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| RESEARCH ARTICLE

Three rules are derived for numerically determining the values of double integrals utilizing the middle point and three-eighths criteria

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| ABSTRACT
By utilizing Romberg acceleration and the two numerical rules (the middle point and the three-eighths) and their corresponding error formulas, we hope to derive three numerical rules that can be used to extract the values of continuous binary integrals during the integration period. Applying the three principles produced results that were accurate to fourteen or fifteen decimal places and had comparatively few subintervals on the integral's inner and outer dimensions inside the region of integration.

| KEYWORDS
Middle point rule, three-eighths rule, Romberg acceleration, double integrals

| ARTICLE INFORMATION
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1. Introduction

Many have worked in the field of numerical integration due to its importance and numerous applications in engineering, physics, and other areas of life. Among them are Newton and Coates, who derived many numerical rules used to find the values of one-dimensional integrals (our research relies on two of these rules) [1], and Fox [2], [3], who enriched this scientific field with important research on finding the error series associated with the Newton-Coates rules, which we still benefit from today. Many other scientists and researchers, both past and present, have also contributed, including Al-Tamimi [4] and Al-Karami [5]. Al-Karami introduced a new rule for calculating one-dimensional integrals numerically, which he named NK1. The values obtained by applying the NK1 rule are highly accurate and often surpass the Newton-Coates integral rules in terms of accuracy, approach speed, and other factors.

As for double integrals, they have received considerable attention in this field due to their importance and the significance of their applications. Therefore, many researchers have worked on deriving numerical rules for finding the values of double integrals or developing existing rules and improving the values resulting from their application to double integrals. In 2010, Agar [6] presented the RMM double rule, which was derived from the midpoint rule and its associated error formula with Rumbreck acceleration. The resulting values showed good accuracy and speed in approaching the exact value. In 2012, Alkaramy [7] presented the RTT double rule, which was derived from the trapezoidal rule with Rumbreck acceleration. This rule is applied to continuous and improper double integrals in the integration region and yielded good results in terms of accuracy and speed in approaching the exact value. Alkaramy also presented the RMT and RTM rules, which he derived using the same method as the RTT rule. The RMT and RTM methods were more accurate than the first, especially for integrals with irregularities across their integration region.

Hilal et al. [8] also presented a study that offered three rules for calculating continuous double integrals. In this study, we will rely on the midpoint rule (which we will denote) and the three-eighths rule (which we will denote) and their associated error formulas [1] and [9] to derive three new rules for numerically calculating the values of double integrals, as follows: 1- Calculate the inner integral on the x-dimensional using the three-eighths rule and its associated error formula, then calculate the value of the outer integral on the y-dimensional by applying the same rule to each term resulting from applying the rule on the x-dimensional, while addressing the associated correction terms, ensuring that the length of the subintervals on the inner dimension is the same as on the outer dimension. We denote this rule by the symbol

2- Calculate the inner integral on the x-dimensional using the midpoint rule and its associated error formula, then calculate the value of the outer integral on the y-dimensional by applying the three-eighths rule to each term resulting from applying the midpoint rule on the x-dimensional, while addressing the associated correction limits. As in the first rule, the length of the subinterval on the inner x-dimensional is equal to the length of the subinterval on the outer y-dimensional. We denote this rule by the symbol

3- Calculate the inner integral on the x-dimensional using the three-eighths rule and its associated error formula, then calculate the value of the outer integral on the y-dimensional by applying the midpoint rule to each term resulting from applying the three-eighths rule on the x-dimensional, while addressing the associated correction limits, as previously stated for the condition of equal lengths of the subintervals on the x and y-dimensionals. We denote this rule by the symbol

To improve the values resulting from applying the above rules, we used the correction limits associated with the rules,

By utilizing the application of Romberg acceleration to the values resulting from applying the three rules to the two integrals.

2- Priorities

Assume for the moment that we have the following integration.

$$J = \int_{x_0}^{x_n} g(x)\,dx$$

The three-eighths rule allows for the numerical calculation of the integral's value.[9]

$$J = \int_{\chi_0}^{\chi_n} g(\chi)\,d\chi = \frac{3h}{8}\left[g_0 + g_n + 3\sum_{i=1,4,7,\cdots}^{n-2} g_i + g_{i+1} + 2\sum_{i=3,6,9,\cdots}^{n-3} g_i \right] + E_T(h)$$

$$E_T(h) = \omega_1 h^2 + \omega_2 h^4 + \omega_3 h^6 + \cdots$$

or applying the middle point principle[1]

$$J = \int_{\chi_0}^{\chi_n} g(\chi)\,d\chi = h\sum_{i=0}^{n-1} g_{i+\frac{h}{2}} + E_{\mathrm{M}}(h)$$

$$E_{\mathrm{M}}(h) = \alpha_1 h^2 + \alpha_2 h^4 + \alpha_3 h^6 + \cdots$$

Note that

$$g_i = g(\chi_i)\ h = \frac{\chi_n - \chi_0}{n}\ ,\ \chi_0, \chi_1, \cdots \chi_n \in [\chi_0, \chi_n]\ \ \chi_i = \chi_0 + ih\ ,$$

