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| RESEARCH ARTICLE

Simultaneous joint universality of Dirichlet L-functions and their derivatives, with applications

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| ABSTRACT
We establish a simultaneous joint universality theorem for Dirichlet L-functions and their derivatives. This result unifies and extends the joint universality theorem for Dirichlet L-functions due to Bagchi, Gonek and Voronin, and the simultaneous universality theorem for the Riemann zeta-function and its derivatives recently obtained by the author. As an application of this generalized universality, we investigate the value-distribution of Dirichlet L-functions and their derivatives.

| KEYWORDS
Riemann zeta-function, Dirichlet L-function, derivatives, joint universality, a-point, value-distribution

| ARTICLE INFORMATION

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1. Introduction and statement of results

The study of value-distribution for the Riemann zeta-function $\zeta(s)$ has a long and rich history, and has been developed by many researchers (see e.g. [16] and [17, Chapter XI]). In 1975, Voronin [18] established the so-called *universality theorem* for $\zeta(s)$. This theorem asserts, roughly, that any non-vanishing analytic function can be uniformly approximated by vertical shifts of $\zeta(s)$. We state it in the modern form (see [8, p. 225, Theorem 5.2]). For a measurable set $A \subset \mathbb{R}$, we denote the Lebesgue measure of A by $\operatorname{meas} A$. Let $\mathcal{D}$ denote the strip $\{s \in \mathbb{C}\colon 1/2 < \operatorname{Re} s < 1\}$.

Theorem A. *Let $\mathcal{K}$ be a compact subset of $\mathcal{D} := \{s \in \mathbb{C}\colon 1/2 < \operatorname{Re} s < 1\}$ such that $\mathbb{C}\backslash\mathcal{K}$ is connected. Let $f(s)$ be a nonvanishing continuous function on $\mathcal{K}$ which is holomorphic in the interior (if any) of $\mathcal{K}$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \max_{s\in\mathcal{K}} |\zeta(s+i\tau) - f(s)| < \epsilon\right\} > 0.$$

Similar universality theorems have been established for various zeta and L-functions, and provide interesting consequences (see e.g. [11], [15] and [16]).

In particular, Bagchi [1] [2], Gonek [5] and Voronin [19] [7] independently obtained the universality theorem for the Dirichlet L-function $L(s,\chi)$ associated to a Dirichlet character χ. Actually, they showed a stronger result, which is called the *joint universality theorem* for Dirichlet L-functions. We now state it in a modern form (see [16, Theorem 1.10]). Let $\mathbb{N}$ denote the set of positive integers.

Theorem B. *Let $n, q_1, \dots, q_n \in \mathbb{N}$. Let $\chi_1 \bmod q_1, \dots, \chi_n \bmod q_n$ be pairwise non-equivalent Dirichlet characters. For each $j = 1, \dots, n$, let $\mathcal{K}_j$ be a compact subset of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}_j$ is connected. For each $j = 1, \dots, n$, let $f_j(s)$ be a nonvanishing continuous function on $\mathcal{K}_j$ which is holomorphic in the interior (if any) of $\mathcal{K}_j$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T]\colon \max_{1\le j\le n} \max_{s\in\mathcal{K}_j} \left|L(s+i\tau,\chi_j) - f_j(s)\right| < \epsilon\right\} > 0.$$

As in Theorem C below, a similar universality property holds for every derivative $\zeta^{(\nu)}(s)$ of the Riemann zeta-function $\zeta(s)$ (see [9], [10] and [12]). Notice that the target function $f(s)$ here is not necessarily nonvanishing.

Theorem C. *Let $\nu \in \mathbb{N}$. Let $\mathcal{K}$ be a compact subset of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}$ is connected. Let $f(s)$ be a continuous function on $\mathcal{K}$ which is holomorphic in the interior (if any) of $\mathcal{K}$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \max_{s\in\mathcal{K}}\left|\zeta^{(\nu)}(s+i\tau) - f(s)\right| < \epsilon\right\} > 0.$$

Recently, the author [14] combined and improved Theorems A and C to establish a *simultaneous* universality theorem for the Riemann zeta-function $\zeta(s)$ and its derivatives $\zeta^{(j)}(s)$, as in Theorem D below. In this paper, subsets $K_1, \dots, K_n \subset \mathbb{C}$ are said to be *disjoint* if $K_j \cap K_{j'}$ is empty for any pair (j, j') with $j \neq j'$. As usual, let $\zeta^{(0)}(s)$ denote $\zeta(s)$.

