

RESEARCH ARTICLE

A Local Laplace-Finite Difference Method for One-Dimensional Kelvin-Voigt Waves with Time-Dependent Viscosity: Stability and Convergence

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ABSTRACT

This Paper develops a local Laplace-finite difference method for a one-dimensional Kelvin-Voigt wave equation with time dependent viscosity. On each time slab, the viscosity is frozen at the endpoint average, which is a second-order approximation of the interval mean, and the resulting constant-coefficient local problem is advanced through a numerical inverse Laplace transform. Second-order centered finite differences are used in space. Both displacement and velocity are recovered from Laplace inversion; in particular, the velocity is inverted from $s\hat{u} - u^n$ rather than reconstructed by a first-order time difference. For the homogeneous semi-discrete problem, an energy estimate shows that the exact frozen-coefficient propagator is dissipative for every time-step size, so it is free of the explicit CFL restriction. The effect of numerical Laplace inversion is treated separately as a quadrature perturbation. Under standard smoothness assumptions, endpoint averaging produces a local coefficient-freezing defect of order $O(\Delta t^3)$ and hence a global temporal error of order $O(\Delta t^2)$; the centered spatial discretization contributes $O(h^2)$. Manufactured-solution tests recover spatial orders 2.001, 2.000, and 2.000 under successive mesh refinement and temporal orders 2.088, 2.018, and 2.004. A modified Talbot inversion with 24 nodes makes the inversion error negligible in the reported tests. For a transient forcing with $\eta(t) = 0.5e^{-t}$, the proposed method with $h = 0.01$ and $\Delta t = 0.05$ agrees with a high-accuracy ODE reference to approximately 10^{-6} in time-history RMSE. These results support the method as an accurate CFL-free time propagator for smooth time-dependent Kelvin-Voigt media, while also identifying contour inversion and coefficient regularity as practical limitations.

KEYWORDS

Kelvin-Voigt wave equation; time-dependent viscosity; numerical Laplace transform; finite differences; Talbot inversion; energy stability; convergence.

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1. Introduction

Wave propagation in viscoelastic media is important in seismology, ultrasonic characterization, polymer mechanics, and structural dynamics because material memory and viscosity attenuate and disperse propagating waves. Classical time-domain algorithms for viscoelasticity include finite-difference and finite-element formulations with internal variables or relaxation mechanisms [1-10]. These methods are mature and effective, but explicit time stepping is subject to a CFL restriction associated with the wave speed and spatial mesh, and variable material parameters introduce an additional source of temporal approximation error.

Laplace-transform methods provide a different route: time evolution is represented by independent boundary-value problems at complex Laplace parameters and the physical solution is reconstructed by contour quadrature. Talbot's contour (Talbot, 1979), optimized or modified contours [11-15], and alternative high-accuracy inversion strategies [16,17] provide a substantial numerical foundation for this approach. Talbot nodes are independent and therefore naturally parallelizable. Related contour-integral time discretizations for evolution equations have rigorous convergence foundations [18-21], and Laplace-spatial-discretization couplings have also been investigated for nonlinear and time-dependent problems [18,22].

The difficulty in the present model is that the viscosity $\eta(t)$ multiplies the strain-rate operator. A global Laplace transform of $\eta(t)u_{xxt}$ does not reduce to multiplication by a scalar function of s unless special structure is available. A direct global transform therefore replaces a simple product in time by convolution in the transform domain. The present work avoids that complication by applying Laplace propagation locally. On each interval $[t_n, t_{n+1}]$, $\eta(t)$ is replaced by a second-order endpoint average and the local constant-coefficient problem is solved in Laplace space.

The contribution is not merely the juxtaposition of Laplace inversion and finite differences, both of which are established techniques. The specific contribution is a consistent local propagator for the time-dependent Kelvin-Voigt model with four elements that must be used together: (i) second-order coefficient freezing; (ii) a sign-consistent transformed formulation including the local initial-data terms; (iii) direct Laplace recovery of both displacement and velocity; and (iv) an energy and error analysis that distinguishes stability of the frozen physical propagator from numerical contour-inversion error. This difference prevents us from asserting “unconditional stability” for a finite-node inversion formula.