$E_{\mathrm{T}}(h)$ The corrective restrictions that go along with the three-eighths rule are referred to by the symbol T

$E_{\mathrm{M}}(h)$ It relates to the midpoint, which we represent with the symbol, and the corrective limits that go along with the base. M

and that $\alpha_1, \alpha_2, \alpha_3, \cdots\ \ \omega_1, \omega_2, \omega_3, \cdots$

Its constants and values do not depend on h

The correction limits associated with each of the two rules can be used to improve the results and bring them closer to the precise values of integration using Romberg acceleration $J = \dfrac{2^k \delta_2 - \delta_1}{2^k - 1}$ [1] .

where δ_2, δ_1 Two distinct values for integration utilizing one of the two techniques(M , T) The infinity is a value that depends on powers in the correction boundary series accompanying the rule.

5- How the three rules are derived ($T-T, M-T, T-M$) To compute binary integrals with numerical methods

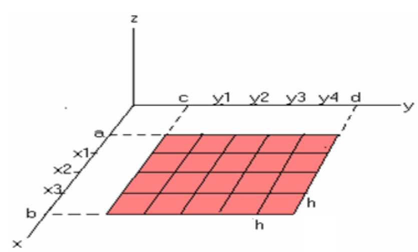
Let's assume the following integral exists.

$$J=\int_{\gamma_0}^{\gamma_n}\int_{\chi_0}^{\chi_n} g(\chi,\gamma)\,d\chi\,d\gamma$$

We undertake a logic breakdown of the integration to get the value of the double integral mentioned above.
$[\chi_0,\chi_n]\times[\gamma_0,\gamma_n]$

Into sub-regions by dividing the period $[\chi_0,\chi_n]$ divided into equal subperiods within a single period's duration h So we have the contract $\chi_0,\chi_1,\cdots\chi_n$ where $\chi_i=\chi_0+ih$ We also divide the period $[\gamma_0,\gamma_n]$

divided into equal subperiods of a single period's duration h So we have the contract
$\gamma_0,\gamma_1,\cdots\gamma_n$ $\gamma_i=\gamma_0+ih$



1- The base $T-T$

We work on calculating internal integration on a dimension χ Using the base T We examine the second independent variable after adding the error formula that goes with the rule. γ Constant, so we have the equation

$$J=\int_{\gamma_0}^{\gamma_n}\int_{\chi_0}^{\chi_n} g(\chi,\gamma)\,d\chi\,d\gamma$$

$$=\int_{\gamma_0}^{\gamma_n}\left[\frac{3h}{8}\left[g_{0,\gamma}+g_{n,\gamma}+3\sum_{i=1,4,7,\cdots}^{n-2} g_{i,\gamma}+g_{i+1,\gamma}+2\sum_{i=3,6,9,\cdots}^{n-3} g_{i,\gamma}\right]+\omega_1h^2+\omega_2h^4+\omega_3h^6+\cdots\right]d\gamma$$

Next, we compute the dimension's external integration. γ Using the base T Adding the error formula accompanying the rule for each term of the equation above

$$1-\ \frac{3h}{8}\int_{\gamma_0}^{\gamma_n}\left[g_{0,\gamma}+g_{n,\gamma}\right]d\gamma=\frac{9h}{64}\left[g_{0,0}+g_{n,0}+g_{0,n}+g_{n,n}\right.$$

$$\left.+2\sum_{i=3,6,9,\cdots}^{n-3} g_{0,i}+g_{n,i}+3\sum_{i=1,4,7,\cdots}^{n-2} g_{0,i}+g_{0,i+1}+g_{n,i}+g_{n,i+1}\right]+\omega_1h^2+\cdots$$

$$2-\int_{\gamma_0}^{\gamma_n}\frac{3h}{8}\left[3\sum_{i=1,4,7,\cdots}^{n-2} g_{i,\gamma}+g_{i+1,\gamma}\right]d\gamma=\frac{9h}{64}\left[3\sum_{i=1,4,7,\cdots}^{n-2}\left[g_{i,0}+g_{i+1,0}+g_{i,n}+g_{i+1,n}\right.\right.$$

