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| RESEARCH ARTICLE

Exponential- Odd Inverted Dagum Distribution: Properties and Difference Estimation Methods

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| ABSTRACT
An inverted distribution are significant statistical tools for analyzing real life data. In such many fields such as medical sciences, computer sciences, economics, and other real-life application. In this work, we utilize the odds-family and Inverted Dagum Distribution to construct a new model namely Exponential-Odd Inverted Dagum Distribution. A variant statistical property of this distribution. Types such as moment, moment generated mapping, hazard rate, and mean deviation are expressed. We presented various approaches, such as moment estimators, maximum likelihood estimators and Jackknife estimator, where Monte-Carlo simulations are used to determine the efficiency proposed estimator methods.

| KEYWORDS

Dagum Distribution, Inverted Dagum Distribution, Odds-Inverted Dagum Distribution, Method of Moment Estimator, Method of Maximum Likelihood Estimator

| ARTICLE INFORMATION

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1. Introduction

Three-parameter Dagum Distribution (Dagum type Ⅰ) [1] was established in (1977) as type Ⅲ of the Burr system of distributions [2]. Later, Domma & Zenga [3], used reliability concept to discuss hazard, and reverse hazard rates for this type of distributions. Klieber& Kotz [4] mentioned to another form of Dagum Distribution known as Dagum type Ⅱ (four parameter) significant when data are zeros. Dagum[5] introduced type Ⅲ for which four parameters are involved. But this type is valid for only bounded variables. Therefore, it barely used in economic applications. Oluyede and Ye [6] introduced Weighed Dagum Distribution and discussed variant types of distributions that related to it. Beta-Dagum Distribution (five parameter) was presented by Domma and Condino [7].

A continuous stochastic variable X is named as Dagum type Ⅰ [1] symbolled by $X \backsim Dgm(x;a,b,p)$. if cumulative density function (CDF) is

$$F(x;a,b,p) = \left[1 + \left(\frac{b}{x}\right)^{a}\right]^{-p} \quad ,x > 0 \qquad (1)$$

Where $b > 0$ is the scale parameter and its two shape parameters $a \; and \; p$ are positive.

And probability density function (PDF) is

$$f(x;a,b,p) = p(b^{a})a(x^{-a-1})\left(1 + \left(\frac{b}{x}\right)^{a}\right)^{-p-1} \qquad (2)$$

The Dagum models occupied as essential parts in the recent studies of income, health system and compensation see [8,9,10].

Osi A. et al [11] proposed a novel distribution related to Dagum Distribution by using inverse transformation namely inverted Dagum Distribution for which a comprehensive study for statistical properties such as Hazard functions quantile mapping and survival mapping were investigated.

The three parameters (a, b, p) CDF of this distribution is given by:

$$F_{ID}(y; a, b, p) = 1 - \left[1 + \left(\frac{b}{y}\right)^{a}\right]^{-p} \text{ for } y > 0 \quad (3)$$

And three parameters (a, b, p) PDF is defined by:

$$f_{ID}(y; a, b, p) = p(b^{a})a(y^{-a-1})\left(1 + \left(\frac{b}{y}\right)^{a}\right)^{-p-1} \quad (4)$$

In addition, the notion of Odds family initiated by Marshal& Olkin [12]. This essential concept inspired literatures to derive many distributions see [13,14] based on form

$$\frac{F(x)}{1-F(x)} \quad , F(x) \text{ is a CDF of baseline distribution.} \quad (5)$$

The main purpose of this article is to propose Exponential odd- Inverted Dagum Distribution for which distributional and statistical aspects are discussed in first section. MOM, MLE and JMLE have been applied to estimate four parameters of E-OIDD. A simulation analysis has been provided to determine flexibility of this presented distribution.

2. Exponential -Odd Inverted Dagum Distribution

Set Inverted Dagum Distribution as a baseline distribution for which CDF and PDF represented in (3) and (4).