Theorem D. *Let $m \in \mathbb{N}$. Let $\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_m$ be disjoint compact subsets of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}_j$ is connected for each $j = 0,1,\dots,m$. Let $f_0(s)$ be a nonvanishing continuous function on $\mathcal{K}_0$ which is holomorphic in the interior (if any) of $\mathcal{K}_0$. For each $j = 1,\dots,m$, let $f_j(s)$ be a continuous function on $\mathcal{K}_j$ which is holomorphic in the interior (if any) of $\mathcal{K}_j$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|\zeta^{(j)}(s+i\tau) - f_j(s)\right| < \epsilon\right\} > 0.$$

The first purpose of this paper is to establish Theorem 1 below, which provides a simultaneous joint universality theorem for Dirichlet L-functions and their derivatives. This generalizes Theorem B to the simultaneous case involving derivatives, and Theorem D to the joint case. Here, as usual, let $L^{(0)}(s,\chi)$ denote $L(s,\chi)$.

Theorem 1. *Let $n, m, q_1, \dots, q_n \in \mathbb{N}$. Let $\chi_1 \bmod q_1, \dots, \chi_n \bmod q_n$ be pairwise non-equivalent Dirichlet characters. For each $j = 1,\dots,n$, let $\mathcal{K}_{j0}, \mathcal{K}_{j1}, \dots, \mathcal{K}_{jm}$ be disjoint compact subsets of $\mathcal{D}$. Suppose that $\mathbb{C}\backslash\mathcal{K}_{jk}$ is connected for each $j = 1,\dots,n$ and $k = 0,1,\dots,m$. For each $j = 1,\dots,n$, let $f_{j0}(s)$ be a nonvanishing continuous function on $\mathcal{K}_{j0}$ which is holomorphic in the interior (if any) of $\mathcal{K}_{j0}$. For each $j = 1,\dots,n$ and $k = 1,\dots,m$, let $f_{jk}(s)$ be a continuous function on $\mathcal{K}_{jk}$ which is holomorphic in the interior (if any) of $\mathcal{K}_{jk}$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \max_{1\le j\le n}\max_{0\le k\le m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j) - f_{jk}(s)\right| < \epsilon\right\} > 0.$$

The second purpose of this paper is to investigate the existence and distribution of simple a-points of Dirichlet L-functions and their derivatives.

For a meromorphic function $f(s)$ on the complex plane $\mathbb{C}$ and a complex number a, the zeros of the function $F(s) := f(s) - a$ are called the a-points of $f(s)$. An a-point ρ of $f(s)$ is said to be *simple* if $F'(\rho)(= f'(\rho)) \neq 0$.

Garunkštis and Steuding [4, Theorem 2] first proved that, for any complex number a, there are infinitely many simple a-points of $\zeta(s)$ with arbitrarily large imaginary part. Later, Gonek, Lester and Milinovich [6, Theorem 4] showed Theorem E below on simple a-points of $\zeta(s)$ by using Voronin's original universality theorem [18].

Theorem E. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $1/2 < \sigma_1 < \sigma_2 < 1$, and let $a \in \mathbb{C}\backslash\{0\}$. For $T > 0$, let $N(T; a, \sigma_1, \sigma_2)$ denote the number of simple a-points of $\zeta(s)$ in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \operatorname{Re} s < \sigma_2,\ 0 < \operatorname{Im} s < T\}$. Then*

$$N(T; a, \sigma_1, \sigma_2) \gg T \quad \text{as } T \to \infty,$$

where the constant implied by $\gg$ may depend on a, σ_1 and σ_2.