$$\left.\left.+2\sum_{j=3,6,9,\cdots}^{n-3} g_{i,j}+g_{i+1,j}+3\sum_{j=1,4,7,\cdots}^{n-2} g_{i,j}+g_{i,j+1}+g_{i+1,j}+g_{i+1,j+1}\right]\right]+\omega_1 h^2+\cdots$$

$$3-\int_{\gamma_0}^{\gamma_n}\frac{3h}{8}\left[2\sum_{i=1,4,\cdots}^{n-2} g_{i,\gamma}\,d\gamma\right]=\frac{9h}{64}\left[2\sum_{i=1,4,\cdots}^{n-2}\left[g_{i,0}+g_{i,n}+2\sum_{j=3,6,9,\cdots}^{n-3} g_{i,j}\right.\right.$$

$$\left.\left.+3\sum_{j=1,4,7,\cdots}^{n-2} g_{i,j}+g_{i,j+1}\right]\right]+\omega_1 h^2+\cdots$$

As for the correction limits associated with the rule, their integration is calculated analytically.

$$4-\int_{\gamma_0}^{\gamma_n}\omega_1 h^2+\omega_2 h^4+\omega_3 h^6+\cdots d\gamma$$

$$=(\gamma_n-\gamma_0)\left[\omega_1 h^2+\omega_2 h^4+\omega_3 h^6+\cdots\right]=\phi_1 h^2+\phi_2 h^4+\phi_3 h^6+\cdots$$

Where $\phi_i=\omega_i(\gamma_n-\gamma_0)$

And from the equations (1),(2),(3),(4) produces the base $\mathrm{T}-\mathrm{T}$

To calculate binary integrals numerically

$$J=\int_{\gamma_0}^{\gamma_n}\int_{\chi_0}^{\chi_n} g(\chi,\gamma)\,d\chi\,d\gamma$$

$$=\frac{9h}{64}\left[g_{0,0}+g_{n,0}+g_{0,n}+g_{n,n}\right.$$

$$+3\sum_{i=1,4,\cdots}^{n-2}\left[g_{0,i}+g_{0,i+1}+g_{n,i}+g_{n,i+1}+g_{i,0}+g_{i+1,0}+g_{i,n}+g_{i+1,n}\right]$$

$$+2\sum_{i=3,6,\cdots}^{n-3}\left[g_{0,i}+g_{n,i}+g_{i,0}+g_{i,n}+2\sum_{j=3,6,\cdots}^{n-3} g_{i,j}+3\sum_{j=1,4,\cdots}^{n-2} g_{i,j}+g_{i,j+1}\right]$$

$$\left.+3\sum_{i=1,4,\cdots}^{n-2}3\sum_{j=1,4,\cdots}^{n-2} g_{i,j}+g_{i,j+1}+g_{i+1,j}+g_{i+1,j+1}+2\sum_{j=3,6,\cdots}^{n-3} g_{i,j}+g_{i+1,j}\right]$$

$$+\phi_1 h^2+\phi_2 h^4+\phi_3 h^6+\cdots$$

2- The base $\mathrm{M}-\mathrm{T}$

To use the rule to determine the double integral's value $\mathrm{M}-\mathrm{T}$ We compute the internal integration on a dimension using the same method as previously X Using the base M With the addition of the error formula accompanying the rule, we consider the second independent variable γ constant

2-

Then we calculate the external integration on the dimension γ Using the base T The result is obtained by adding the error formula related to the rule for each term in the above equation.

$$J=\int_{\gamma_0}^{\gamma_n}\int_{\chi_0}^{\chi_n} g(\chi,\gamma)\,d\chi\,d\gamma$$

$$=\frac{9h}{64}\sum_{j=0}^{n-1}\left[\left[g_{j+\frac{h}{2},0}+g_{j+\frac{h}{2},n}+3\sum_{i=1,4,\cdots}^{n-2} g_{j+\frac{h}{2},i}+g_{j+\frac{h}{2},i+1}+2\sum_{i=3,6,\cdots}^{n-3} g_{j+\frac{h}{2},i}\right]+\Phi_1h^2+\Phi_2h^4+\Phi_3h^6+\cdots\right]$$

3- The Best $\mathrm{T-M}$

To derive the rule $\mathrm{T-M}$ We use the same technique used in deriving the two bases mentioned above, with the difference being calculating the internal integration over a distance χ Using the three-eighths rule T

Including the corresponding correction limits and taking the variable into account γ

