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| RESEARCH ARTICLE

Atom-Bond Connectivity Index on a Graph of Finite Free Semilattices

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| ABSTRACT

Let SL_n be the finite free semilattice on $X_n = \{1,2,\dots,n\}$ and Γ_n be the zero-divisor graph of SL_n. In this paper, we find the atom-bond connectivity index, Randić index and harmonic index of Γ_n for $n \geq 3$.

| KEYWORDS

Finite free semilattice, ABC index, Randić index, Harmonic index.

| ARTICLE INFORMATION

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1. Introduction

The zero-divisor graph of a commutative ring is defined by Beck; however, the zero element of the ring is a vertex in Beck's definition [2]. Subsequently, Anderson and Livingston redefined the zero-divisor graph of a commutative ring which is the standard zero-divisor graph of a commutative ring [1]. Let R be a commutative ring and $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph of R is denoted by $\Gamma(R)$. It is an undirected graph with vertex set $Z(R)^* = Z(R)\backslash\{0\}$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. Let S be a commutative semigroup with zero and $Z(S)$ be the set of zero-divisors of S. Demeyer et al. defined the zero-divisor graph of S [3]. The zero-divisor graph of S is denoted by $\Gamma(S)$ and it is an undirected graph with vertex set $Z(S)^* = Z(S)\backslash\{0\}$ and distinct two vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$.

Let G be a graph. A graph G consists of two pairs $(V(G), E(G))$, where the vertex set of G is denoted by $V(G)$, and the edge set of G is denoted by $E(G)$.

Definition 1.1. For any $n+1$ different vertices $u = v_1 - \cdots - v_n - v_{n+1} = v$ in $V(G)$, if there exists an edge $v_i - v_{i+1}$ in $E(G)$for each $1 \leq i \leq n$, then $u = v_1 - \cdots - v_n - v_{n+1} = v$ is called a path between u and v.

Definition 1.2. A graph G is called a connected graph if there exists a path from u to v for every pair of distinct vertices $u, v \in V(G)$. If G does not have any loops or multiple edges, then G is called a simple graph.

In this paper, we only consider simple graphs.

Definition 1.3. The eccentricity of a vertex v in G is denoted by $ecc(G)$ and defined by

$$ecc(v) = \max\{d_G(u,v) : u \in V(G)\}.$$

The diameter of G is denoted by $diam(G)$ and defined by

$$diam(G) = \max\{ecc(v) : v \in V(G)\}.$$

Definition 1.4. The length of the shortest cycle contained in a graph G is called the girth of G and it is denoted by $gr(G)$. Moreover, if G does not contain any cycles, then its girth is defined as infinity.

Definition 1.5. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v, denoted by $deg_G(v)$. Among all degrees, the maximum degree is denoted by $\Delta(G)$, and the minimum degree is denoted by $\delta(G)$.

Definition 1.6. Let C be a non-empty subset of $V\ (G)$. If every two distinct vertices in C are adjacent, then C is called a clique in G. The number of all the vertices in any maximal clique of G is called clique number of G.

Definition 1.7. If we color all the vertices in G with the rule that no two adjacent vertices have the same color, then the minimum number of colors needed to color of G is called chromatic number of G.

Definition 1.8. Let G be a graph and $|V(G)| = n$. The adjacency matrix of G is a binary matrix of order n. The entries of the adjacency matrix are 1 if two vertices are neighbors of each other and 0 otherwise.

Definition 1.9. Let D be a nonempty subset of the vertex set $V(G)$ of G. If, for each $u \in V\ (G)\backslash D$, there exists $v_u \in D$ such that $u - v_u \in E(G)$, then D is called a dominating set. The cardinality of minimum cardinality of a dominating set of G is called dominating number of G.

Definition 1.10. An independent set of a graph G is a subset of vertices $V(G)$ such that no two vertices in the subset represent an edge of G. The cardinality of maximum cardinality of a independent set of G is called independence number of G.

Definition 1.11. An edge coloring of a graph is an assignment of colors to the edges of G such that no two adjacent edges have the same color. The minimum required number of colors for and the edge coloring of G is called the chromatic index of G.