By utilizing Theorem 1, we can establish the next theorem on the simultaneous distribution of simple a-points for Dirichlet L-functions and their derivatives. For $s_0 \in \mathbb{C}$ and $r > 0$, let $B(s_0, r)$ denote the open disk $\{s \in \mathbb{C}\colon |s - s_0| < r\}$.

Theorem 2. *Let $n, m, m', M \in \mathbb{N}$. Let $\chi_1 \bmod q_1, \dots, \chi_n \bmod q_n$ be pairwise non-equivalent Dirichlet characters. Suppose that $a_{j0l} \in \mathbb{C}\backslash\{0\}$ for any $j = 1,\dots,n$ and $l = 1,\dots,m'$, and that $a_{jkl} \in \mathbb{C}$ for any $j = 1,\dots,n$, $k = 1,\dots,m$, and $l = 1,\dots,m'$. Suppose that $s_{jkl} \in \mathcal{D}$ for any $j = 1,\dots,n$, $k = 0,1,\dots,m$, and $l = 1,\dots,m'$. Let $r > 0$. Let $\mathcal{A} = \mathcal{A}(\{\chi_j\}, m, m', M, \{a_{jkl}\}, \{s_{jkl}\}, r)$ denote the set of real numbers τ such that $L^{(k)}(s,\chi_j)$ has at least M simple a_{jkl}-points in $B(s_{jkl} + i\tau, r)$ for any $j = 1,\dots,n$, $k = 0,1,\dots,m$, and $l = 1,\dots,m'$. Then*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\{\tau \in [0,T]\colon \tau \in \mathcal{A}\} > 0.$$

From this general theorem, we immediately deduce the following corollary concerning the distribution of simple a-points for an individual Dirichlet L-function and its derivative, which extends Theorem E.

Corollary 1. *Let $q \in \mathbb{N}$ and let χ be a Dirichlet character modulo q. Let k be a nonnegative integer. Suppose that $a \in \mathbb{C}\setminus\{0\}$ if $k = 0$ and $a \in \mathbb{C}$ if $k \neq 0$. Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $1/2 < \sigma_1 < \sigma_2 < 1$. For $T > 0$, let $N(T; k, a, \sigma_1, \sigma_2, \chi)$ denote the number of simple a-points of $L^{(k)}(s,\chi)$ in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \operatorname{Re} s < \sigma_2,\ 0 < \operatorname{Im} s < T\}$. Then*

$$N(T; k, a, \sigma_1, \sigma_2, \chi) \gg T \quad \text{as } T \to \infty,$$

where the constant implied by $\gg$ may depend on q, χ, k, a, σ_1, and σ_2.

Notation 1. In this paper, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote the sets of all complex numbers, real numbers, and positive integers, respectively. For a subset S of $\mathbb{C}$, let $\overline{S}$ denote the closure of S and let ∂S denote the boundary of S. Let $\emptyset$ denote the empty set.

2. Preliminary results

In the proof of Theorem 1, we will use some results concerning topology on the complex plane $\mathbb{C}$. Lemmas 1, 3 and 4 below are given and proved in [13] (see also [14]).

Lemma 1. *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $K_1, \dots, K_n$ be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\setminus K_j$ is connected for each $j = 1, \dots, n$. Then $\mathbb{C}\setminus\left(\bigcup_{j=1}^{n} K_j\right)$ is connected.*

The next result is the well-known Jordan curve theorem.

Lemma 2. *Let C be a Jordan curve (i.e., a simple closed curve) in the complex plane $\mathbb{C}$. Then its complement $\mathbb{C}\setminus C$ consists of exactly two connected components. One of these components is bounded and the other is unbounded, and the curve C is the boundary of each component.*

As in Section 1, let $\mathcal{D}$ denote the set $\{s \in \mathbb{C}\colon 1/2 < \operatorname{Re} s < 1\}$.

