Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Topological Features of Locally Compact Hypergroups

Baraa Abdulhussein Bdaiwi Alhayyawi
The General Directorate of Education in Al-Qadisiyah Governorate. Ministry of Education, Iraq
Corresponding Author: Baraa Abdulhussein Bdaiwi Alhayyawi, **E-mail**: azheralzamily2@gmail.com

| ABSTRACT
In this paper, we give some sufficient conditions for that a continuous function $f: K \to K$ to be an open mapping, where K is a locally compact hypergroup. The obtained results are improvements of some recent facts on locally compact groups.

| KEYWORDS
Open functions, locally compact groups, locally compact hypergroups, convolution, morphisms

| ARTICLE INFORMATION
ACCEPTED: 20 May 2026 **PUBLISHED:** 06 June 2026 **DOI:** 10.32996/jmss.2026.7.5.1

1. Introduction and Preliminaries
Locally compact hypergroups were introduced in the 70s decade in several papers [3, 4, 6, 7], as a generalization of locally compact groups in such way that also in this structure we can study Fourier transforms and harmonic analysis. Indeed, the structure of hypergroups in general is so much more complicated than the structure of locally compact groups, because although one can define the convolution between Dirac measures and measurable functions, but there is not necessarily some action between elements. See the monograph [1] for some important classes of hypergroups with the basic properties of them. In definition of hypergroups two topologies are very important and applicable: (1) the cone topology which is defined on the space of all non-negative regular measures, and (2) the Michael topology which is defined on the space of all non-empty compact subsets of the hypergroup. In this paper, we intend to improve some recent results regarding the open mappings on locally compact groups by extending them to locally compact hypergroups. More precisely, recently the authors in [2] proved that if G, H are locally compact group, and $T: G \to H$ is a continuous homomorphism such that for every open subgroup J of $G, T(J)$ is open in H, then the mapping T is open. Recall that a function f from a topological space X into a topological space Y is called open if the image $f(A)$ of any open subset A of X to be open in Y. They prove this fact by some measure theoretic method, and show that if f is not open, then $\mu(f(U)) = 0$ for all σ-compact $U \subseteq G$, where μ is the left Haar measure of H. Motivated by this very interesting result, in this paper we prove that if K is a locally compact hypergroup with a left Haar measure μ, and $T: K \to K$ is continuous such that

$$T(V)^- = T(V^-), T(x) * T(V) \subseteq T(x * V)$$

for all $x \in K$ and all open set $V \subseteq K$ containing of e in K, then for every open relatively compact set $\emptyset \neq U \subseteq K$ with $\mu(T(U)) > 0$, the image $T(U) \subseteq K$ is open in K. This fact would be an improvement of [2, Lemma 6].

At this part of the paper, for convenience of readers, we recall the definition and some related notations and concepts which we use them throughout this paper.

In the sequel, K is a locally compact and Hausdorff topological space equipped with an involution $x \mapsto x^-$ from K onto K, and has some convolution $*: M(K) \to M(K)$, where $M(K)$ denotes the set of all complex regular measures on K, with the following properties:

1. $(M(K), +, *)$ is a Banach algebra,
2. For every $x, y \in K, \delta_x * \delta_y$ is a probability measure, and $\operatorname{supp}(\delta_x * \delta_y)$ is compact,
3. $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $M^+(K)$ is continuous,
4. $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ on $K \times K$ is continuous,

5. For every $x, y \in K$ we have $(x^-)^- = x$ and $\left(\delta_x * \delta_y\right)^- = \delta_{y^-} * \delta_{x^-}$,
6. There is some $e \in K$ (which is called the identity of K) that for every $x \in K$ we have $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$. Also, $e \in \operatorname{supp}\left(\delta_x * \delta_y\right)$ whenever $x = y^-$for all $x, y \in K$.

