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| RESEARCH ARTICLE

Exact Connected Domination in a Proper Enhanced Power Graph Component

Luis Alfredo Dupont García[1]✉, Daniel Gualtiero Mendoza Ramírez[2], José Juan Muñoz León[3]
[1] *Universidad Veracruzana, Xalapa, Veracruz, Mexico*
[2] *Universidad Veracruzana, Xalapa, Veracruz, Mexico*
[3] *Universidad Veracruzana, Xalapa, Veracruz, Mexico*
Corresponding Author: Luis Alfredo Dupont García, **E-mail**: ldupont@uv.mx

| ABSTRACT
The k-connected m-dominating set problem combines multiple domination with fault-tolerant connectivity. This paper investigates the problem on a structured graph family arising naturally as a component of a proper enhanced power graph. We first study core-branch graphs H(r;a1,...,as) = K_r join (K_a1 disjoint union ... disjoint union K_as) and obtain an exact feasibility characterization. For m > r and k <= r, the minimum-cardinality problem reduces to the one-dimensional discrete optimization F(z) = z + sum_i min{a_i,m-z}. We then apply the reduction to G = C_p x C_(p^2). The identity is shown to be the unique dominating vertex of its enhanced power graph, and the proper enhanced power graph decomposes as [K_(p-1) join p K_(p(p-1))] disjoint union p K_(p-1). For the distinguished component H_p = K_(p-1) join p K_(p(p-1)), we determine the complete feasibility region and the exact minimum k-connected m-domination number through closed formulas or a one-dimensional minimization. In the only non-closed-form regime, a vertex-level problem on Theta(p^3) vertices is reduced to at most p-k scalar evaluations.

| KEYWORDS
Enhanced power graph; connected domination; multiple domination; finite groups; graph optimization; algebraic graph theory

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1. Introduction

Graphs associated with groups provide a natural interface between algebra and combinatorics. Power graphs, commuting graphs, enhanced power graphs, and related constructions encode algebraic relations as graph adjacency and allow algebraic structure to be investigated through graph-theoretic invariants.

The enhanced power graph of a group G has vertex set G, with two distinct elements adjacent whenever they belong to a common cyclic subgroup. Introduced and systematically developed in the modern literature by Aalipour et al. (2017), the enhanced power graph lies naturally between the power graph and the commuting graph. Subsequent studies have developed its structural theory for several classes of finite groups (Bera & Bhuniya, 2018; Zahirovic et al., 2020; Panda et al., 2021; Ma et al., 2022).

Domination is particularly relevant in this context. Enhanced power graphs may contain dominating vertices, and ordinary domination can therefore conceal much of their internal structure. The proper enhanced power graph, obtained by deleting all dominating vertices, provides a more informative setting for domination-related questions. Connectivity, domination number, and related properties have been investigated by Bera and Dey (2022), while strong domination was subsequently considered by Bera (2024).

A separate graph-optimization literature concerns k-connected m-dominating sets. Such a set combines internal fault tolerance with redundant external domination: selected vertices must induce a k-connected graph, and every unselected vertex must have at least m selected neighbors. The problem has been investigated through approximation and optimization techniques in disk graphs and general graphs (Zhang et al., 2007, 2018; Nutov, 2018, 2020).

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The present paper connects these two research directions. We first study the abstract core-branch family K_r join (K_a1 disjoint union ... disjoint union K_as) and derive an exact feasibility characterization. In the high-domination regime, symmetry inside the branch cliques reduces the original vertex-selection problem to a one-variable integer optimization.

We then consider G = C_p x C_(p^2). Its proper enhanced power graph is disconnected. We identify a distinguished component H_p = K_(p-1) join p K_(p(p-1)) containing all elements of order p^2, together with the common nonidentity elements of their order-p subgroup. Accordingly, the main domination parameter of this paper is evaluated on H_p, not globally on the disconnected proper enhanced power graph.

The main contributions are: (1) an exact feasibility characterization for k-connected m-domination on core-branch graphs; (2) an exact one-dimensional compression theorem; (3) an explicit decomposition of the proper enhanced power graph of C_p x C_(p^2); (4) a complete feasibility classification for H_p; and (5) exact formulas throughout the parameter space, except for a finite one-dimensional minimization in the intermediate regime. To the best of our knowledge, k-connected m-domination has not previously been systematically investigated in this setting.