ثابت ومن ثم حساب التكامل الخارجي على البعد γ بتطبيق قاعدة النقطة الوسطى على كل حد من الحدود الناتجة من التكامل الداخلي , فيكون لدينا

$$J=\int_{\gamma_0}^{\gamma_n}\int_{\chi_0}^{\chi_n} g(\chi,\gamma)\,d\chi\,d\gamma$$

$$=\frac{9h}{64}\sum_{j=0}^{n-1}\left[\left[g_{0,j+\frac{h}{2}}+g_{n,j+\frac{h}{2}}+3\sum_{i=1,4,\cdots}^{n-2} g_{i,j+\frac{h}{2}}+g_{i+1,j+\frac{h}{2}}+2\sum_{i=3,6,\cdots}^{n-3} g_{i,j+\frac{h}{2}}\right]+\delta_1h^2+\delta_2h^4+\delta_3h^6+\cdots\right]$$

3. Examples and conclusion

We have implemented the three guidelines. $\mathrm{T-T}$, $\mathrm{M-T}$, $\mathrm{T-M}$ In the tables below, we completed a variety of binary integrals and noted the final values as well as the number of subintervals connected to each value. To spare the reader time and effort, we did not record every resultant value.

It is important to note that we worked on the following number of sub-periods inside the integration region: (3,6,12,24,48,96,192,384,...)

This is because applying the three-eighths rule requires a number of partitions that are multiples of 3, while Rumbreck acceleration ensures that the length of the partition for the second value (derived using one of the rules) is half the length of the first value. Also, to save time, we started with 48 partitions when calculating the integral value using one of the three methods, then doubled the number to obtain the second value, and so on until we obtained the approximate, accurate values shown in the example tables below.

	Integration with its precise value	**Number of whole decimal places along with the number of subintervals associated with them**		
		$\frac{\mathrm{T-T}}{n}$	$\frac{\mathrm{M-T}}{n}$	$\frac{\mathrm{T-M}}{n}$
1	$\int_{0.5}^{1}\int_{1}^{1.5} x+x/y\ dxdy$ **0.745716987849966**	$\frac{15}{768}$ **0.745716987849966**	$\frac{15}{384}$ **0.745716987849966**	$\frac{14}{192}$ **0.745716987849966**
2	$\int_{1}^{1.5}\int_{1}^{1.5} e^x y\ dxdy$ **1.10212952617439**	$\frac{14}{192}$ **1.10212952617439**	$\frac{14}{192}$ **1.10212952617439**	$\frac{14}{192}$ **1.10212952617439**

3	$\int_{1}^{1.5}\int_{2.5}^{3}\sqrt{yx^3}\,dxdy$ **1.27383349936379**	$\frac{14}{192}$ **1.27383349936379**	$\frac{14}{192}$ **1.27383349936379**	$\frac{14}{192}$ **1.27383349936379**
4	$\int_{2}^{2.5}\int_{1}^{1.5}e^{x-y}y\,dxdy$ **0.209331172043030**	$\frac{15}{192}$ **0.209331172043030**	$\frac{15}{192}$ **0.209331172043030**	$\frac{15}{192}$ **0.209331172043030**
5	$\int_{1}^{1.5}\int_{2}^{2.5}x^{\frac{y}{2}}\,dxdy$ **0.415521277582249**	$\frac{15}{192}$ **0.415521277582249**	$\frac{15}{192}$ **0.415521277582249**	$\frac{15}{192}$ **0.415521277582249**
6	$\int_{0.5}^{1}\int_{1}^{1.5}x\ln(x+y)\,dxdy$ **0.618402621820461**	$\frac{14}{192}$ **0.618402621820461**	$\frac{15}{192}$ **0.618402621820461**	$\frac{14}{192}$ **0.618402621820461**
7	$\int_{0.5}^{1}\int_{1}^{1.5}\sin(x+y^2)\,dxdy$ **0.23335088728081**	$\frac{14}{192}$ **0.23335088728081**	$\frac{14}{192}$ **0.23335088728081**	$\frac{15}{192}$ **0.23335088728081**

Table (1) This represents the values resulting from applying the three rules with Romberg acceleration to seven binary integrals and the accuracy of those results. (The number of decimal places divided by the number of subintervals associated with them)

We can see from the outcomes listed in the table (1) which employ rules to represent the numerical values of particular integrals($M-T$, $T-T$, $T-M$,) With Romberg's acceleration (1) the following

Regarding the first integration $\int_{0.5}^{1}\int_{1}^{1.5}x+x/y\,dxdy$ When applying the two rules (T-T, M-T)

The results obtained with Romberg acceleration were accurate to fifteen decimal places when the number of subintervals (768,384) In other words, the two rules provided the same accuracy in sequence, but the number of sub-periods linked to each rule's result varied , But when applying the rule T-M

Romberg acceleration produced a result that was accurate to fourteen decimal places when the number of subintervals n=192 Partial period
However, the results were accurate to fourteen decimal places and at the same number of subintervals when the three criteria were applied with Romberg acceleration to the second and third integrals. n=192
This applies equally to all three applications. This indicates that the efficiency of the three rules is identical with respect to these two examples Similarly, we observe a match in the efficiency of the rules.