Now, substitute (3) in (5)

That is

$$t(y) = \frac{F_{ID}(y)}{1 - F_{ID}(y)}$$

This implies that $t(y) = \frac{1-\left[1+\left(\frac{b}{y}\right)^{a}\right]^{-p}}{1-\left(1-\left[1+\left(\frac{b}{y}\right)^{a}\right]^{-p}\right)}$

Consequently, $t(y) = [1 + (by)^{a}]^{-p} - 1 \qquad y > 0 \quad (6)$

Since a, b, and $p > 0$, then $t(y)$ is an increasing function

It is remaining to replace x with $t(y)$ in Exponential Cumulative Distribution Function as showing in the proceeding step:

$$F_{E}(t(y)) = 1 - e^{-\alpha t(y)} \qquad y > 0$$

In turn,

$$F_{E-OIDD}(y; \alpha, a, b, p) = 1 - exp[-\alpha(1 + (by)^{a})^{p} + \alpha] \qquad y > 0$$

So,

$$F_{E-OIDD}(y; \alpha, a, b, p) = 1 - exp[-\alpha(1 + (by)^{a})^{p} + \alpha] \exp(\alpha) \quad y > 0 \quad (7)$$

For which α, b are scale parameters and $a \; and \; p$ are shape parameters.

To verify (7) is our proposed CDF:

Now, $\alpha > 0$, so are $a, b, p > 0$

$$\lim_{y\to 0^{+}} F_{E-OIDD}(y; \alpha, a, b, p) = \lim_{y\to 0^{+}} \left(1 - \frac{\exp(\alpha)}{exp[\alpha(1+(by)^{a})^{p}]}\right)$$

$$= 1 - \frac{\exp(\alpha)}{exp(\alpha(1+0)^{p})} = 1 - \frac{\exp(\alpha)}{\exp(\alpha)} = 0$$

And,

$$\lim_{y\to\infty} F_{E-OIDD}(y;\alpha,a,b,p) = \lim_{y\to\infty}\left(1-\frac{\exp(\alpha)}{exp[\alpha(1+(by)^a)^p]}\right)$$

Since, $y \to \infty$

$\Rightarrow (by)^{\alpha} \to \infty$

$\Rightarrow (1+(by)^a)^p \to \infty$

$exp[\alpha(1+(by)^a)^p] \to \infty$

$$\lim_{y\to\infty}\left(1-\frac{\exp(\alpha)}{exp[\alpha(1+(by)^a)^p]}\right) = 1-0 = 1$$

That is, $F_{E-OIDD}(y;\alpha,a,b,p)$ is an increasing function from 0 to 1.

$F_{E-OIDD}(y;\alpha,a,b,p)$ is our proposed cumulative distribution function.

Drive PDF for (E-OIDD)

Since

$$F_{E-OIDD}(y;\alpha,a,b,p) = 1 - exp[-\alpha(1+(by)^a)^p + \alpha]\; exp\,(\alpha)$$

Then

$$f_{E-OIDD}(y;\alpha,a,b,p) = \frac{d}{dy}F_{E-OIDD}(y;\alpha,a,b,p)$$

$$= -exp\,(\alpha)\big[exp[-\alpha(1+(by)^a)^p]\big]\cdot(-\alpha)p(1+(by)^a)^{p-1}(ab^a)y^{a-1}$$

$$= \alpha p(ab^a)\; exp(\alpha)\; exp\,(-\alpha(1+(by)^a)^p)y^{a-1}$$

Clearly; $f_{E-OIDD}(y;\alpha,a,b,p) \geq 0$

Finally, to verify

$$\int_0^{\infty} f_{E-OIDD}(y;\alpha,a,b,p) = F(\infty) - F(0) = 1 - 0 = 1$$

Hence, $f_{E-OIDD}(y;\alpha,a,b,p) = \alpha p(ab^a)\; exp(\alpha)\; exp\,(-\alpha(1+(by)^a)^p)y^{a-1}$ (8)

Is a PDF.

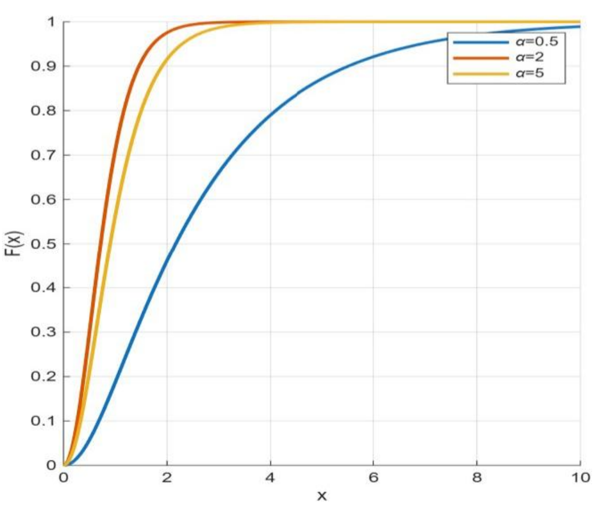


FIGURE (1): $F_{E-OIDD}(y;\alpha,a,b,p)$ of E-OIDD with fixed $a = 2, b = 1, p = 0.5$ and different values of α