The numerical study has three goals. First, a manufactured solution decouples the spatial and temporal convergence and validates the expected second-order rates. Second, the dependence on the number of modified Talbot nodes is quantified directly. Third, a transient variable-viscosity benchmark is contrasted with an independent high-accuracy ODE solution and, at the same time step, with the classical average-acceleration Newmark method [23].

The remainder of the paper is organized in the following way. The model and assumptions are given in Section 2. In Section 3 we derive the local Laplace-finite difference scheme. Section 4 is devoted to the energy estimate and the global error structure. Section 5 shows the numerical verification and the transient benchmark. Sections 6 and 7 present limitations and conclusions. 2.

Mathematical model

Consider the one-dimensional Kelvin-Voigt wave equation

$$\rho u_{tt}(x, t) - \frac{\partial}{\partial x} (E u_x(x, t) + \eta(t) u_{xt}(x, t)) = f(x, t), \quad 0 < x < L, \quad 0 < t \leq T, \quad \dots(1)$$

with initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x),$$

and Dirichlet boundary conditions $u(0, t) = g_0(t)$ and $u(L, t) = g_L(t)$. Here $\rho > 0$ is density, $E > 0$ is the elastic modulus, and $\eta(t) \geq 0$ is a time-dependent Kelvin-Voigt viscosity. The form is consistent with the usual use of viscous stress relative to strain rate in linear viscoelastic wave models [1-3] [6] [8-10].

For the analysis, we assume $\eta \in C^2([0, T])$ and $0 \leq \eta(t) \leq \eta_{\max}$, with sufficiently smooth f , boundary data, and exact solution. The energy proof uses homogeneous Dirichlet data; nonhomogeneous data can be treated by the usual boundary lifting.

Let $x_i = ih$, $i = 0, \dots, M$, with $h = L/M$, and define the positive Dirichlet difference operator

$$(K_h \mathbf{w})_i = \frac{2w_i - w_{i-1} - w_{i+1}}{h^2}, \quad i = 1, \dots, M-1. \quad \dots(2)$$

K_h is symmetric positive definite for homogeneous boundary conditions. The semi-discrete equation is

$$\rho \ddot{\mathbf{u}}(t) + EK_h \mathbf{u}(t) + \eta(t) K_h \dot{\mathbf{u}}(t) = \mathbf{f}(t). \quad \dots(3)$$

3.local Laplace – finite difference method

3.1. Second-order local coefficient freezing

Let $t_n = n\Delta t$ and introduce local time $\tau = t - t_n \in [0, \Delta t]$. On this slab, define

$$\bar{\eta}_n = \frac{\eta(t_n) + \eta(t_{n+1})}{2}. \quad \dots(4)$$

The Taylor expansion near the middle point $t_{n+1/2}$ yields

$$\bar{\eta}_n = \eta(t_{n+1/2}) + O(\Delta t^2),$$

and the integral mean of η over the interval equals its value at the middle plus $O(\Delta t^2)$. So, (2) is a second order approximation to the coefficient. More accurate than left-endpoint freezing which is usually first-order global temporal error.

The locally frozen problem is

$$\rho u_{\tau\tau} - E u_{xx} - \bar{\eta}_n u_{x\tau} = f(x, t_n + \tau), \quad \dots(5)$$

with $u(x, 0) = u^n(x)$ and $u_\tau(x, 0) = v^n(x)$.

3.2. Correct transformed equation

Let $\hat{u}_n(x, s) = \mathcal{L}_\tau\{u(x, \tau)\}$ and $\hat{f}_n(x, s) = \mathcal{L}_\tau\{f(x, t_n + \tau)\}$. Using

$$\mathcal{L}\{u_{\tau\tau}\} = s^2 \hat{u}_n - s u^n - v^n, \quad \mathcal{L}\{u_{x\tau}\} = s \hat{u}_{n,xx} - u_{xx}^n, \quad \dots(6)$$

the local transformed equation is

$$\rho(s^2 \hat{u}_n - s u^n - v^n) - E \hat{u}_{n,xx} - \bar{\eta}_n (s \hat{u}_{n,xx} - u_{xx}^n) = \hat{f}_n. \quad \dots(7)$$

Equivalently,

$$(E + s \bar{\eta}_n) \hat{u}_{n,xx} - \rho s^2 \hat{u}_n = -\hat{f}_n - \rho s u^n - \rho v^n + \bar{\eta}_n u_{xx}^n. \quad \dots(8)$$

The sign of the last initial-data term in (3) follows trivially from the transform of $u_{xx\tau}$ and is essential for consistency.

