 1,\ c = -1,\ \ d = -1,\ \ L\varphi(y,t) = \frac{\partial\varphi(y,t)}{\partial y},\ \ r = 1,\ s = 2$ and $g(y,t) = 0$ in (17) to have

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - \frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} - \frac{\partial\varphi(y,t)}{\partial y} = 0,\ \ 0 < \alpha \le 1,\ 1 < \delta \le 2,\ y,t \ge 0 \qquad (47)$$

with ICs $\varphi(y,0) = f_0(y)$ and BCs $\varphi(0,t) = h_0(t),\ \ \frac{\partial\varphi(0,t)}{\partial y} = \varphi_y(0,t) = h_1(t)$

So, (24) provides the required solution to (47)

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{uv^{\alpha}u^{\delta}}{uu^{\delta} - uv^{\alpha} - u^{\delta}v^{\alpha}}\left[v^{-\alpha+2}F_0(u) - u^{-\delta+2}H_0(v) - u^{-\delta+3}H_1(v) - uH_0(v)\right]\right] \qquad (48)$$

Example 4: Taking $\delta = 2$ in (47), we get

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} = \frac{\partial^2\varphi(y,t)}{\partial y^2} + \frac{\partial\varphi(y,t)}{\partial y} = 0,\quad 0 < \alpha \le 1 \qquad (49)$$

with $\varphi(y,0) = f_0(y) = y,\quad \varphi(0,t) = h_0(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad \varphi_y(0,t) = h_1(t) = 1 \qquad (50)$

Taking single Elzaki transform in (50),

$$F_0(u) = u^3,\quad H_0(v) = v^{\alpha+2},\quad H_1(v) = v^2 \qquad (51)$$

Substituting (51) in (48) and rearranging, we obtain

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}[u^3v^2 + u^2v^{\alpha+2}] = y + \frac{t^{\alpha}}{\Gamma(1+\alpha)} \qquad (52)$$

When $\alpha = 1$, we obtain the closed form solution

$$\varphi(y,t) = y + t \qquad (53)$$

Table 4: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $t = 2.5$ for Eq. (49).

y	$\alpha = .75$	$\alpha = .85$	$\alpha = .95$	$\alpha = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .95$
1	3.163266	3.304286	3.437080	3.500000	0.336734	0.195714	0.06292
2	4.163266	4.304286	4.437080	4.500000	0.336734	0.195714	0.06292
3	5.163266	5.304286	5.437080	5.500000	0.336734	0.195714	0.06292

y	$\alpha = .75$	$\alpha = .85$	$\alpha = .95$	$\alpha = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .95$
4	6.163266	6.304286	6.437080	6.500000	0.336734	0.195714	0.06292
5	7.163266	7.304286	7.437080	7.500000	0.336734	0.195714	0.06292
6	8.163266	8.304286	8.437080	8.500000	0.336734	0.195714	0.06292
7	9.163266	9.304286	9.437080	9.500000	0.336734	0.195714	0.06292

Figure 4 shows the approximate 2D and 3D solution of equation (52) for various fractional order values.