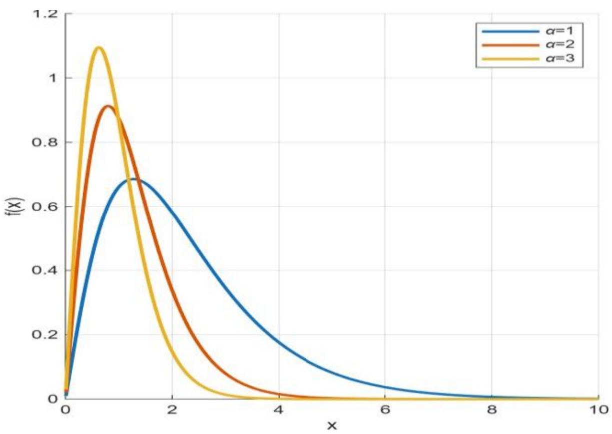


FIGURE (2): $f_{E-OIDD}(y;\alpha,a,b,p)$ of E-OIDD with fixed $\mathrm{a}=2, \mathrm{b}=1, \mathrm{p}=0.5$ and different values of α

2. Some Statistical Properties

In this section, some important mathematical and statistical properties (E-OIDD) like reliability function, hazard function, the moments, moment generating function, the coefficient of variation (CV), skewness (CS) and kurtosis (CK) are discussed.

2.1. Hazard function:

Through the basic kit for studying the longevity and reliability characteristics of a system is the hazard average (HR) function. The HR give the average of failure of the system promptly after time t, the hazard function of the (E-OIDD) is given by:

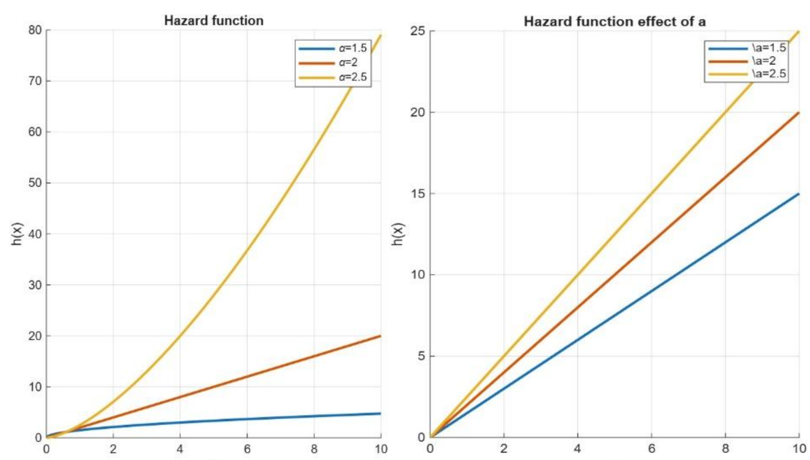
$$h_{E-OIDD}(y;\alpha,a,b,p)=\frac{f_{E-OIDD}(y;\alpha,a,b,p)}{1-F_{E-OIDD}(y;\alpha,a,b,p)}$$

Assume that $R_{E-OIDD}(y;\alpha,a,b,p)=1-F_{E-OIDD}(y;\alpha,a,b,p)$, then

$$R_{E-OIDD}(y;\alpha,a,b,p)=\frac{exp\ (\alpha)}{exp[\alpha(1+(by)^{a})^{p}]}$$

Accordingly;