Lemma 3. *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $K_1, \dots, K_n$ be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\setminus K_j$ is connected for each $j = 1, \dots, n$. Suppose that $K_j \subset \mathcal{D}$ for each $j = 1, \dots, n$. Then there exist integers $N_1, \dots, N_n \geq 1$ and Jordan polygons $C_{j,k}$ ($j = 1, \dots, n$; $k = 1, \dots, N_j$) in $\mathcal{D}$ such that for each $j = 1, \dots, n$, $K_j \subset \bigcup_{k=1}^{N_j} U_{j,k} \subset \bigcup_{k=1}^{N_j} \overline{U_{j,k}} \subset \mathcal{D}$ and $K_j \cap U_{j,k} \neq \emptyset$ for all $k = 1, \dots, N_j$, and such that all the closures $\overline{U_{j,k}}$ ($j = 1, \dots, n$; $k = 1, \dots, N_j$) are disjoint. Here, each $U_{j,k}$ denotes the bounded connected component of $\mathbb{C}\setminus C_{j,k}$ (see Lemma 2).*

The next result is a celebrated theorem of Mergelyan (see e.g. [5, p. 97, Theorem 1]).

Lemma 4. *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C}\setminus K$ is connected. Let f be a continuous function on K which is holomorphic in the interior (if any) of K. Then, for any $\epsilon > 0$, there exists a polynomial P such that*

$$\max_{s \in K} |f(s) - P(s)| < \epsilon.$$

3. Proof of Theorem 1

By modifying suitably an argument in [14] and [13], we shall prove Theorem 1. According to Lemma 3 (in which we take $K_1 = \mathcal{K}_{j0}, \dots, K_{m+1} = \mathcal{K}_{jm}$ for each $j = 1, \dots, n$ individually), there exist integers $N_{j1}, \dots, N_{jm} \geq 1$ ($j = 1, \dots, n$) and Jordan polygons C_{jkl} ($j = 1, \dots, n$; $k = 1, \dots, m$; $l = 1, \dots, N_{jk}$) in $\mathcal{D}$ satisfying the following two conditions for each j: (i) for each $k = 1, \dots, m$, we have $\mathcal{K}_{jk} \subset \bigcup_{l=1}^{N_{jk}} U_{jkl} \subset \bigcup_{l=1}^{N_{jk}} \overline{U_{jkl}} \subset \mathcal{D}$ and $\mathcal{K}_{jk} \cap U_{jkl} \neq \emptyset$ for all $l = 1, \dots, N_{jk}$, and (ii) $\mathcal{K}_{j0}$ and the sets $\overline{U_{jkl}}$ ($k = 1, \dots, m$; $l = 1, \dots, N_{jk}$) are disjoint. Here, each U_{jkl} denotes the bounded connected component of $\mathbb{C}\setminus C_{jkl}$ (see Lemma 2).

Let $\epsilon > 0$ be arbitrary. By virtue of Lemma 4, we have polynomials $p_{j,k}(s)$ such that

$$\max_{s \in \mathcal{K}_{jk}} \left|f_{jk}(s) - p_{j,k}(s)\right| < \frac{\epsilon}{3} \quad \text{for each } j = 1, \dots, n \text{ and } k = 1, \dots, m. \tag{1}$$

Every $p_{j,k}(s)$ is defined and holomorphic on $\mathbb{C}$. Hence, for every j and k, there exist holomorphic functions (polynomials) $p_{j,k,k-1}(s), p_{j,k,k-2}(s), \dots, p_{j,k,0}(s)$ on $\mathbb{C}$ such that

$$\frac{d}{ds} p_{j,k,k-1}(s) = p_{j,k}(s), \quad \cdots, \quad \frac{d}{ds} p_{j,k,0}(s) = p_{j,k,1}(s). \tag{2}$$

For each $j = 1, \dots, n$, $k = 1, \dots, m$ and $l = 1, \dots, N_{jk}$, we define the constant

$$b_{jkl} := \max_{s \in \overline{U_{jkl}}} |p_{j,k,0}(s)| \tag{3}$$

and the function

$$g_{jkl}(s) := p_{j,k,0}(s) + b_{jkl} + 1, \quad s \in \mathbb{C}. \tag{4}$$

Then by (2),

$$g_{jkl}^{(k)}(s) = p_{j,k}(s). \tag{5}$$

For each $j = 1, \dots, n$, we put

$$K_j := \mathcal{K}_{j0} \cup \left(\bigcup_{k=1}^{m} \bigcup_{l=1}^{N_{jk}} \overline{U_{jkl}} \right) \subset \mathcal{D}. \tag{6}$$