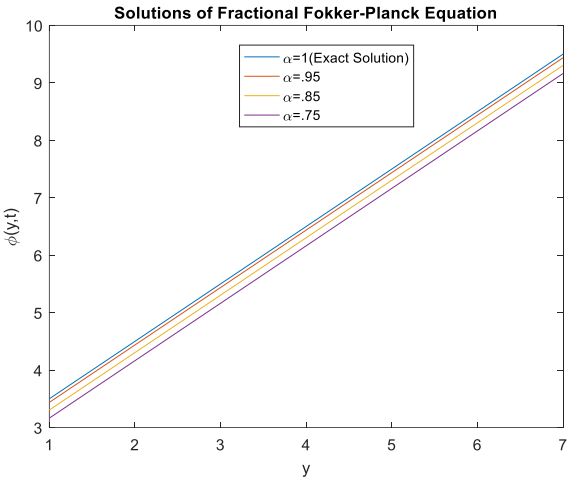


Figure 4- 2D solutions of fractional Fokker-Planck equation

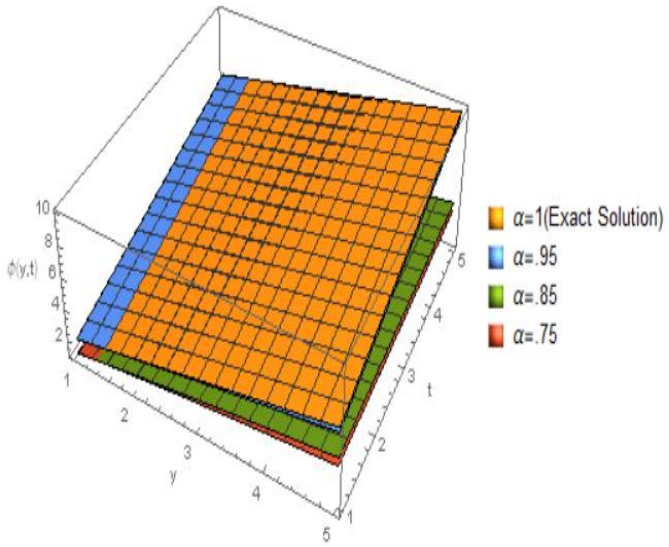


Figure 4 - 3D solutions of fractional Fokker-Planck equation

4.5. Linear Fractional burger's Equation

If we choose, $d = 1, b = 1, c = -1, r = 1,\ s = 2$ and $L\varphi(y,t) = \frac{\partial\varphi(y,t)}{\partial y}$ in (17), then the question becomes

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - \frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} + \frac{\partial\varphi(y,t)}{\partial y} = g(y,t), \quad y,t \geq 0, \quad 0 < \alpha \leq 1,\ 1 < \delta \leq 2 \tag{54}$$

With ICs $\varphi(y,0) = f_0(y)$ and BCs $\varphi(0,t) = h_0(t), \quad \frac{\partial\varphi(0,t)}{\partial y} = \varphi_y(0,t) = h_1(t)$

Therefore, equation (24) provides the required result to (54),

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{uv^{\alpha}u^{\delta}}{uu^{\delta} - uv^{\alpha} + u^{\delta}v^{\alpha}}\left[G(u,v) + v^{-\alpha+2}F_0(u) - u^{-\delta+2}H_0(v) - u^{-\delta+3}H_1(v) + uH_0(v)\right]\right] \tag{55}$$

Example 5: Taking $\delta = 2$ and $g(y,t) = 0$in (54), we obtain

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - \frac{\partial^{2}\varphi(y,t)}{\partial y^{2}} + \frac{\partial\varphi(y,t)}{\partial y} = 0, \quad 0 < \alpha \leq 1 \tag{56}$$

With $\varphi(y,0) = f_0(y) = e^{-y}, \varphi(0,t) = h_0(t) = E_{\alpha}(2t^{\alpha}), \frac{\partial\varphi(0,t)}{\partial y} = \varphi_y(0,t) = h_1(t) = -E_{\alpha}(2t^{\alpha})$ (57)

Taking single Elzaki transform in (57),

$$F_0(u) = \frac{u^2}{1+u}, \quad H_0(v) = \frac{v^2}{1-2v^{\alpha}}, \quad H_1(v) = -\frac{v^2}{1-2v^{\alpha}} \tag{58}$$

Substituting (58) in (55) and rearranging, we get

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{u^2v^2}{(1+u)(1-2v^{\alpha})}\right] = e^{-y}E_{\alpha}(2t^{\alpha}) \quad (59)$$

Which is the required solution of the equation (54)

When $\alpha = 1$, we get the exact solution

$$\varphi(y,t) = e^{2t-y} \quad (60)$$

Table 5: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $t = .6$ for Eq. (56).