$$h_{E-OIDD}(y;\alpha,a,b,p)=\alpha p(ab^{a})y^{\alpha-1} \qquad (9)$$



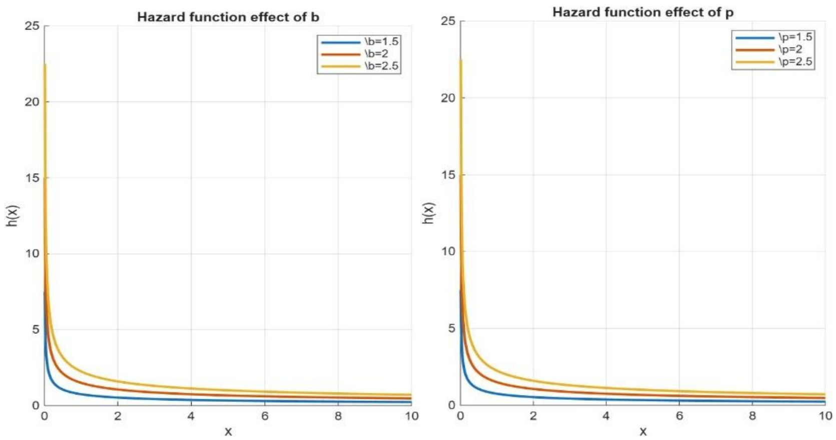


FIGURE (3): $h_{E-OIDD}(y;\alpha,a,b,p)$ of E-OIDD for various parameters

2.2. Reliability function:

The reliability function is a mathematical function which standardize the probability for a system, component, or agent will proceed its purposed function successfully with out defeat for a specific interval of time t declared operating condition.

Furthermore, famous as the survival function and symbolize by R(y) or S(y).

Let Y be a random variable having an (E-OIDD), then its reliability function is given by

$R_{E-OIDD}(y;\alpha,a,b,p) = 1 - F_{E-OIDD}(y;\alpha,a,b,p)$

$$R_{E-OIDD}(y;\alpha,a,b,p) = exp[-\alpha(1+(by)^{a})^{p}+\alpha]\ \exp(\alpha) \tag{10}$$

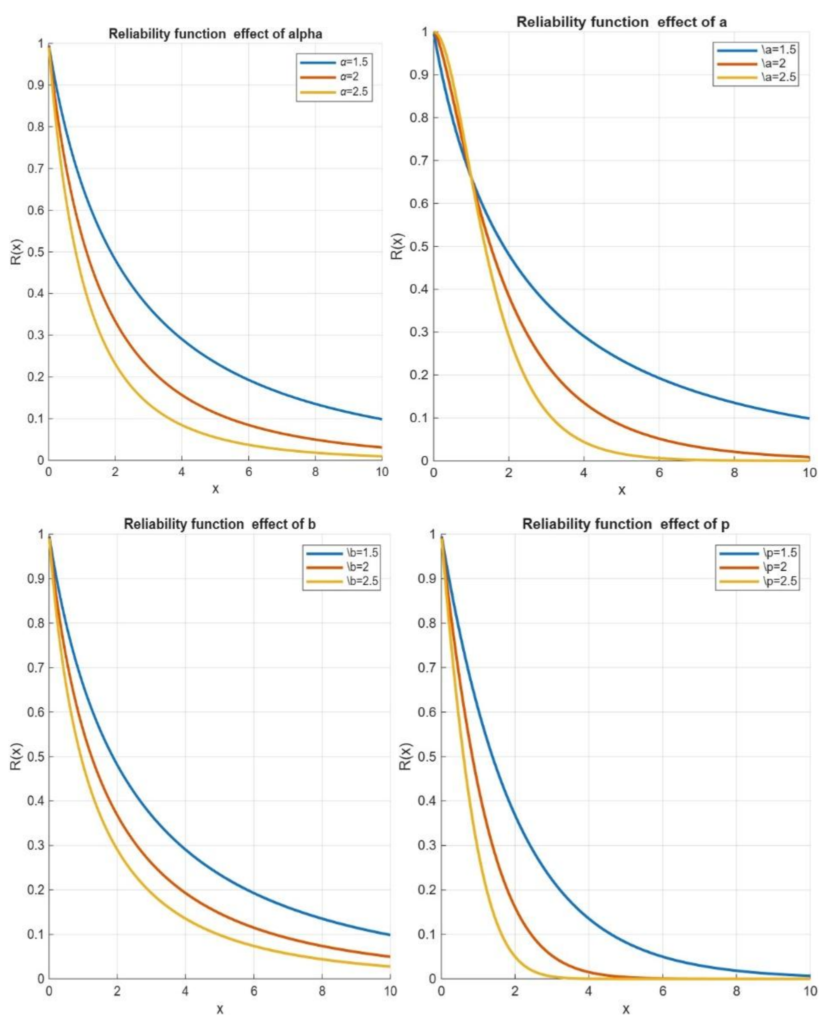


FIGURE (4): $R_{E-OIDD}(y;\alpha,a,b,p)$ of E-OIDD for various parameter

2.3. Moments

The moments of all distributions tell as a lot about its lineaments like mean, variance, skewness, kurtosis, etc. can be spotted out of moments.