2. Literature Review

Aalipour et al. (2017) studied the relationship among power graphs, enhanced power graphs, and commuting graphs and established foundational structural properties. Bera and Bhuniya (2018), Zahirovic et al. (2020), and Panda et al. (2021) developed further results for finite groups. Ma et al. (2022) provided a systematic survey collecting results concerning connectivity, domination, metric parameters, perfectness, forbidden subgraphs, and relationships with other graphs associated with groups.

Bera and Dey (2022) studied proper enhanced power graphs of finite nilpotent groups, including dominating vertices, connectivity, diameter, domination number, and spectral properties. Strong domination was subsequently studied by Bera (2024). The present problem differs by imposing both prescribed connectivity of the selected subgraph and multiple domination of exterior vertices.

Zhang et al. (2007) studied approximation algorithms for k-connected m-dominating sets in disk graphs. Subsequent work treated minimum k-connected m-fold domination on general graphs (Zhang et al., 2018). Nutov (2018, 2020) obtained further approximation results. These works motivate identifying structured graph classes for which exact solutions are possible.

The research gap is therefore the connection between algebraically structured proper enhanced power graphs and exact k-connected m-domination. The central observation here is that a proper enhanced power graph component can exhibit a highly symmetric core-branch clique geometry, allowing exact compression of the vertex-level optimization.

3. Methodology and Preliminaries

The methodology is structural. First, connectivity and domination are translated into integer constraints recording the numbers of selected vertices in the core and branches. Second, clique symmetry is exploited to eliminate individual vertex decisions. Third, the cyclic-subgroup structure of C_p x C_(p^2) is analyzed to identify the corresponding core-branch component. Computational enumeration is used only as an independent consistency check and is not used in any proof.

Definition 3.1. For k,m >= 1, a subset D of V(X) is a k-connected m-dominating set, or (k,m)-CDS, if X[D] is k-connected and |N(v) intersect D| >= m for every v outside D. Whenever such a set exists, gamma_(k,m)(X) is its minimum cardinality.

Consider H(r;a1,...,as) = K_r join (K_a1 disjoint union ... disjoint union K_as), s >= 2. Let U denote the core and B_i the i-th branch. For D subset V(H), define z = |D intersect U| and y_i = |D intersect B_i|.

4. Results and Findings

4.1 Core-branch connectivity

Lemma 4.1. If D intersects at least two branches, then H[D] is k-connected if and only if z >= k.

Proof. If z < k, deleting all selected core vertices disconnects the occupied branches. Conversely, if z >= k, deleting fewer than k vertices leaves at least one selected core vertex, which connects all remaining selected branch vertices.

Lemma 4.2. If D intersects at most one branch, then H[D] is a complete graph. Therefore H[D] is k-connected if and only if |D| >= k+1.

4.2 Feasibility

If y_i < a_i, an unselected vertex in B_i has z+y_i selected neighbors, hence z+y_i >= m. If z < r, an unselected core vertex is adjacent to every selected vertex, so z + sum_i y_i >= m.

Theorem 4.3 - Exact Feasibility Characterization. A vector (z,y_1,...,y_s) represents a (k,m)-CDS if and only if 0 <= z <= r, 0 <= y_i <= a_i, the preceding domination inequalities hold whenever applicable, and either z >= k when at least two branches are occupied, or z + sum_i y_i >= k+1 when at most one branch is occupied.

4.3 Exact compression

Suppose m > r and 1 <= k <= r. Every branch must be occupied. For fixed z define F(z) = z + sum_i min{a_i,m-z}.

Theorem 4.4 - Exact Compression Theorem. If m > r and 1 <= k <= r, then gamma_(k,m)(H) = min F(z), where the minimum is taken over k <= z <= r subject to z = r or F(z) >= m.

Proof. Because m > r, an empty branch would contain an exterior vertex with fewer than m selected neighbors. Thus every branch is occupied and z >= k. For fixed z, an incompletely selected branch requires at least m-z selected vertices; if a_i < m-z, the whole branch must be selected. Hence its minimum contribution is min{a_i,m-z}. If z < r, an exterior core vertex requires F(z) >= m. Conversely, selecting min{a_i,m-z} vertices from each branch for any admissible z yields a feasible set. Minimization proves the formula.

4.4 The group C_p x C_(p^2)

Let G be isomorphic to C_p x C_(p^2), generated by elements x and y with |x|=p and |y|=p^2. Put L=<y^p>. Then G[p] is isomorphic to C_p^2.