y	$\alpha = .75$	$\alpha = .85$	$\alpha = .95$	$\alpha = 1$ (Exact Solution)	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .75$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .85$	$\lvert\varphi_{exact} - \varphi_{DETM}\rvert$ $\alpha = .95$
1	2.181751	1.656499	1.337227	1.221403	0.960348	0.435096	0.115824
2	0.802621	0.609392	0.491938	0.449329	0.353292	0.160063	0.042609
3	0.295268	0.224183	0.180974	0.165299	0.129969	0.058884	0.015675
4	0.108623	0.082472	0.066577	0.060810	0.047813	0.021662	0.005767
5	0.039960	0.030340	0.024492	0.022371	0.017589	0.007969	0.002121
6	0.014701	0.011161	0.009010	0.008229	0.006471	0.002932	0.007804
7	0.005408	0.004106	0.003315	0.003028	0.002380	0.000108	0.000287

Figure 5 shows the approximate 2D and 3D solution of equation (59) for various fractional order values.

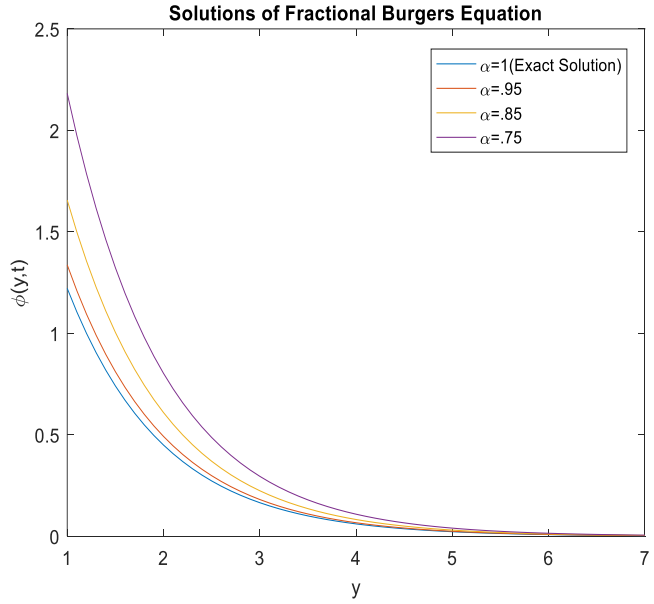


Figure 5- 2D solutions of fractional burger's equation

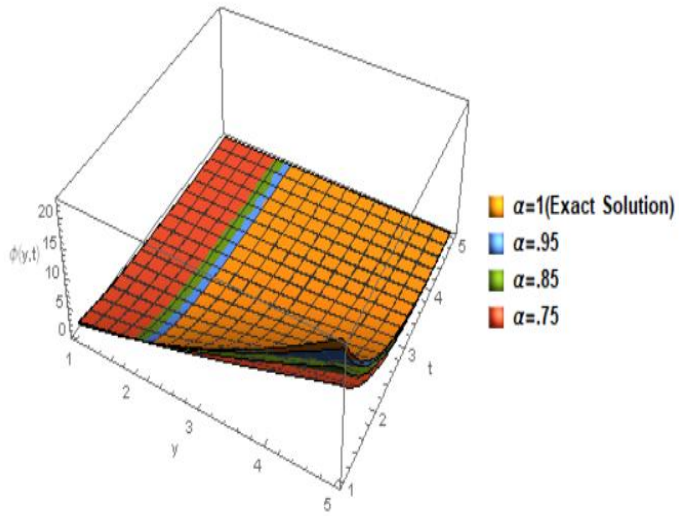


Figure 5- 3D solutions of fractional burger's equation

4.6. Linear Fractional Korteweg de- Vries Equation

Substitute $b = 1,\ \ r = \ s = 1$ and $L\varphi(y,t) = \frac{\partial^3\varphi(y,t)}{\partial y^3}$ in (17) to get,

$$\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} + c\frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} + d\frac{\partial^3\varphi(y,t)}{\partial y^3} = g(y,t),\ 0 < \alpha,\delta \leq 2,\ \ y,t \geq 0 \quad (61)$$

with ICs $\varphi(y,0) = f_0(y)$

and BCs $\varphi(0,t)=h_0(t),\quad \frac{\partial\varphi(0,t)}{\partial y}=\varphi_y(0,t)=h_1(t),\quad \frac{\partial^2\varphi(0,t)}{\partial y^2}=\varphi_{yy}(0,t)=h_2(t)$