In this case, K is called a locally compact hypergroup. Note that in the above sentences, $\operatorname{supp}(\mu)$ is the support of the measure $\mu \in M(K)$, and δ_x is the Dirac measure at the point $x \in K$. Note that a hypergroup K is a locally compact group if and only if $\operatorname{supp}\left(\delta_x * \delta_y\right)$ is singletone for all $x, y \in K$.

In this paper, the convolution of sets is important for us: if $A, B \subseteq K$, then we define

$$A * B := \bigcup_{x \in A, y \in B} \operatorname{supp}\left(\delta_x * \delta_y\right).$$

Also, we denote $A^- := \{x^- : x \in A\}$. Any nonzero nonnegative Radon measure μ on K with

$$\int_K h(t) d\mu(t) = \int_K h(x * y) d\mu(y)$$

for all $f \in C_0(K)$ and $x \in K$, is called a left Haar measure of K, where

$$h(x * y) := \int_K h(t) d\left(\delta_x * \delta_y\right)(t)$$

Thanks to [4, Theorem 4.3C], any hypergroup admits a left sub invariant measure, while this still remains a conjecture that every hypergroup has a left Haar measure. However, by [4] and [6] every commutative, discrete or compact hypergroup possesses a left Haar measure. In fact, in [4, Theorem 7.1A] we see that if K is a discrete hypergroup, then the measure μ defined by

$$\mu(\{x\}) := \frac{1}{(\delta_{x^-} * \delta_x)(\{e\})}, (x \in K)$$

for all $x \in K$, would be a left Haar measure on K.

By [4, Lemma 5.1A], if μ is a left Haar measure for a locally compact hypergroup K, then we have $\operatorname{supp}(\lambda) = K$. This means that for every open non-empty subset U of K we have $\mu(U) > 0$. The convolution of two measurable functions $f, g: K \to \mathbb{C}$ is defined by

$$(f * g)(x) := \int_K f(x * y) g(y) d\mu(y)$$

for all $x \in K$, while this integral exists.

2. Open mappings on hypergroups

For giving the main result of this paper, we need some properties of hypergroups which we mention about them in the following lemmas.

Lemma 2.1. For every open set $W \subseteq K$ containing e there is an open neighborhood V of e in K such that $\bar{V}$ is compact and $V * V \subseteq W$.

Proof. See [4, Lemma 3.2D] and use the locally compactness of K.

For every function $f: K \to \mathbb{C}$ we define the map $f^-: K \to \mathbb{C}$ as $f^-(x) := f(x^-)$for all $x \in K$.

Lemma 2.2. For every compact $E, F \subseteq K$ with $F = F^-$, the function $\chi_E * \chi_F$ is continuous.

Proof. Assume that $E, F \subseteq K$ are compact and F is symmetric. In this case, for each $x \in K$,

$$\chi_F^-(x) = \chi_{F^-}(x) = \chi_F(x)$$

i.e. $\chi_F^- = \chi_F$. On the other hand, $\|\chi_F\|_2^2 = \int_K |\chi_F|^2 d\mu = \int_K \chi_F d\mu = \mu(F) < \infty$, and similarly, $\mu(E) < \infty$, because by compactness of E, F and since μ is regular, the measure of them is finite. Hence, $\chi_E, \chi_F \in L^2(K, \mu)$. Now, by [4, Theorem 5.5D], $\chi_E * \chi_F^- = \chi_E * \chi_F$ is continuous.

Lemma 2.3. For every $A, B, C \subseteq K$ we have $A \cap (B * C) \neq \emptyset$ if and only if $C \cap (B^- * A) \neq \emptyset$.

Proof. See [4, Lemma 4.1B].

Now, we are ready to give the main result of this paper.

1. Theorem 2.4. Assume that K is a locally compact hypergroup which possesses a left Haar measure μ, and $T: K \to K$ is a continuous function such that for each open neighborhood V of e in K, $T(V)^- = T(V^-)$, and
2. $\{T(x)\} * T(V) \subseteq T(\{x\} * V)$ for all $x \in K$.

If $\emptyset \neq U \subseteq K$ is open and relatively compact with $\mu(T(U)) > 0$, then $T(U) \subseteq K$ is open.