3.3. Finite-difference system in the Laplace domain

The centered second-order difference is applied to the spatial derivative, which is equivalent to a transforming (1) after substituted $\eta(t)$ by $\bar{\eta}_n$. With $\mathbf{U}_n(s)$ denoting the vector of interior Laplace-domain unknowns,

$$\mathbf{A}_n(s) \mathbf{U}_n(s) = \mathbf{F}_n(s) + \rho s \mathbf{u}^n + \rho \mathbf{v}^n + \bar{\eta}_n K_h \mathbf{u}^n, \quad \dots(9)$$

where

$$\mathbf{A}_n(s) = \rho s^2 I + (E + s \bar{\eta}_n) K_h. \quad \dots(10)$$

Boundary contributions are added to the right-hand side when g_0 or g_L is nonzero. Because K_h is tridiagonal, so is $\mathbf{A}_n(s)$, and a banded direct solver requires $O(M)$ work per Laplace node. More importantly, the systems associated with distinct complex values of s are independent and can be computed concurrently.

3.4. Modified Talbot inversion and state transfer

Talbot-type inversion deforms the Bromwich contour into the left half-plane so that the factor e^{st} damps much of the contour [11,14,15,24]. We use the optimized modified Talbot parameterization of Dingfelder and Weideman [11].

$$z(\theta) = \frac{N}{\tau} \zeta(\theta), \quad \zeta(\theta) = -0.6122 + 0.5017 \theta \cot(0.6407 \theta) + 0.2645 i \theta, \quad -\pi < \theta < \pi. \quad \dots(11)$$

With the midpoint nodes $\theta_k = -\pi + (k - 1/2)2\pi/N$, numerical inversion at $\tau = \Delta t$ is

$$q_N(\Delta t) = \frac{1}{N i} \sum_{k=1}^N e^{z_k \Delta t} \hat{q}(z_k) z_k'(\theta_k), \quad z_k = z(\theta_k). \quad \dots(12)$$

Complex-conjugate symmetry can be used to halve the work for real-valued solutions. The optimized convergence analysis in (Dingfelder & Weideman, 2015) is strongest for transforms whose singularities lie on the negative real axis. Damped-wave resolvents can possess complex poles, so the universal spectral rate derived for that restricted setting is not asserted here. Instead, Section 5 reports a direct node-convergence study for the present wave problem.

State transfer between slabs is performed consistently by

$$\mathbf{u}^{n+1} = \mathcal{L}^{-1}\{\mathbf{U}_n(s)\}(\Delta t), \quad \dots(13)$$

$$\mathbf{v}^{n+1} = \mathcal{L}^{-1}\{s \mathbf{U}_n(s) - \mathbf{u}^n\}(\Delta t). \quad \dots(14)$$

Equation (9) is important: replacing it by $(\mathbf{u}^{n+1} - \mathbf{u}^n)/\Delta t$ would introduce a first-order velocity approximation and would contaminate the nominal second-order temporal behavior.

3.5. Algorithm

For $n = 0, \dots, N_t - 1$, the implementation performs the following operations: compute $\bar{\eta}_n$ from (2); generate the N modified Talbot nodes for the local interval; solve the tridiagonal system (4) independently at each node; invert (8) and (9) using (7); then pass $(\mathbf{u}^{n+1}, \mathbf{v}^{n+1})$ to the next time slab. The cost is $O(NMN_t)$ with a direct tridiagonal solver, and the loop over the N complex nodes is embarrassingly parallel.