Thus, the required solution of equation (61) is found by (24)

$$\varphi(y,t)={E_y}^{-1}{E_t}^{-1}\left[\frac{u^3v^\alpha u^\delta}{u^3u^\delta+cu^3v^\alpha+du^\delta v^\alpha}\left[G(u,v)+v^{-\alpha+2}F_0(u)+cu^{-\delta+2}H_0(v)+\frac{d}{u}H_1(v)+duH_2(v)\right]\right] \quad (62)$$

Example 6: If we choose $\delta=1,\ c=2,\ d=1$, and $g(y,t)=0$ in (61), the equation becomes

$$\frac{\partial^\alpha\varphi(y,t)}{\partial t^\alpha}+\frac{\partial\varphi(y,t)}{\partial y}+\frac{\partial^3\varphi(y,t)}{\partial y^3}=0,\quad 0<\alpha\le 1 \quad (63)$$

With $\varphi(y,0)=f_0(y)=\sin y,\ \ \varphi(0,t)=h_0(t)=-t^\alpha E_{2\alpha,\alpha+1}(-t^{2\alpha}),$
$\varphi_y(0,t)=h_1(t)=E_{2\alpha,1}(-t^{2\alpha}),\ \ \varphi_{yy}(0,t)=h_2(t)=t^\alpha E_{2\alpha,\alpha+1}(-t^{2\alpha})$ (64)

Using single Elzaki transform in (64),

$$F_0(u)=\frac{u^3}{1+u^2},\quad H_0(v)=-\frac{v^{\alpha+2}}{1+v^{2\alpha}},\quad H_1(v)=\frac{v^2}{1+v^{2\alpha}},\quad H_2(v)=\frac{v^{\alpha+2}}{1+v^{2\alpha}} \quad (65)$$

By replacing (65) in (62) and reducing, we obtain

$$\varphi(y,t)={E_y}^{-1}{E_t}^{-1}\left[\frac{u^3v^2}{(1+u)(1+v^{2\alpha})}-\frac{u^2v^{\alpha+2}}{(1+u)(1+v^{2\alpha})}\right]$$
$$=\sin y\,E_{2\alpha,1}(-t^{2\alpha})-\cos y\,t^\alpha E_{2\alpha,\alpha+1}(-t^{2\alpha}) \quad (66)$$

Which is the required solution of equation (63)

When $\alpha=1$,we obtain the exact solution

$\varphi(y,t)=\sin y\cos t-\cos y\sin t=\sin(y-t)$

Table 6: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders $t=.9$ for Eq. (63)

Y	$\alpha=.75$	$\alpha=.85$	$\alpha=.95$	$\alpha=1$ (Exact Solution)	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\alpha=.75$	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\alpha=.85$	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\alpha=.95$
1	0.027598	0.026487	0.072371	0.099833	0.072235	0.073346	0.027462
2	0.730853	0.795330	0.860489	0.891207	0.160354	0.095877	0.030717
3	0.792279	0.832951	0.857477	0.863209	0.070930	0.030258	0.005731
4	0.125288	0.104760	0.066105	0.041581	0.083707	0.063179	0.024524
5	-0.656893	-0.719747	-0.786044	-0.818277	0.161384	0.098530	0.032232
6	-0.835129	-0.882521	-0.915508	-0.925815	0.090685	0.043293	0.010306
7	-0.245552	-0.233910	-0.203258	-0.182163	0.063389	0.051747	0.021095

Figure 6 shows the approximate 2D and 3D solution of equation (66) for various fractional order values.