Noting that for each j, $\mathcal{K}_{j0}$ and the $\overline{U_{jkl}}$'s are disjoint, we define the function h_j on K_j by

$$h_j(s) := \begin{cases} f_{j0}(s) & \text{for } s \in \mathcal{K}_{j0}, \\ g_{jkl}(s) & \text{for } s \in \overline{U_{jkl}} \ (k = 1, \dots, m;\ l = 1, \dots, N_{jk}). \end{cases}$$

By (4) and (3) we see that the function g_{jkl} has no zeros on $\overline{U_{jkl}}$. This and the condition on f_{j0} imply that

$$\text{the function } h_j \text{ has no zeros for each } j = 1, \dots, n. \tag{7}$$

Let us take a small positive real number ϵ_0 such that

$$\frac{k!}{2\pi} \operatorname{length}(C_{jkl})\, \epsilon_0 \max_{z \in C_{jkl}, s \in \mathcal{K}_{jk} \cap U_{jkl}} |(z-s)^{-(k+1)}| < \frac{\epsilon}{3} \tag{8}$$

for all $j = 1, \dots, n$, $k = 1, \dots, m$ and $l = 1, \dots, N_{jk}$, and such that $\epsilon_0 < \epsilon/3$.

According to Lemma 2 and the definition of U_{jkl}, the set $\mathbb{C} \backslash \overline{U_{jkl}}$ is connected for any j, k, l. By condition, $\mathbb{C} \backslash \mathcal{K}_{j0}$ is connected for each j. For each j, $\mathcal{K}_{j0}$ and the $\overline{U_{jkl}}$'s are disjoint compact subsets of $\mathbb{C}$. Therefore, (6) and Lemma 1 imply that

$$\mathbb{C} \backslash K_j \text{ is connected for each } j = 1, \dots, n. \tag{9}$$

Using (9), (7) and Theorem D, we obtain the following approximation on $K_1, \dots, K_n$:

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T]\colon \max_{1 \le j \le n} \max_{s \in K_j} |L(s + i\tau, \chi_j) - h_j(s)| < \epsilon_0 \right\} > 0. \tag{10}$$

Suppose that τ is a nonnegative real number satisfying

$$\max_{1 \le j \le n} \max_{s \in K_j} |L(s + i\tau, \chi_j) - h_j(s)| < \epsilon_0. \tag{11}$$

Then by definition (6), for each $j = 1, \dots, n$,

$$\max_{s \in \mathcal{K}_{j0}} |L(s + i\tau, \chi_j) - f_{j0}(s)| < \epsilon_0 \tag{12}$$

and

$$\max_{1 \le k \le m, 1 \le l \le N_{jk}} \max_{s \in C_{jkl}} |L(s + i\tau, \chi_j) - g_{jkl}(s)| < \epsilon_0, \tag{13}$$

since $C_{jkl} = \partial U_{jkl} \subset \overline{U_{jkl}}$ by Lemma 2.

By (5), Cauchy's integral formula for derivatives, (13) (in which we replace s by z) and (8), we see that τ satisfies the following: for any j, k and l, and uniformly for $s \in \mathcal{K}_{jk} \cap U_{jkl}$ ($\neq \emptyset$),

$$\begin{aligned}
&\left|L^{(k)}(s+i\tau,\chi_j)-p_{j,k}(s)\right| = \left|L^{(k)}(s+i\tau,\chi_j)-g_{jkl}^{(k)}(s)\right| \\
&= \left|\frac{k!}{2\pi \mathrm{i}}\oint_{C_{jkl}} \frac{L(z+i\tau,\chi_j)-g_{jkl}(z)}{(z-s)^{k+1}}dz\right| \\
&\leq \frac{k!}{2\pi}\mathrm{length}(C_{jkl})\max_{z\in C_{jkl}}\left|L(z+i\tau,\chi_j)-g_{jkl}(z)\right|_{z\in C_{jkl}, s\in\mathcal{K}_{jk}\cap U_{jkl}}\max\left|(z-s)^{-(k+1)}\right| \\
&< \frac{\epsilon}{3}.
\end{aligned} \tag{14}$$