4. Stability and convergence

4.1. Frozen-step energy stability

For a homogeneous boundary-value problem, consider one slab with $\eta(t)$ replaced by the nonnegative constant $\bar{\eta}_n$. Define the discrete mechanical energy

$$\mathcal{E}_h(t) = \frac{\rho}{2} \|\dot{\mathbf{u}}(t)\|_2^2 + \frac{E}{2} \mathbf{u}(t)^T K_h \mathbf{u}(t). \quad \dots(15)$$

Multiplying the homogeneous semi-discrete equation by $\dot{\mathbf{u}}^T$ yields

$$\frac{d\mathcal{E}_h}{dt} = -\bar{\eta}_n \dot{\mathbf{u}}^T K_h \dot{\mathbf{u}} \leq 0. \quad \dots(16)$$

Proposition 1. If $E > 0$, $\rho > 0$, and $\bar{\eta}_n \geq 0$, the exact propagator of every frozen local semi-discrete problem is energy dissipative for any $\Delta t > 0$.

This establishes a CFL-free property of the exact local propagation: the time step is selected by accuracy and by the validity of coefficient freezing, not by the explicit stability limit $\Delta t \leq h/\sqrt{E/\rho}$. It does not imply that an arbitrary finite-node numerical inverse Laplace formula is error-free. The latter contributes the independent quadrature error quantified below.

For nonzero forcing, the same calculation gives

$$\frac{d\mathcal{E}_h}{dt} = -\bar{\eta}_n \dot{\mathbf{u}}^T K_h \dot{\mathbf{u}} + \dot{\mathbf{u}}^T \mathbf{f}, \quad \dots(17)$$

and standard Cauchy–Schwarz/Young estimates yield a finite-time stability bound in terms of the initial energy and forcing norm.

4.2. Error decomposition

Let Φ_n denote the exact flow of the variable-coefficient semi-discrete first-order system and $\tilde{\Phi}_n$ the exact flow obtained after replacing $\eta(t)$ by $\bar{\eta}_n$. Since $\bar{\eta}_n = \eta(t_{n+1/2}) + O(\Delta t^2)$ and the coefficient is a bounded operator on the finite dimensional grid, a variation-of-constants expansion around the midpoint gives

$$\|\Phi_n - \tilde{\Phi}_n\| \leq C\Delta t^3, \quad \dots(18)$$

if $\eta \in C^2$ and the solution is smooth enough. Accumulation over $T/\Delta t$ stable steps thus leads to an $O(\Delta t^2)$ global coefficient-freezing error.

The centered difference is $O(h^2)$. Let δ_N be the error in the state produced by the numerical inverse Laplace quadrature on a single slab. So we come to the practical estimate combining the consistency with the energy stability of the frozen flow

$$\max_{0 \leq n \leq N_t} \|\mathbf{e}^n\| \leq C_T \left(h^2 + \Delta t^2 + \frac{T}{\Delta t} \delta_N \right). \quad \dots(19)$$

Proposition 2. Under the stated smoothness assumptions, and with Talbot inversion accurate enough that $(T/\Delta t)\delta_N$ The local Laplace-finite difference method is second order in space and second order in time with N asymptotically below the discretization error.

The form of (13) is such that contour inversion is deliberately explicit. It is better to increase N until the Talbot error plateaus below the spatial and temporal errors, rather than to use without verification a universal fixed N .

5. Numerical Experiments

All calculations reported are of double precision.. Unless otherwise stated, $L = 1$, $\rho = 1$, $E = 10$, homogeneous Dirichlet conditions, and $N = 24$ modified Talbot nodes per local step. Tridiagonal complex systems are solved directly. The discrete error measures are

$$\|e\|_{2,h} = \left(h \sum_{i=1}^{M-1} |e_i|^2 \right)^{1/2}, \quad \|e\|_{\infty} = \max_i |e_i|. \quad \dots(20)$$

Observed orders are computed as $p = \log(e_{mcoarse}/e_{mfine})/\log 2$.

5.1. Manufactured-solution verification

Choose

$$u(x, t) = q(t)\sin(\pi x), \quad q(t) = t^2 e^{-t}. \quad \dots(21)$$

Then $q'(t) = e^{-t}(2t - t^2)$ and $q''(t) = e^{-t}(t^2 - 4t + 2)$. The forcing is generated analytically as

$$f(x, t) = [q''(t) + \pi^2(Eq(t) + \eta(t)q'(t))]\sin(\pi x). \quad \dots(22)$$

This test is useful because it exercises both the inertial and viscosity terms while retaining an exact solution.