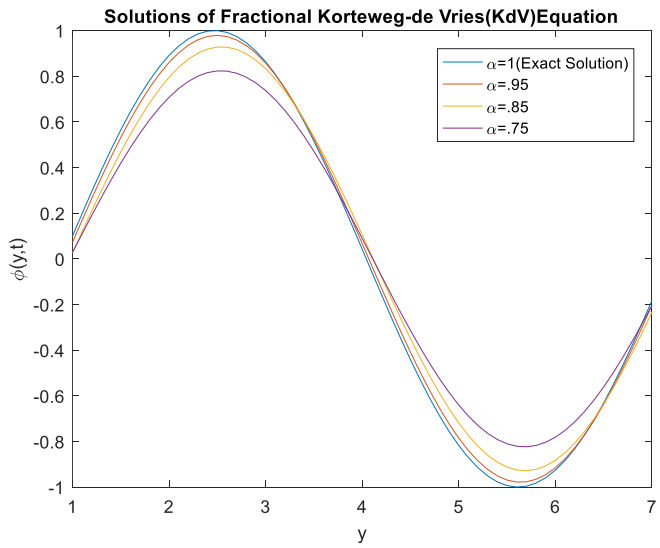


Figure 6- 2D solutions of linear fractional Kdv equation

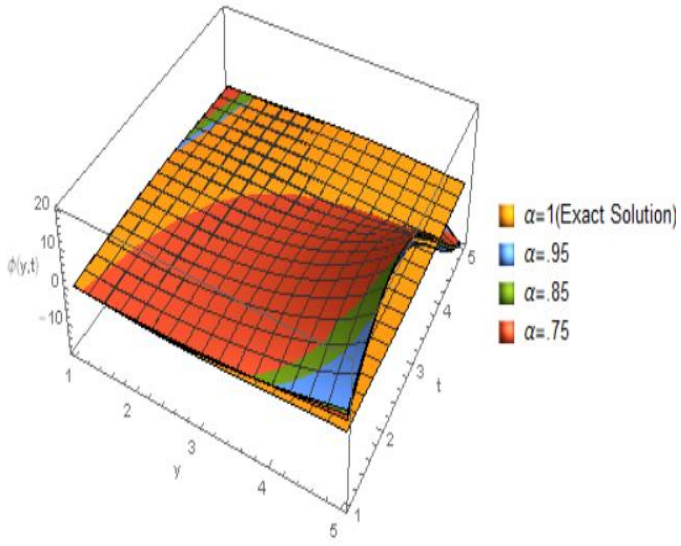


Figure 6- 3D solutions of linear fractional Kdv equation

4.7. Linear Fractional Telegraph Equation

Taking $c = -p,\ d = 1,\ r = \ s = 2,\ \ g(y,t) = 0$ and $L = A_0 + A_1\frac{\partial}{\partial t}$in (17) to get

$$b\frac{\partial^{\alpha}\varphi(y,t)}{\partial t^{\alpha}} - p\frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}} + A_0\varphi(y,t) + A_1\frac{\partial\varphi(y,t)}{\partial t} = 0,\ \ 1 < \alpha,\delta \le 2\ ,\ \ y,t \ge 0 \qquad (67)$$

With ICs $\varphi(y,0) = f_0(y),\ \ \varphi_t(y,0) = f_1(y)$ and BCs$\varphi(0,t) = h_0(t),\ \ \ \varphi_y(0,t) = h_1(t)$

Consequently, (24) provides the solution to (67) in the following form

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{v v^{\alpha}u^{\delta}}{Avu^{\delta} - pvv^{\alpha} + A_1u^{\delta}v^{\alpha} + A_0vu^{\delta}v^{\alpha}}\begin{bmatrix} b\{v^{-\alpha+2}F_0(u) + v^{-\alpha+3}F_1(u)\} \\ -p\{u^{-\delta+2}H_0(v) + u^{-\delta+3}H_1(v)\} + A_1vF_0(u)\end{bmatrix}\right] \qquad (68)$$

Example 7: If we choose $b = A_0 = A_1 = \ p = 1$, and $\alpha = 2$in (67), then the equation becomes

$$\frac{\partial^2\varphi(y,t)}{\partial t^2} + \frac{\partial\varphi(y,t)}{\partial t} + \varphi(y,t) = \frac{\partial^{\delta}\varphi(y,t)}{\partial y^{\delta}},\ \ \ \ 1 < \delta \le 2 \qquad (69)$$