Since $\mathcal{K}_{jk}\subset\bigcup_{l=1}^{N_{jk}}U_{jkl}$ for any j and k, it follows from (1), (12), (14) and $\epsilon_0<\epsilon/3$ that τ satisfies

$$\begin{aligned}
&\max_{1\leq j\leq n}\max_{0\leq k\leq m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j)-f_{jk}(s)\right| \\
&\leq \max_{1\leq j\leq n}\max\{\max_{s\in\mathcal{K}_{j0}}\left|L(s+i\tau,\chi_j)-f_{j0}(s)\right|, \\
&\qquad \max_{1\leq k\leq m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j)-f_{jk}(s)\right|\} \\
&\leq \max_{1\leq j\leq n}\max\left\{\epsilon_0,\ \max_{1\leq k\leq m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j)-p_{j,k}(s)\right|+\frac{\epsilon}{3}\right\} \\
&< \max\left\{\epsilon_0,\frac{\epsilon}{3}+\frac{\epsilon}{3}\right\}<\epsilon.
\end{aligned}$$

This and (11) imply that the set of such τ has positive lower density. Theorem 1 is proved.

4. Proof of Theorem 2

We shall now prove Theorem 2. Let us take complex numbers z_{jklv} $(j=1,\dots,n;\ k=0,1,\dots,m;\ l=1,\dots,m';\ v=1,\dots,M)$ and a small positive real number $\delta<\min_{1\leq j\leq n,1\leq l\leq m'}\left|a_{j0l}\right|$ such that

$$\overline{B(z_{jklv},\delta)}\subset B(s_{jkl},r) \quad \text{and} \quad \overline{B(z_{jklv},\delta)}\subset\mathcal{D} \quad \text{for any } j,k,l,v \tag{15}$$

and such that

$$\text{the sets } \overline{B(z_{jklv},\delta)} \text{ are disjoint.} \tag{16}$$

For each $j=1,\dots,n$ and $k=0,1,\dots,m$, we put

$$\mathcal{K}_{jk}:=\bigcup_{l=1}^{m'}\bigcup_{v=1}^{M}\overline{B(z_{jklv},\delta)}\subset\mathcal{D} \tag{17}$$

and define the function f_{jk} on $\mathcal{K}_{jk}$ by

$$f_{jk}(s):=s-z_{jklv}+a_{jkl} \quad \text{for } s\in\overline{B(z_{jklv},\delta)} \tag{18}$$

in view of (16).

Since $\delta<\min_{1\leq j\leq n,1\leq l\leq m'}\left|a_{j0l}\right|$, the function $f_{j0}(s)$ has no zeros on $\mathcal{K}_{j0}$. By (16), the sets $\mathcal{K}_{jk}$ $(j=1,\dots,n;\ k=0,1,\dots,m)$ are disjoint. Thus it follows from Theorem 1 that

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\ \max_{1\leq j\leq n}\max_{0\leq k\leq m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j)-f_{jk}(s)\right|<\delta\right\}>0. \tag{19}$$

Suppose that τ is a real number satisfying

$$\max_{1\leq j\leq n}\max_{0\leq k\leq m}\max_{s\in\mathcal{K}_{jk}}\left|L^{(k)}(s+i\tau,\chi_j)-f_{jk}(s)\right|<\delta. \tag{20}$$

We fix any integers j,k,l,v with $j=1,\dots,n$, $k=0,\dots,m$, $l=1,\dots,m'$, and $v=1,\dots,M$. Then by (17), (20) and (18), for any $s\in\partial B(z_{jklv},\delta)$, we have

$$\begin{aligned}
\left|\left(L^{(k)}(s+i\tau,\chi_j)-a_{jkl}\right)-\left(f_{jk}(s)-a_{jkl}\right)\right| &= \left|L^{(k)}(s+i\tau,\chi_j)-f_{jk}(s)\right| \\
&< \delta=\left|s-z_{jklv}\right|=\left|f_{jk}(s)-a_{jkl}\right|.
\end{aligned}$$