Spatial convergence

To isolate the spatial discretization, set $\eta(t) = 0.5$ and use $\Delta t = 0.1$ to $T = 0.5$; for this single-mode manufactured problem the local Laplace step integrates the constant-coefficient time dependence to the accuracy of the contour inversion. Table 1 shows a clean second-order spatial rate.

Table 1. Spatial convergence at $T = 0.5$ for the manufactured solution.

M	h	$\ e\ _{2,h}$	$\ e\ _{\infty}$	order
20	0.05000	2.1907×10^{-4}	3.0981×10^{-4}	—
40	0.02500	5.4723×10^{-5}	7.7390×10^{-5}	2.0012
80	0.01250	1.3678×10^{-5}	1.9344×10^{-5}	2.0003
160	0.00625	3.4193×10^{-6}	4.8356×10^{-6}	2.0001

Temporal convergence with variable viscosity

Next take $\eta(t) = 0.5e^{-t}$ and $M = 100$. To isolate temporal error from spatial error, the reference is the exact-in-time solution of the same semi-discrete sine eigenmode. Table 2 shows second-order convergence as Δt decreases.

Table 2. Temporal convergence at $T = 1$ with $\eta(t) = 0.5e^{-t}$ and $M = 100$.

Δt	$\ e\ _{2,h}$	$\ e\ _{\infty}$	order
0.200	4.0860×10^{-5}	5.7784×10^{-5}	—
0.100	9.6085×10^{-6}	1.3588×10^{-5}	2.0883
0.050	2.3715×10^{-6}	3.3538×10^{-6}	2.0185
0.025	5.9107×10^{-7}	8.3590×10^{-7}	2.0044

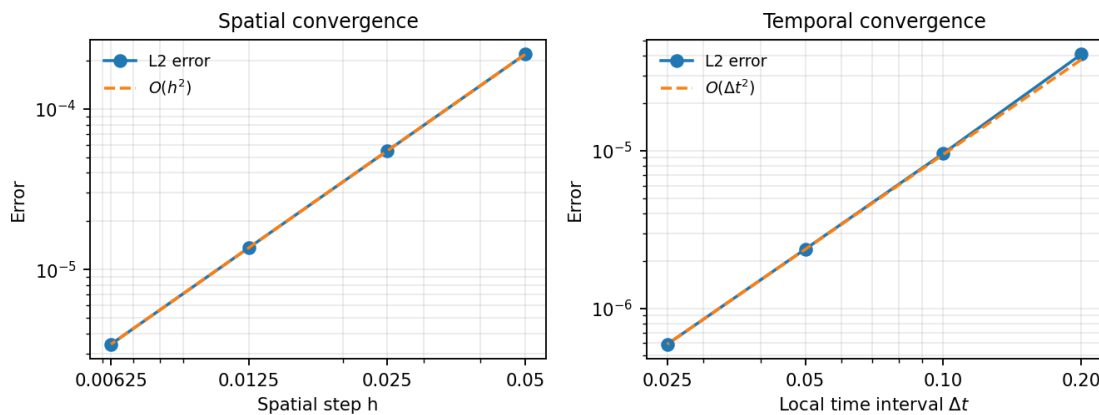


Figure 1. Observed second-order convergence in space and time. The dashed reference slopes are proportional to h^2 and Δt^2 , respectively.

5.2. Sensitivity to the number of Talbot nodes

The Laplace inversion must be converged independently of h and Δt . For the manufactured single mode at $t = 0.5$ with constant $\eta = 0.5$, Table 3 reports the absolute amplitude error as the number of modified Talbot nodes is increased. The error falls by more than nine orders of magnitude from $N = 8$ to $N = 20$ and reaches the floating-point regime by $N = 28$. We therefore use $N = 24$ elsewhere; this is an empirical choice for the present resolvents rather than a universal prescription.

Table 3. Modified Talbot node-convergence test at $t = 0.5$.