Proof. Assume that $\emptyset \neq U \subseteq K$ is open, $\bar{U}$ is compact, and $\mu(T(U)) > 0$. Pick some arbitrary element $x \in U$, and set $z := T(x)$. Since U is open, by [4, Lemma 3.2D], there exists some open set $W \subseteq K$ containing e such that $\{x\} * W \subseteq U$. Now, thanks to Lemma 2.1, there is an open symmetric neighborhood V of e in K such that $\bar{V}$ is compact and $V * V \subseteq W$. This implies that

$$\{x\} * V * V \subseteq U \tag{2.1}$$

Note that $\overline{T(V)} \subseteq T(\bar{V})$, and $T(\bar{V})$ is compact because T is continuous and $\bar{V}$ is compact. Hence, we conclude that $\overline{T(V)}$ is compact. This implies that $\mu(T(V)) \leq \mu(\overline{T(V)}) < \infty$, and so, $\chi_{T(V)} \in L^2(K)$.

Define $f: K \to \mathbb{C}$ as

$$f(t):=\chi_{T(V)} * \chi_{T(V)}(t)=\int_K \chi_{T(V)}(t*u)\chi_{T(V)}(u)d\mu(u)$$

for all $t \in K$. Note that thanks to Lemma 2.2, f is continuous, and we have

$$\{t \in K: f(t)>0\} \subseteq T(V*V) \tag{2.2}$$

In fact, if $t \in K$ and $f(t)>0$, then there exists some $y \in T(V)$ such that

$$(\{t\}*\{y\}) \cap T(V) \neq \emptyset$$

which by Lemma 2.3 implies that

$$x \in T(V)*\{y^-\} \subseteq T(V)*T(V)^- = T(V)*T(V^-) = T(V)*T(V)$$

Hence, $x \in T(V)*T(V)$, and so the inclusion (2.2) holds. We have

$$\begin{aligned} f(e) &= \int_K \chi_{T(V)}(e*u)\chi_{T(V)}(u)d\mu(u) \\ &= \int_K \chi^2_{T(V)}(u)d\mu(u) \\ &= \int_K \chi_{T(V)}(u)d\mu(u) \end{aligned}$$

thus, $f(e)=\mu(T(V))>0$. Therefore, by continuity of f, there exists a neighborhood Y of e in K such that $f(t)>0$ for all $t \in Y$. Then, by the above conclusion, for every $t \in Y$ we have $t \in T(V)*T(V)$, i.e. $Y \subseteq T(V)*T(V)$. Also, by the condition (2) in the hypothesis we have $T(V)*T(V) \subseteq T(V*V)$. This implies that:

$$\{z\}*Y \subseteq \{T(x)\}*T(V*V) \subseteq T(\{x\}*V*V) \subseteq T(U)$$

because $V*V$ is a neighborhood of e too, so, $T(U)$ contains a neighborhood of z. Therefore, $T(U)$ is an open set.

Theorem 2.5. Assume that K is a locally compact hypergroup, and $T: K \to K$ with $T(e)=e$ such that for each open neighborhood V of e in K,

$$\{T(x)\}*T(V) \subseteq T(\{x\}*V)$$

for all $x \in K$. If for every open relatively compact neighborhood U of e in $K, T(U)$ is also open in K, then T is an open mapping.

Proof. Let O be an arbitrary open subset of K, consider some $x \in O$ and set $y:=T(x)$. Since O is open, by [4, Lemma 3.2D], there is an open relatively compact neighborhood V of e in K such that $\{x\}*V \subseteq O$. Hence, by the hypothesis we have

$$\{y\}*T(V)=\{T(x)\}*T(V) \subseteq T(\{x\}*V) \subseteq T(O)$$

i.e. $T(O)$ contains the open neighborhood $y*T(V)$ because $T(V)$ is open and contains e. This means that T is an open mapping.