The function $f_{jk}(s) - a_{jkl} = s - z_{jkl\nu}$ has only one zero, counted with multiplicity, in $B(z_{jkl\nu}, \delta)$. Hence by Rouché's theorem, the function $L^{(k)}(s + i\tau, \chi_j) - a_{jkl}$ has a zero of multiplicity one in $B(z_{jkl\nu}, \delta)$, which implies that the function $L^{(k)}(s, \chi_j)$ has a simple a_{jkl}-point in $B(z_{jkl\nu} + i\tau, \delta)$. Therefore by definition, (15) and (16), we have $\tau \in \mathcal{A}$. This and (19) yield the theorem.

5. Proof of Corollary 1

We shall now prove Corollary 1. In Theorem 2, we choose $n = 1$, $m = \max\{k, 1\}$, $m' = 1$, $M = 1$, $a_{1k1} = a$, and $s_{1j1} = s_0 := \frac{1}{2}(\sigma_1 + \sigma_2)$ for all $j = 0, \dots, m$. (For $j \neq k$, the choices of a_{1j1} are arbitrary, provided $a_{101} \neq 0$.) We set r to be a positive real number with $r < \frac{1}{2}(\sigma_2 - \sigma_1) < \frac{1}{4}$. Note that $B(s_0 + i\tau, r) \subset \mathcal{D}_{\sigma_1, \sigma_2}$ for any $\tau \in \mathbb{R}$, where $\mathcal{D}_{\sigma_1, \sigma_2} := \{s \in \mathbb{C} : \sigma_1 < \operatorname{Re} s < \sigma_2\}$.

By Theorem 2, the set $\mathcal{A}$ of real numbers τ such that $L^{(k)}(s, \chi)$ has a simple a-point in $B(s_0 + i\tau, r)$ has positive lower density. This means that there exists a constant $c > 0$ such that

$$\operatorname{meas}(\mathcal{A} \cap [r, T - r]) \geq cT \tag{21}$$

for all sufficiently large T.

From the set $\mathcal{A} \cap [r, T - r]$, we can extract a sequence of points $\tau_1 < \tau_2 < \cdots < \tau_E$ satisfying $\tau_{\nu+1} - \tau_\nu \geq 2r$ for $\nu = 1, \dots, E - 1$. Indeed, we can choose $\tau_1 = \inf\{\tau \in \mathcal{A} \cap [r, T - r]\}$, and recursively choose $\tau_{\nu+1} = \inf\{\tau \in \mathcal{A} \cap [r, T - r] : \tau \geq \tau_\nu + 2r\}$. Each chosen point τ_ν eliminates an interval $[\tau_\nu, \tau_\nu + 2r)$ of length at most $2r$ from $\mathcal{A} \cap [r, T - r]$ for the next choice. Thus, the number E of such points we can extract must satisfy $E \cdot 2r \geq \operatorname{meas}(\mathcal{A} \cap [r, T - r])$. In view of (21), this implies that $E \geq \frac{c}{2r} T$.

For these chosen points τ_ν, the open disks $B(s_0 + i\tau_\nu, r)$ are disjoint since the distance between any two centers is at least $2r$. Moreover, since $\tau_\nu \in [r, T - r]$, these disks are strictly contained in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \operatorname{Re} s < \sigma_2,\ 0 < \operatorname{Im} s < T\}$. Since $\tau_\nu \in \mathcal{A}$, the function $L^{(k)}(s, \chi)$ has at least one simple a-point in $B(s_0 + i\tau_\nu, r)$ for each $\nu = 1, \dots, E$. Therefore, the total number of simple a-points of $L^{(k)}(s, \chi)$ in this rectangle is at least E. We conclude that

$$N(T; k, a, \sigma_1, \sigma_2, \chi) \geq E \geq \frac{c}{2r} T \gg T \quad \text{as } T \to \infty.$$

This completes the proof.

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