Talbot nodes N	absolute amplitude error
8	2.6267×10^{-4}
12	2.2135×10^{-6}
16	1.3374×10^{-8}
20	2.2467×10^{-11}
24	4.7001×10^{-13}
28	1.8319×10^{-15}

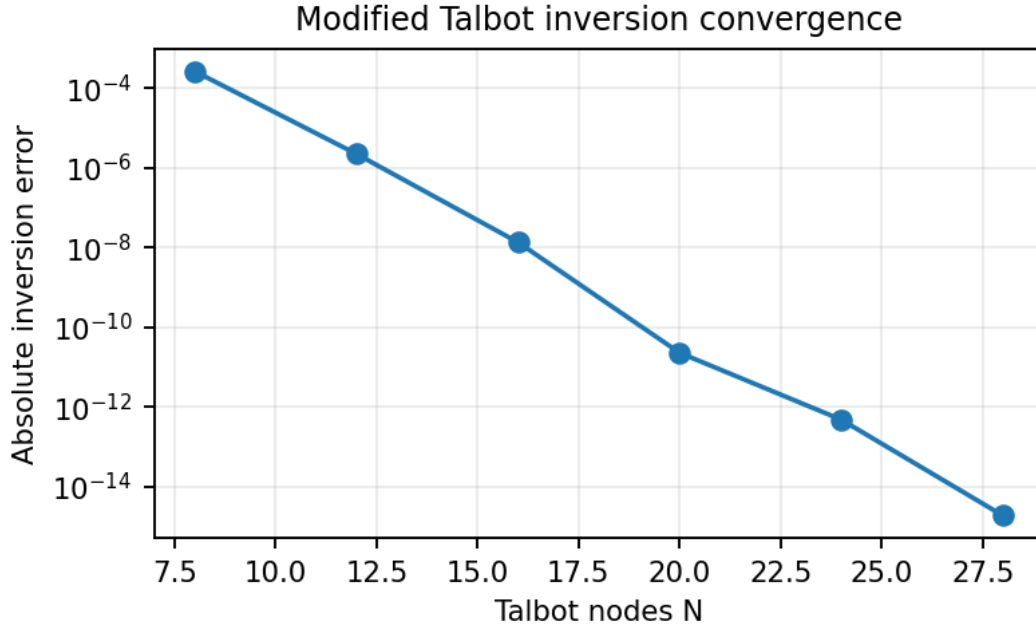


Figure 2. Empirical convergence of the modified Talbot inversion for the damped-wave test. The plateau reflects double-precision arithmetic.

As an independent implementation sanity check, inversion of $\hat{q}(s) = 1/(s + 1)$ at $t = 1$ produced absolute errors 2.81×10^{-5} , 1.16×10^{-7} , 1.42×10^{-10} , 2.93×10^{-12} , and 8.72×10^{-15} for $N = 8, 12, 16, 20, 24$, respectively, against e^{-1} .

5.3. Variable-viscosity transient benchmark

We now consider a physically interpretable transient with

$$\begin{aligned} u(x, 0) = u_t(x, 0) = 0, \quad u(0, t) = u(1, t) = 0, \quad \dots(23) \\ \eta(t) = 0.5e^{-t}, \quad f(x, t) = e^{-t}\sin(\pi x). \end{aligned}$$

Because the source and boundary data excite only the first sine mode, $u(x, t) = a(t)\sin(\pi x)$ and the continuous PDE reduces exactly to

$$a''(t) + 0.5\pi^2 e^{-t}a'(t) + 10\pi^2 a(t) = e^{-t}, \quad a(0) = a'(0) = 0. \quad \dots(24)$$

Equation (18) is solved independently with an adaptive high-accuracy ODE solver (10^{-12} relative tolerance and 10^{-14} absolute tolerance) and is used as the reference. The local Laplace-FDM calculation uses $M = 100$ ($h = 0.01$), $\Delta t = 0.05$, and $N = 24$.