Theorem:

The r^{th} moment around zero for (E-OIDD), If the random variable y is distributed as (E-OIDD) with parameters (α, a, b, p), then it defines as:

$$E(y^r) = \alpha P(ab^a)exp(\alpha)\ \sum_{k=0}^{\infty}\binom{\frac{r}{a}}{k}(-1)^{\frac{r}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^k\Gamma\left(\frac{k}{p}+1\right) \quad (11)$$

Proof:

By using equ. (8) the r^{th} moment as follows:

Assume that $W = \alpha P(ab^a)exp(\alpha)$

$E(y^r) = W\int_0^{\infty} y^{r+a-1}\ exp[\ -\alpha(1+(by)^a)^p]dy$

Let $u = \alpha(1+(by)^a)^b$

$\Rightarrow \frac{u}{\alpha} = (1+(by)^a)^p$

$\Rightarrow \sqrt[b]{\frac{u}{\alpha}}=1+(by)^a$

$\Rightarrow \frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}$ -1=$(by)^a$

$\Rightarrow (\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}\ -1)^{\frac{1}{a}} = by$

$\Rightarrow y = \frac{(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}-1)^{\frac{1}{a}}}{b}$

$\Rightarrow dy = \frac{1}{ab}(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}-1)^{\frac{1-a}{\alpha}}\frac{1}{\sqrt[p]{\alpha}}\frac{1}{p}u^{\frac{1-p}{p}}\ du$

$\Rightarrow dy = \frac{1}{abp}\frac{1}{\sqrt[p]{\alpha}}(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}-1)^{\frac{1-a}{\alpha}}\ du$

$= \int_0^{\infty} y^{r+a-1}exp[-\alpha(1+(by)^a)^p]dy$

$= \int_0^{\infty}\left[(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}\ -1)^{\frac{1}{a}}\right]^{r+a-1} exp\ (-u)\frac{1}{abp}\frac{1}{\sqrt[p]{\alpha}}(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}-1)^{\frac{1-a}{\alpha}}\ du$

It follows that:

$\int_0^{\infty}(\frac{1}{\sqrt[p]{\alpha}}\sqrt[p]{u}-1)^{\frac{r}{\alpha}}exp\ (-u)\frac{1}{abp}\frac{1}{\sqrt[p]{\alpha}}du$

Since $s = \frac{r}{a} > 0$, then

$= \int_0^{\infty}\sum_{k=0}^{\infty}\binom{s}{k}\left(\frac{1}{\sqrt[p]{\alpha}}(u)^{\frac{1}{p}}\right)^k(-1)^{s-k}e^{-u}du$

$= \sum_{k=0}^{\infty}\binom{s}{k}(-1)^{s-k}\int_0^{\infty}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^k(u)^{\frac{k}{p}}e^{-u}du$

$= \sum_{k=0}^{\infty}\binom{s}{k}(-1)^{s-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^k\int_0^{\infty}(u)^{\frac{k}{p}}e^{-u}du$

Since, $\int_0^{\infty}(u)^{\frac{k}{p}}e^{-u}du = \Gamma\left(\frac{k}{p}+1\right)$

$= \sum_{k=0}^{\infty}\binom{s}{k}(-1)^{s-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^k\Gamma\left(\frac{k}{p}+1\right)$

This implies that

$$E(y^r) = W\ \sum_{k=0}^{\infty} \binom{\frac{r}{a}}{k} (-1)^{\frac{r}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)$$

$$E(y^r) = \alpha P(ab^a) exp(\alpha)\ \sum_{k=0}^{\infty} \binom{\frac{r}{a}}{k} (-1)^{\frac{r}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)$$

when $r = 1$

$$E(y) = \alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{1}{a}}{k} (-1)^{\frac{1}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)$$

when $r = 2$

$$E(y^2) = \alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{2}{a}}{k} (-1)^{\frac{2}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)$$

when $r = 3$

$$E(y^3) = \alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{3}{a}}{k} (-1)^{\frac{3}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)$$

And the variance of E-OIDD

$$\mathrm{V(x)} = \mu_2 - \mu^2$$

$$= \alpha P(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{2}{a}}{k} (-1)^{\frac{2}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right) - \left(\alpha P(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{1}{a}}{k} (-1)^{\frac{1}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)\right)^2$$

2.4. The Moment Generating Function

Many of the motivating features and advantages of a distribution can be gained its moments and moment generated function like mean, variance, skewness, kurtosis, etc.