With $\ \varphi(y,0) = f_0(y) = E_{\delta}(y^{\delta}) + yE_{\delta,2}(y^{\delta}),\ \ \varphi_t(y,0) = f_1(y) = -[E_{\delta}(y^{\delta}) + yE_{\delta,2}(y^{\delta})],$
$\varphi(0,t) = h_0(t) = e^{-t},\ \ \varphi_y(0,t) = h_1(t) = e^{-t}$ (70)

Using single Elzaki transform in (70),

$$F_0(u) = \frac{u^2}{1-u^{\delta}} + \frac{u^3}{1-u^{\delta}},\ \ \ F_1(u) = -\frac{u^2}{1-u^{\delta}} - \frac{u^3}{1-u^{\delta}},\ \ \ H_0(v) = \ \ H_1(v) = \frac{v^2}{1+v} \qquad (71)$$

Substituting (71) in (68) and rearranging, we find the required result of the equation (69)

$$\varphi(y,t) = {E_y}^{-1}{E_t}^{-1}\left[\frac{u^2v^2}{(1+v)(1-u^{\delta})} + \frac{u^3v^2}{(1+v)(1+u^{\delta})}\right]$$
$$= e^{-t}\left[E_{\delta}(y^{\delta}) + yE_{\delta,2}(y^{\delta})\right] \qquad (72)$$

When $\delta = 2$, the exact solution is-

$$\varphi(y,t) = e^{-t}[coshy + sinhy] = e^{y-t}$$

Table 7: Approximate and exact solutions of $\varphi(y,t)$ with absolute error for various fractional orders at $y=2$ for Eq. (69).

T	$\delta = 1.75$	$\delta = 1.85$	$\delta = 1.95$	$\delta = 2$ (Exact Solution)	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\delta = 1.75$	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\delta = 1.85$	$\lvert\varphi_{exact}-\varphi_{DETM}\rvert$ $\delta = 1.95$
1	3.061989	2.914015	2.780392	2.718282	0.343707	0.195733	0.062110
2	1.126443	1.072006	1.022849	1.000000	0.126443	0.072006	0.022849
3	0.414395	0.394369	0.376285	0.367879	0.046515	0.026490	0.008406
4	0.152447	0.145080	0.138428	0.135335	0.017112	0.009745	0.003093
5	0.056082	0.053372	0.050925	0.049787	0.006295	0.003585	0.001138
6	0.020632	0.019634	0.018734	0.018316	0.002316	0.001318	0.000418
7	0.007590	0.007223	0.006892	0.0067380	0.000852	0.000485	0.000154

Figure 7 shows the approximate 2D and 3D solution of equation (72) for various fractional order values.