Table 4. Center displacement $u(0.5, t)$ for the variable-viscosity transient.

t	local Laplace-FDM	high-accuracy reference	absolute error
0.2	0.0101927311	0.0101932357	5.0464×10^{-7}
0.6	0.0026386353	0.0026366364	1.9989×10^{-6}
1.0	0.0058744170	0.0058745848	1.6785×10^{-7}

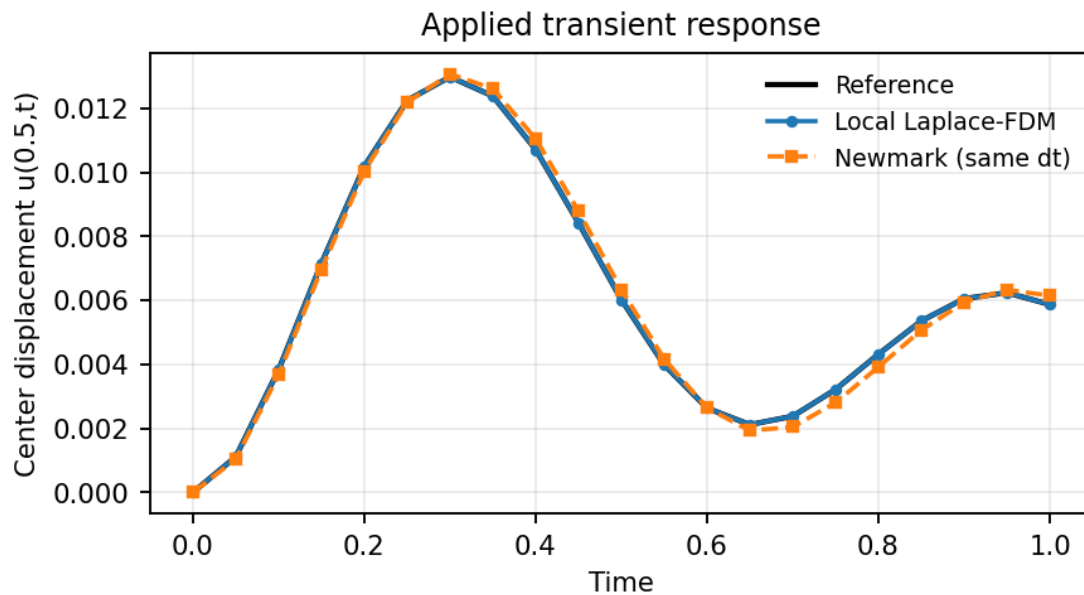


Figure 3. Center-point displacement history. The local Laplace–FDM curve is plotted against the independent reference solution.

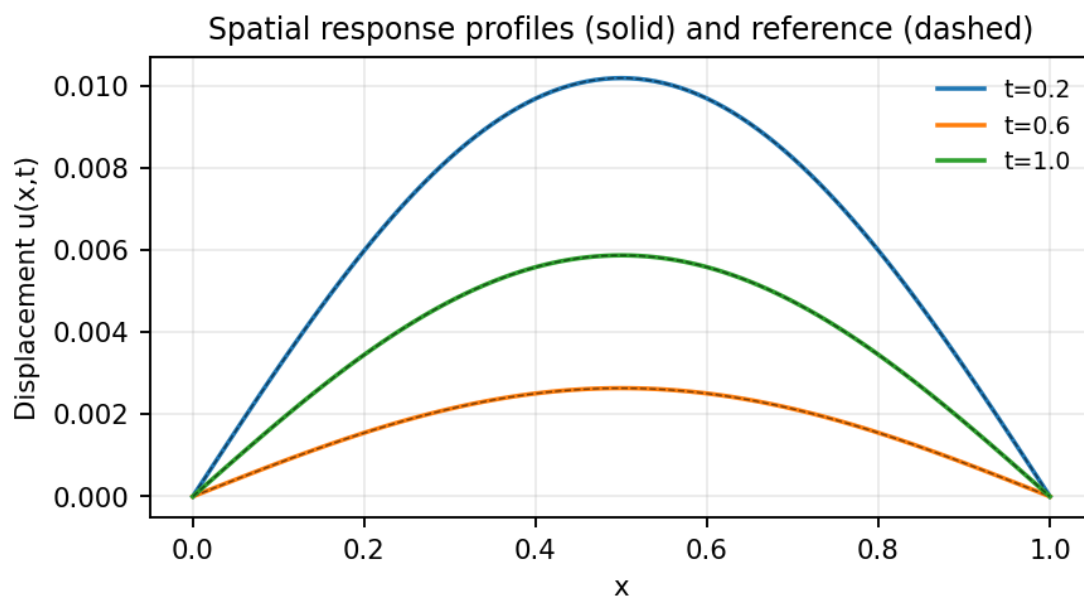


Figure 4. Spatial displacement profiles at selected times. The amplitude evolution reflects the combined elastic, viscous, and forcing effects.