Theorem:

The moment generated function for (E-OIDD), If the random variable y is distributed as (E-OIDD) with parameters (α, a, b, p), then it defines as:

$$M_y(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \left[\alpha P(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{i}{a}}{k} (-1)^{\frac{i}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)\right] \quad (12)$$

Proof:

By eq. (8) and definition of moment generating function, we have:

$$M_y(t) = E(e^{ty}) = \int_0^{\infty} e^{ty} \alpha p(ab^a) \exp(\alpha) \exp\left(-\alpha(1+(by)^a)^p\right) y^{\alpha-1}\, dy$$

And by Tayler series

$$M_y(t) = \int_0^{\infty} \left(1 + \frac{ty}{1!} + \frac{(ty)^2}{2!} + \dots\dots + \frac{(ty)^n}{n!} + \cdots\right) \cdot \alpha p(ab^a) \exp(\alpha) \exp\left(-\alpha(1 + (by)^a)^p\right) y^{\alpha-1} dy$$

Then

$$M_y(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \left[\alpha P(ab^a) exp(\alpha)\ \sum_{k=0}^{\infty} \binom{\frac{i}{a}}{k} (-1)^{\frac{i}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)\right]$$

2.5 Coefficients

Coefficient of variation CV

$$CV = \left(\alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{2}{a}}{k} (-1)^{\frac{2}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right) - \left(\alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{1}{a}}{k} (-1)^{\frac{1}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)\right)^2\right)^{\frac{1}{2}} \cdot$$

$$\left(\alpha p(ab^a) exp(\alpha) \sum_{k=0}^{\infty} \binom{\frac{1}{a}}{k} (-1)^{\frac{1}{a}-k} \left(\frac{1}{\sqrt[p]{\alpha}}\right)^k \Gamma\left(\frac{k}{p}+1\right)\right)^{-1}$$

Coefficient of skewness CS:

$$CS = \alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{3}{a}}{k}(-1)^{\frac{3}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)\cdot\left(\alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{2}{a}}{k}(-1)^{\frac{2}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)-\left(\alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{1}{a}}{k}(-1)^{\frac{1}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)\right)^{2}\right)^{\frac{3}{2}}$$

Coefficient of Kurtosis CK

$$CK = \alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{4}{a}}{k}(-1)^{\frac{4}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)\cdot\left(\alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{2}{a}}{k}(-1)^{\frac{2}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)-\left(\alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{1}{a}}{k}(-1)^{\frac{1}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right)\right)^{2}\right)\text{-}3$$

2.6 Order Statistic of E-OIDD

$Z_1 = Min(Y_1, Y_2,\ldots, Y_n)$

$Z_2 = the\ 2nd\ smallest\ of\ Z_1, Z_2, \ldots, Z_n$

$Z_n = Max\ (Y_1, Y_2,\ldots, Y_n)$

The min cdf order statistic

$$F_{z_1}(y) = 1 - (exp[-\alpha(1+(by)^{a})^{p}+\alpha]\ \exp(\alpha))^{n}$$

And the min of pdf order statistic

$$f_{z_1}(y) = n[exp[-\alpha(1+(by)^{a})^{p}+\alpha]\ \exp(\alpha)]^{n-1}\alpha p(ab^{a})\ exp(\alpha)\ exp\ (-\alpha(1+ (by)^{a})^{p})y^{a-1}$$

Then the max cdf order statistic

$$F_{z_n}(y) = [1 - exp[-\alpha(1+(by)^{a})^{p}+\alpha]\ exp\ (\alpha)\ ]^{n}$$

And the max of pdf order statistic

$$f_{z_n}(y) = n[1 - exp[-\alpha(1+(by)^{a})^{p}+\alpha]\ exp\ (\alpha)\ ]^{n-1}\alpha p(ab^{a})\ exp(\alpha)\ exp\ (-\alpha(1+ (by)^{a})^{p})y^{a-1}$$

3. Estimation for Parameters of (E-OIDD) Distribution:

In this section, we depict three estimation methods for estimating the parameters (α, p, a, b) of (E-OIDD). For everyone of methods we take in to account condition when all parameters $\alpha, p, a\ and\ b$ are unknown.

3.1 Method of Moment Estimator (MOM)

The method of moment estimation of (E-OIDD) with four parameters (α, p, a, b) can be gained by equal first three notional moments with the sample moments to estimate three parameters $\alpha, p, and\ b$.