Theorem 2.6. Assume that K is a locally compact hypergroup which possesses a left Haar measure μ, and $T: K \to K$, with $T(e)=e$, is a continuous function such that for each open neighborhood V of e in K,

(1) $T(V)^- = T(V^-)$, and

(2) $\{T(x)\}*T(V) \subseteq T(\{x\}*V)$ for all $x \in K$.

If there is a compact set $E \subseteq K$ which $\mu(T(E))>0$, then for every open relatively compact neighborhood U of e in K, there exists some $a \in K$ such that $T(\{a\}*U)$ is open in K.

Proof. Assume that $E \subseteq K$ is compact, and $\mu(T(E))>0$. Let $U \subseteq K$ be an arbitrary open relatively compact neighborhood of e in K. We have

$$E \subseteq \bigcup_{x \in E} (\{x\}*U)$$

Hence, since E is compact and for each $x \in E, \{x\}*U$ is open, there are $x_1, \dots, x_n \in E$ such that

$$E \subseteq \bigcup_{j=1}^{n} (\{x_j\}*U)$$

thus,

$$T(E) \subseteq T\left(\bigcup_{j=1}^{n} \{x_j\}*U\right) \subseteq \bigcup_{j=1}^{n} T(\{x_j\}*U)$$

Therefore,

$$0<\mu(T(E)) \leq \mu\left(\bigcup_{i=1}^{n} T(\{x_j\}*U)\right) \leq \sum_{i=1}^{n} \mu\left(T(\{x_j\}*U)\right)$$

Then, for some $k \in \{1, \dots, n\}$ we have $\mu(T(\{x_k\}*U))>0$. Since $\{x_k\}*U$ is open in K, by Theorem 2.4 we conclude that $T(\{x_j\}*U)$ is open in K too, and the proof is complete.

For giving the next result we need to recall the following definition.

Definition 2.7. Define

$$Z(K):=\{x \in K: \operatorname{supp}(\delta_x * \delta_{x^-}) \text{is singleton }\}.$$

This set is called the center of the hypergroup K.

For more details regarding the center of hyoergroups we refer to [4, Section 10.4] and [5]. In fact, one can prove that for every $x \in Z(K)$ and $y \in K, \operatorname{supp}(\delta_x * \delta_y)$ and $\operatorname{supp}(\delta_y * \delta_x)$ are singletons.

Corollary 2.8. Assume that K is a locally compact hypergroup which possesses a left Haar measure μ, and $T: K \to Z(K)$, with $T(e) = e$, is a continuous function such that for each open neighborhood V of e in K,
(1) $T(V)^- = T(V^-)$, and
(2) $T(x) * T(V) = T(x * V)$ for all $x \in K$.
If there is a compact set $E \subseteq K$ which $\mu(T(U)) > 0$, then T is an open map.
Proof. Let $U \subseteq K$ be an arbitrary open relatively compact neighborhood of e in K. Then by Theorem 2.6, there is some $a \in K$ such that $T(\{a\} * U)$ is open in K. But, note that by the hypothesis, $T(\{a\} * U) = \{T(a)\} * T(U)$, and so since $T(a) \in Z(K)$, we have $T(U) = \{T(a)^-\} * T(\{a\} * U)$. Therefore, $T(U)$ is open, and by Theorem 2.5 the proof is complete.
The following fact which was proved in [2] can be concluded directly from the above theorem too.
Corollary 2.9. Assume that K is a locally compact group with a left Haar measure μ, and $T: K \to K$, with $T(e) = e$, is a continuous function such that for each open neighborhood V of e in K,
(1) $T(V)^{-1} = T(V^{-1})$, and
(2) $T(x)T(V) = T(xV)$ for all $x \in K$.
If there is a compact set $E \subseteq K$ which $\mu(T(U)) > 0$, then T is an open map.
Proof. Just note that if K is a locally compact group we have $Z(K) = K$, and in this case the inverse mapping play the role of involution, and the convolution product of two sets would be the product of them.

Funding: This research received no external fundin.
Conflicts of Interest: The authors declare no conflict of interest.

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