For additional context, the average-acceleration Newmark method ($\beta = 1/4$, $\gamma = 1/2$) [23] was applied to the same semi-discrete first sine mode and the same $\Delta t = 0.05$. Over $0 \leq t \leq 1$, the time-history RMSE was 1.06×10^{-6} for the local Laplace–FDM method and 2.39×10^{-4} for Newmark in this particular oscillatory test. This comparison should not be interpreted as universal superiority: Newmark is also an implicit, well-established second-order method, and its relative accuracy depends on frequency content, time step, damping treatment, and implementation. The comparison instead illustrates that the local Laplace propagator can resolve the constant-coefficient dynamics within each slab very accurately while using the step primarily to resolve the variation of $\eta(t)$.

For reference, an undamped explicit centered time update on the same grid would have the wave-speed stability scale

$$\Delta t_{\text{CFL}} \approx \frac{h}{\sqrt{E/\rho}} = 3.16 \times 10^{-3}, \quad \dots(25)$$

which is much smaller than $\Delta t = 0.05$. This observation is consistent with the CFL-free property of the implicit Laplace propagation established by Proposition 1, and it is not a statement that the accuracy is independent of Δt .

6. Discussion

The results provide insight into the accuracy of the method and its limitations. The spatial operator is still the usual second order centered difference, so the observed $O(h^2)$ rate is not surprising. The time mechanism is less standard: local constant-coefficient dynamics is solved by inverse Laplace propagation, while the dominant time-discretization error is introduced by the variation of $\eta(t)$ in a slab. Endpoint averaging makes that coefficient approximation second-order symmetric and the velocity state is directly inverted, avoiding a hidden first-order update.

This method is attractive when the spatial mesh is fine enough so that an explicit CFL step would be much smaller than the time scale on which material properties vary. It is also attractive if the independent complex shifted systems can be solved in parallel. Conversely, if $\eta(t)$ varies rapidly or discontinuously, or on a time scale comparable to the wave period, a smaller adaptive local step or a higher order coefficient representation is needed. Similarly, the modified Talbot contour is not uniformly optimal for any possible pole geometry. Thus, node convergence should be checked for a new parameter regime rather than blindly inherited from a parabolic problem.

The current study is intentionally one-dimensional to allow a clean separation of temporal and transform errors. The extension to two and three dimensions is conceptually straightforward at the semi-discrete level, replacing the tridiagonal K_h by the appropriate sparse stiffness operator, but then the main computational issue is efficient complex shifted solves. Established multidimensional viscoelastic finite-difference and finite-element literature [1,2,5,6,9,25,26] provide natural benchmark families for such an extension.

7. Conclusions

We propose a local Laplace–finite difference method and test it for a one-dimensional Kelvin-Voigt wave equation with smooth time-dependent viscosity. The corrected transformed system retains all local initial data contributions and Laplace inversion recovers both components of the state. The frozen local problem satisfies a discrete energy dissipation law for arbitrary time-step size. The total numerical error is decomposed into spatial, coefficient-freezing and contour-inversion components. Second order convergence in both space and time is verified by manufactured tests. A dedicated node study confirms that 24 modified Talbot nodes are sufficient for the range of parameters reported. The variable-viscosity transient benchmark agrees with an independent high-accuracy reference to about 10^{-6} RMSE for $h=0.01$ and $\Delta t=0.05$.

Thus the method should be viewed as a CFL-free second order local propagator, whose time step is governed by variation of coefficients and accuracy, not explicit wave stability. Future work should address adaptive slab selection, higher-order treatment of rapidly varying coefficients, multi-dimensional sparse shifted solvers, and systematic contour selection for strongly underdamped pole configurations.

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