Using eq. (11).

$$\frac{\sum_{i=1}^{n} y_i}{n} = \alpha P(ab^{a})exp(\alpha)\ \sum_{k=0}^{\infty}\binom{\frac{1}{a}}{k}(-1)^{\frac{1}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right) \quad (13)$$

$$\frac{\sum_{i=1}^{n} {y_i}^{2}}{n} = \alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{2}{a}}{k}(-1)^{\frac{2}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right) \quad (14)$$

$$\frac{\sum_{i=1}^{n} {y_i}^{3}}{n} = \alpha P(ab^{a})exp(\alpha)\sum_{k=0}^{\infty}\binom{\frac{3}{a}}{k}(-1)^{\frac{3}{a}-k}\left(\frac{1}{\sqrt[p]{\alpha}}\right)^{k}\Gamma\left(\frac{k}{p}+1\right) \quad (15)$$

3.2 Method of Maximum Likelihood Estimator (MLE):

Maximum likelihood estimation is very accurate method for parameter estimation. For the (E-OIDD) with parameters $\alpha, p, a\ and\ b$, take the size n random sample $(y_1, y_2,\ldots, y_n)$, then the likelihood function to estimate three parameters $\alpha, p, and\ b$. given by:

Using eq. (8).

$$Lf(y,\alpha,p,a,b) = \prod_{i=1}^{n}[\alpha p(ab^{a})\exp(\alpha)\exp(-\alpha(1+(by)^{a})^{p})y^{\alpha-1}]$$

$$= (\alpha p(ab^{a})\exp(\alpha))^{n}\cdot\prod_{i=1}^{n}\alpha p(ab^{a})\exp(\alpha)\exp(-\alpha(1+(by)^{a})^{p})\cdot(\prod_{i=1}^{n} y_i)^{a-1}$$

$$lnLf(y,\alpha,p,a,b) = nln(\alpha pa + b^{a}) + n\alpha - \alpha\sum_{i=1}^{n}(1+(by_i)^{a})^{p} + (a-1)\sum_{i=1}^{n} lny_i$$

$\frac{lnLf(y,\alpha,p,a,b)}{\partial\alpha} = \frac{npa}{\alpha pa+b^a} + n - \sum_{i=1}^{n}(1+(by_i)^a)^p$ (16)

$\frac{lnLf(y,\alpha,p,a,b)}{\partial p} = \frac{n\alpha a}{\alpha pa+b^a} - \alpha\sum_{i=1}^{n}(1+(by_i)^a)^p\, ln(1+(by_i)^a)$ (17)

$\frac{lnLf(y,\alpha,p,a,b)}{\partial b} = \frac{nab^{a-1}}{\alpha pa+b^a} - \alpha pab^{a-1}(y_i)^a\sum_{i=1}^{n}(1+(by_i)^a)^{p-1}$ (18)

3.3 Jackknife Maximum Likelihood Estimator (JMLE):

Joins Maximum likelihood estimation together with jackknife criteria. It is an idea used to decrease or remove the small-sample current bias in criterion Maximum likelihood estimators.

To estimate three parameters $\alpha, p, and\ b$.

Using eq. s. (16), (17), and (18).

$\hat{\alpha}_{JMLE} = n\hat{\alpha}_{MLE} - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{\alpha}_{MLE}(j)$ (19)

s.t. $\hat{\alpha}_{MLE}(j) = \frac{npa}{\alpha pa+b^a} + n - \sum_{i=1,i\neq j}^{n}(1+(by_i)^a)^p = 0$

$\hat{p}_{JMLE} = n\hat{p}_{MLE} - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{p}_{MLE}(j)$ (20)

That is $\hat{p}_{MLE}(j) = \frac{n\alpha a}{\alpha pa+b^a} - \alpha\sum_{i=1,i\neq j}^{n}(1+(by_i)^a)^p\, ln(1+(by_i)^a) = 0$

And

$\hat{b}_{JMLE} = n\hat{b}_{MLE} - \frac{(n-1)}{n}\sum_{j=1}^{n}\hat{b}_{MLE}(j)$ (21)

For

$\hat{b}_{MLE}(j) = \frac{nab^{a-1}}{\alpha pa+b^a} - \alpha pab^{a-1}(y_i)^a\sum_{i=1,i\neq j}^{n}(1+(by_i)^a)^{p-1} = 0$