($T-T, M-T, T-M$) By extracting the value of the integral in the fourth and fifth integrals, where the values resulting from applying the three rules were correct for fifteen decimal places and at the same number of subintervals, subinterval n=192..

However, in examples six and seven, we observe the accuracy of the values resulting from the application of the three rules.) ($T-T, M-T, T-M$ With the Romberg acceleration, the range is between (15 -14)

Decimal order when the number of subperiods used is fixed at n=192 subperiods

The effectiveness of the regulations is seen above.
($T-T, M-T, T-M$)

Continuous numerical integrals are calculated by placing the integral within the region of integration and applying the correction restrictions related to the previously described principles. As a result, these guidelines can be used to determine the values of functions that lack a known mathematical formula and are instead tables of values for the independent and dependent variables, or integrals whose values cannot be found analytically or can only be found with difficulty and hardship. The integration is an illustration of such integrals.

$\int_{1}^{1.5}\int_{1}^{1.5} x^{xy}\, dxdy$ Whose analytical value is unknown, so we resorted to using rules.($\mathrm{T-T, M-T, T-M}$)

Using Romberg's acceleration to determine the integral's value, we may see from the outcomes listed in the tables (2), (3), (4) It is the outcome of using the guidelines for Romberg acceleration ($\mathrm{T-T, M-T, T-M}$)

Integration has a value of, accordingly.(0.36735511650139)

It is a correct value for fourteen decimal places, and this is achieved by the results approaching a certain value and then remaining at that value resulting from applying the rules horizontally, which indicates that it is the exact value of integration. We should not forget that the series of results recorded in the tables is a Cauchy convergent series whose goal is the exact value of integration.

n	$\mathrm{T-T}$	K=2	K=4	K=6	K=8
48	0.36725857296969				
96	0.36733098040467	0.36735511621632			
192	0.36734908246385	0.36735511648357	0.36735511650139		
384	0.36735360799117	0.36735511650028	0.36735511650139	0.36735511650139	
768	0.36735473937378	0.36735511650132	0.36735511650139	0.36735511650139	0.36735511650139

Table (2) Represents the values resulting from the integration procedure. $\int_{1}^{1.5}\int_{1}^{1.5} x^{xy}\, dxdy$ Using the base $\mathrm{T-T}$

	$\mathrm{M-T}$	K=2	K=4	K=6	K=8
48	0.36734932914532				
96	0.36735366964129	0.36735511647328			
192	0.36735475478505	0.36735511649964	0.36735511650139		
384	0.36735502607222	0.36735511650128	0.36735511650139	0.36735511650139	
768	0.36735509389409	0.36735511650139	0.36735511650139	0.36735511650139	0.36735511650139

Table (3) Represents the values resulting from the integration procedure. $\int_{1}^{1.5}\int_{1}^{1.5} x^{xy}\, dxdy$ Using the base $\mathrm{M-T}$

	$\mathrm{T-M}$	K=2	K=4	K=6	K=8
48	0.36735488942482				
96	0.36735505951875	0.36735511621672			
192	0.36735510224238	0.36735511648360	0.36735511650139		
384	0.36735511293581	0.36735511650028	0.36735511650139	0.36735511650139	
768	0.36735511560994	0.36735511650132	0.36735511650139	0.36735511650139	0.36735511650139

Table (4) Represents the values resulting from the integration procedure. $\int_{1}^{1.5}\int_{1}^{1.5} x^{xy}\, dxdy$ Using the base $\mathrm{T-M}$

From the above, we conclude that the three rules $\mathrm{T-T, M-T, T-M}$

It is highly efficient at finding the values of continuous binary integrals within the integration region, where the accuracy of the values obtained using the rules with Romberg acceleration ranges between

(15 -14) Decimal order, and often with simple sub-intervals n=192 Partial period .

It is worth noting the importance of rules $T-T, M-T, T-M$ In order to determine the values of binary integrals whose values cannot be determined analytically, the resulting values are approached to a specific quantity and then stabilized at a specific value, which is the value of the integral being studied.

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