Lemma 4.5. The identity is the unique dominating vertex of the enhanced power graph of G.

Proof. The identity is adjacent to every element. If g has order p^2, choose a nonidentity element from an order-p subgroup different from the unique order-p subgroup of <g>. If g has order p, choose a nonidentity element from another order-p subgroup. In either case the two elements cannot lie in a common cyclic subgroup. Thus every nonidentity vertex has a nonneighbor. This is consistent with the general characterization of Bera and Dey (2022).

Theorem 4.6 - Component Decomposition. The proper enhanced power graph of C_p x C_(p^2) is isomorphic to H_p disjoint union p K_(p-1), where H_p = K_(p-1) join p K_(p(p-1)).

Proof. The nonidentity elements of L form K_(p-1). The p cyclic subgroups of order p^2 have common intersection L; after removing L each contributes a clique of p(p-1) vertices, with no cross edges. The remaining p order-p subgroups each contribute a K_(p-1). The count (p-1)(p^2+1)+p(p-1)=p^3-1 accounts for every nonidentity element.

Remark 4.1. The proper enhanced power graph is disconnected. Consequently, the connected domination parameter studied below is evaluated on the distinguished component H_p rather than globally. This component contains all elements of order p^2 together with the nonidentity elements of their common order-p subgroup and exhibits the nontrivial core-branch geometry required by Theorem 4.4.

4.5 The remaining components

Proposition 4.7. For p >= 3 and 1 <= k <= p-2, gamma_(k,m)(K_(p-1)) = min{p-1, max{k+1,m}}. Equivalently, if max{k+1,m} <= p-2, an optimal proper subset has that cardinality; otherwise, for k <= p-2, the whole vertex set is optimal and the optimum is p-1, since the external domination condition is vacuous when D=V(K_(p-1)). No (k,m)-CDS exists in K_(p-1) for k >= p-1. For p=2 the additional components are K_1 and no (k,m)-CDS exists under k>=1.

Proof. Any induced subgraph on t selected vertices is K_t and is k-connected exactly when t >= k+1. If D is a proper subset, every exterior vertex is adjacent to all |D| selected vertices, so |D| >= m is also necessary and sufficient for m-domination. Hence the smallest proper solution has size max{k+1,m} whenever this does not exceed p-2. If that threshold is p-1 or larger while k <= p-2, taking D=V(K_(p-1)) gives a k-connected set and leaves no exterior vertex to dominate, so the optimum is p-1. If k >= p-1, K_(p-1) itself is not k-connected, and no feasible set exists.

4.6 Main theorem for H_p

Define A_p=p(p-1), B_p=p^2(p-1), and N_p=(p-1)(p^2+1). For p <= m <= p^2-2 put F_(p,m)(z)=z+p min{A_p,m-z}.

Theorem 4.8 - Complete k-Connected m-Domination Theorem.

If 1 <= k <= p-1, a (k,m)-CDS exists in H_p for every m>=1 and:

gamma_(k,m)(H_p) = max{k+1,m}, for 1 <= m <= p-1;

gamma_(k,m)(H_p) = min F_(p,m)(z), over k <= z <= p-1 with z=p-1 or F_(p,m)(z)>=m, for p <= m <= p^2-2;

gamma_(k,m)(H_p) = B_p + max{k,m-B_p}, for p^2-1 <= m <= N_p-1;

gamma_(k,m)(H_p) = N_p, for m >= N_p.

If p <= k <= p^2-2 and 1 <= m <= p-1, then gamma_(k,m)(H_p)=k+1. No (k,m)-CDS exists in the remaining parameter region.

Proof. For m <= p-1, every feasible set has cardinality at least t=max{k+1,m}: the bound k+1 is forced by k-connectivity, while m is forced whenever an exterior vertex is present. If t<=p-1, choose t vertices from the core K_(p-1); they induce K_t, hence a k-connected graph, and every exterior vertex has at least t>=m selected neighbors. The only remaining boundary case in this regime is t=p, which can occur when k=p-1; then choose all p-1 core vertices together with one branch vertex. The selected subgraph is K_p, and every exterior vertex has at least p-1>=m selected neighbors. Thus gamma_(k,m)(H_p)=max{k+1,m}. For p <= m <= p^2-2, Theorem 4.4 applies with r=p-1, s=p, and a_i=A_p. For a branch vertex v, deg_(H_p)(v)=[p(p-1)-1]+(p-1)=p^2-2. Hence, for p^2-1 <= m <= N_p-1, no branch vertex can remain outside D, so all B_p branch vertices must be selected; the minimum number of core vertices is max{k,m-B_p}. For m >= N_p, only the whole vertex set can work, making the external