4.Results and Discussion

4.1. simulation

Here, we proposed numerical computations to contrast the achievement of the estimators conduct in the preceding sections. The generation of the (E-OIDD) with parameters $(\alpha, p, a\ and\ b)$ can be absolutely gained over the transformation $\mathrm{F(y) = U}$ where U is a uniform distribution and $\mathrm{F(y)}$ is cumulative distribution function of (E-OIDD). We investigate three choices of the parameters $(\alpha, p, and\ b)$, the default value for the scale parameter $\alpha = 1.5, 2.5, 3\ when\ a = 4, b = 2, p = 3$, the default values for the shape parameter $p = 1,2,3\ when\ a = 2, b = 0.5, \alpha = 1.5$, and the default values for the scale parameter $b = 1, 3.5, 4, when\ a = 1.5,\ \alpha = 3, p = 2$ with sample size n=10,50,100 for all cases considered. The simulation experiment was repeated (R=100, 500, 1000) we calculate MSE for estimate the parameters with the following form

$$MSE(\hat{\varepsilon}) = \frac{1}{n}\sum_{i=1}^{n}(\hat{\varepsilon}-\varepsilon)^2$$

For each estimate $\hat{\varepsilon} = (\hat{\alpha}, \hat{b}, \hat{p})$

4.2. Results

we present the results of three estimation methods such as maximum likelihood estimator, method of moment estimator and jackknife maximum likelihood estimator to determine three parameters $\alpha, p, and\ b$.The results obtain by using equations (13), (14), (15), (16), (17), (18), (19), (20) and (21).

Results appear in tables (1), (2) and (3).

Table (1) MSE values of scale parameter α

when $a = 4, b = 2, and\ p = 3$ for (E-OIDD)

R	n	MLE	MOM	JMLE	Best
100	10	0.0551423	0.0069107	0.0044561	JMLE
	50	0.0543788	0.0062530	0.0037881	JMLE
	100	0.0456670	0.0061888	0.0022201	JMLE
500	10	0.4322105	0.0881530	0.0421109	JMLE
	50	0.3300971	0.0845108	0.0379104	JMLE

	100	0.3224459	0.0611207	0.0247188	JMLE
1000	10	0.4988801	0.4008137	0.2513299	JMLE
	50	0.4888001	0.3290136	0.2078659	JMLE
	100	0.3998801	0.1880356	0.0887012	JMLE

Table (2) MSE values of shape parameter p

when $a = 2, b = 0.5, and\ \alpha = 1.5$ for (E-OIDD)

R	n	MLE	MOM	JMLE	Best
100	10	0.0547701	0.1567019	0.3498100	MLE
	50	0.0546710	0.0998112	0.1553298	MLE
	100	0.0098760	0.0732345	0.1148013	MLE
500	10	0.1813245	0.4580133	0.5561220	MLE
	50	0.0332451	0.2278012	0.5441809	MLE
	100	0.0313245	0.2226709	0.3784213	MLE
1000	10	0.3569014	0.7023166	0.7390876	MLE
	50	0.2301725	0.5760554	0.3245709	MLE
	100	0.2167890	0.5489076	0.3018067	MLE

Table (3) MSE values of scale parameter b

when $a = 1.5,\ \alpha = 3, p = 2$ for (E-OIDD)

R	n	MLE	MOM	JMLE	Best
100	10	0.0793322	0.0456543	0.4322100	MOM
	50	0.0090676	0.0447321	0.0345489	MLE
	100	0.0088765	0.0239880	0.1754330	MLE
500	10	0.3890876	0.5242378	0.5178001	MLE
	50	0.2467055	0.2909123	0.4566709	MLE
	100	0.0109888	0.0123456	0.0344422	MLE
1000	10	0.1398077	0.2224599	0.4456643	MLE
	50	0.1198700	0.1098988	0.2433866	MOM
	100	0.0119087	0.0999126	0.0908711	MLE

4.3. Discussion

1-From table (1) (the results of scale parameter α showing that (MLE) is the better than all methods in all sample size (n=10,50,100) with all replicated (100, 500, 1000).

2-1-From table (2) (the results of shape parameter p showing that (JMLE) is the better than all methods in all sample size (n=10,50,100) with all replicated (100, 500, 1000).

3-From table (3) (the results of scale parameter b showing that (MLE) is the better than all methods in all sample size (n=10,50,100) with all replicated (100, 500, 1000), unless in (n=10) with replicated (R=100) and in (n=50) with replicated (R=500)

The order priority of methods for tables (1), (2) and (3)

Order	1	2	3
Method	MLE	JMLE	MOM

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