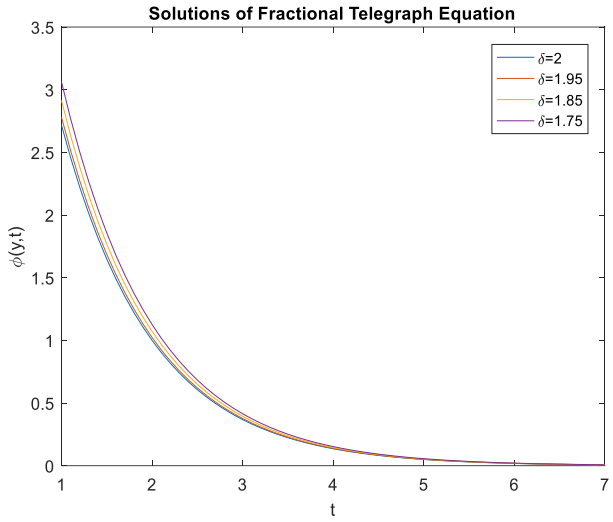


Figure 7- 2D solutions of linear fractional telegraph equation

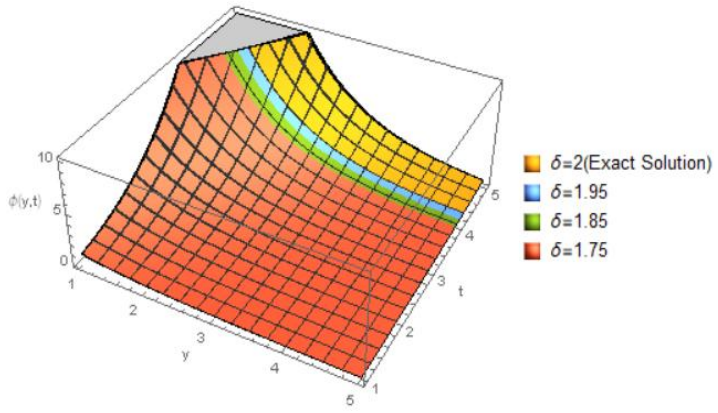


Figure 7- 3D solutions of linear fractional telegraph equation

5. Result and Discussion

This section presents a numerical analysis of the exact solution for the cases (α, δ=1 and δ=2), as well as the approximate solutions obtained for various fractional orders α and δ. Numerical tables and graphical illustrations are provided to demonstrate the accuracy and effectiveness of the proposed DET-based method for solving LFPDEs.

Figures 1–6 show the 2D and 3D plots of the approximate solutions obtained using the present approach, together with a comparison against the corresponding exact solutions for Examples 1–7. For Examples 1–6, the fractional orders $\alpha = \delta = 0.95,0.85,0.75$are considered, while for Example 7, the values $\delta = 1.95,1.85,1.75$are examined to evaluate the numerical performance of the method. The graphical results clearly demonstrate the close agreement between the exact solutions and the DET-based approximations, confirming the accuracy, reliability, and convergence of the proposed approach. Tables 1–7 provide a detailed comparison of exact and approximate solutions across various fractional orders for each example, including an analysis of the resulting errors. The numerical result confirms the rapid convergence behavior of the method's approximation.

Moreover, the obtained results represent wave propagation in media with memory, where the fractional order governs the degree of hereditary effects. The Mittag–Leffler function captures the fractional dynamics, reflecting nonlocal temporal effects, long-tail relaxation and subdiffusive temporal behavior, indicating that the wave persists longer and decays more slowly than in the classical case.

6. Conclusion

In this paper, the method of DET has been successfully utilized to find the approximate solutions of LFPDEs under starting and boundary conditions regarding Mittag-Leffler function. The proposed DEL transform is highly effective for Caputo fractional derivatives, enabling direct solutions to LFPDEs without perturbation, discretization, or repeated inversions and it seamlessly handles mixed space-time fractional derivatives compared to other conventional methods such as double Laplace transform (DLT) or Laplace-Sumudu transform (LST) or double Sumudu transform (DST).

The DET inversion provides an explicit solution in terms of the Mittag–Leffler function without analytical complexities. This key advantage distinguishes the proposed approach from existing methods, offering a straightforward and efficient framework for solving fractional-order problems in mathematical physics. While analytical inversion of DET performs well for simple linear FPDEs, its numerical inversion for more complex functions is less developed than Laplace inversion. DET is mainly effective for linear problems and certain special functions but faces challenges with nonlinear terms unless combined with decomposition techniques such as Adomian polynomials or homotopy perturbation methods.

Overall. DET provides a simpler, highly effective and more powerful method with fewer computational hurdles to find the approximate solutions for a wide class of LFPDEs of mathematical physics. In future work, we aim to extend this analysis to nonlinear LFPDEs by hybridizing the DET with appropriate decomposition techniques. Since many real-world phenomena such as coupled fractional differential systems, fluid dynamics, quantum mechanics, and bioengineering are governed by fractional PDEs, further extending the DET framework to coupled and multi-dimensional models has the potential to unlock new applications across multidisciplinary areas of physics and engineering.

Data Availability

No datasets were utilized to support this study.

Conflicts of interest

The authors don't have any conflict of interest.

Funding

There is no funding for this research work.

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