domination condition vacuous. If k >= p, a feasible set cannot occupy two branches: removing the p-1 core vertices separates any two occupied distinct branches, so such a set cannot be k-connected. Thus it must lie in the core and one branch, forcing k <= p^2-2 and m <= p-1. Under these conditions a K_(k+1) construction attains the lower bound.

4.7 Feasibility map

The feasible region for H_p is exactly {1,...,p-1} x N together with {p,...,p^2-2} x {1,...,p-1}. The only feasible regime not represented immediately by a closed formula is 1 <= k <= p-1 and p <= m <= p^2-2; even there only p-k values of z need be examined.

4.8 Consistency checks

For p=2, H_2 = K_1 join (K_2 disjoint union K_2), and gamma_(1,1)=2, gamma_(1,2)=3, and gamma_(1,m)=5 for m>=3. For p=3, H_3=K_2 join 3K_6 with N_3=20; for k=1, gamma_(1,m)=19 for 8<=m<=19 and gamma_(1,m)=20 for m>=20.

The computations used here are independent consistency checks and are not part of the proof of Theorem 4.8. For p=2 and p=3, the checks were performed by exhaustive enumeration of the symmetry-reduced selection profiles (z,y_1,...,y_s), followed by direct verification of the induced-subgraph connectivity and the external m-domination condition. For the larger instances p=5 and p=7, compressed profile-based evaluations were used to test the values predicted by the regime formulas and the one-dimensional reduction, rather than an exhaustive enumeration of all vertex subsets. No discrepancies were found in these checks. The analytical arguments above, rather than the computations, establish Theorem 4.8.

5. Discussion

The central contribution is the exact compression of the (k,m)-CDS problem on core-branch graphs. A direct formulation operates on r + sum_i a_i individual vertices. Once the number z of selected core vertices is fixed, however, branch symmetry determines the minimum contribution exactly as min{a_i,m-z}, eliminating individual branch-vertex decisions.

For H_p, N_p=(p-1)(p^2+1)=Theta(p^3). In the only regime without an immediate closed formula, at most p-k <= p-1 scalar evaluations are required. Thus a vertex-level optimization on Theta(p^3) vertices is reduced to an exact O(p) search.

The feasibility map reveals a sharp structural transition at k=p. For k<=p-1, multiple branches may participate because the core can provide the required connectivity. For k>=p, a feasible set cannot occupy multiple branches and must be confined to the core and one branch, imposing k<=p^2-2 and m<=p-1.

The full proper enhanced power graph is disconnected. The distinguished component H_p contains the nontrivial overlap geometry, whereas the other components are complete graphs whose parameters follow immediately from Proposition 4.7.

Extensions to C_p x C_(p^b), b>=3, constitute a natural direction for future work. In that setting, the longer chains of nested cyclic subgroups suggest a hierarchy of cyclic-subgroup intersections and, consequently, a multi-level core-branch geometry rather than the single core-branch layer occurring for b=2. Establishing an exact decomposition and determining whether a comparable low-dimensional compression survives at each level are separate structural problems and are not asserted here.

6. Conclusion

We developed an exact structural approach to k-connected m-domination on core-branch graphs and applied it to a distinguished component of the proper enhanced power graph of C_p x C_(p^2). For K_r join (K_a1 disjoint union ... disjoint union K_as), we obtained an exact feasibility characterization and reduced the high-domination problem to a one-dimensional discrete optimization.

We proved that the proper enhanced power graph of C_p x C_(p^2) decomposes as [K_(p-1) join pK_(p(p-1))] disjoint union pK_(p-1). For its distinguished component, we completely determined the feasible (k,m)-region and the corresponding minimum k-connected m-domination number. The results illustrate how cyclic-subgroup structure can transform a vertex-level combinatorial optimization problem into an exact low-dimensional model.

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ORCID iD: Luis Alfredo Dupont García: 0000-0002-9044-3547; Daniel Gualtiero Mendoza Ramírez: 0000-0003-3924-8129; José Juan Muñoz León: 0000-0003-3557-8251.

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