Let $n \in Z^+$ and $X_n = \{1,2, \dots, n\}$. Let SL_n be the set consisting of all subsets of X_n, except the empty set. SL_n is a commutative semigroup of idempotents with the multiplication $A.B = A \cup B$ for $A, B \in SL_n$, and it is called the free semilattice on X_n. The zero-divisor graph of SL_n is denoted by $\Gamma(SL_n)$, and Toker investigated many properties of $\Gamma(SL_n\ )$ for $n \geq 3$ in 2016 [9]. For $n \geq 3$, Toker proved that $\Gamma(SL_n)$ is a connected graph and $diam(\Gamma(SL_n)) = 3$. Moreover, the degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_n)$ have been determined [9]. In this paper, we use Γ_n instead of $\Gamma(SL_n)$ for convenience.

In 1998, Estrada et al. defined the atom-bond connectivity index of a connected graph [4]. Let G be a graph. The atom-bond connectivity index of G is denoted by $ABC(G)$, where

$$ABC(G) = \sum_{u-v \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

In 1975, Milan Randić introduced the Randić index of a graph [7]. For a graph G, the Randić index of G is denoted by $R(G)$ and

$$R(G) = \sum_{u-v \in E(G)} \sqrt{\frac{1}{d_u d_v}}.$$

The harmonic index of a graph was first observed in a paper in 1987 [5].The harmonic index of G is denoted by $H(G)$ and

$$H(G) = \sum_{u-v \in E(G)} \frac{2}{d_u + d_v}.$$

We refer to [6, 8] for other terms in semigroup and graph theories, which are not explained here. In this paper, we calculate the first atom-bond connectivity index, Randić index and harmonic index of Γ_n for $n \geq 3$. In addition, we give an example.

2. Main Results

Let G be graph and $v \in V(G)$. Moreover, let

$$x(v) = \sum_{u\in N(v)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}},$$

$$y(v) = \sum_{u\in N(v)} \sqrt{\frac{1}{d_u d_v}},$$

$$z(v) = \sum_{u\in N(v)} \frac{2}{d_u+d_v}.$$

Thus, it is easy to see that

$$ABC(G) = \sum_{u-v\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}} = \frac{1}{2}\sum_{v\in V(G)} x(v),$$

$$R(G) = \sum_{u-v\in E(G)} \sqrt{\frac{1}{d_u d_v}} = \frac{1}{2}\sum_{v\in V(G)} y(v),$$

$$H(G) = \sum_{u-v\in E(G)} \frac{2}{d_u+d_v} = \frac{1}{2}\sum_{v\in V(G)} z(v).$$

For $n \geq 3$, let $A \in V(\Gamma_n)$ and $|A| = k$ where $1 \leq k \leq n-1$. Then, $deg_{\Gamma_n}(v) = 2^k - 1$, and there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$ [9].

In the following theorem, we give the atom-bond connectivity index of Γ_n for $n \geq 3$.

Theorem 2.1. For $n \geq 3$, we have $ABC(\Gamma_n) = \frac{1}{2}\sum_{k=1}^{n-1}\sum_{r=0}^{k-1}\binom{n}{k}\binom{k}{r}\sqrt{\frac{(2^{n-k+r}-1)+(2^k-1)-2}{(2^{n-k+r}-1).(2^k-1)}}$.

Proof. Let $n \geq 3$ and $A \in V(\Gamma_n)$ where $|A| = k$ for $1 \leq k \leq n-1$. Moreover, let

$$W_r = \{\bar{A} \cup B : A - \bar{A} \cup B \in E(\Gamma_n)\, , \emptyset = B \subsetneq A \ \ and \ \ |B| = r\}$$

for $0 \leq r \leq k-1$. Let $N(A)$ be the set of all adjacent vertices of A in Γ_n. It is easy to see that $W_a \cap W_b = \emptyset$ where $a \neq b$ and $N(A) = \cup_{r=0}^{k-1} W_r$. It is clear that $|W_r| = \binom{k}{r}$, and if $Q \in W_r$, then $deg_{(\Gamma_n)}(Q) = 2^{n-k+r} - 1$. It follows that

$$x(A) = \sum_{r=0}^{k-1}\binom{k}{r}.\sqrt{\frac{(2^{n-k+r}-1)+(2^k-1)-2}{(2^{n-k+r}-1).(2^k-1)}}$$

Since there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$, we conclude that

$$ABC(\Gamma_n) = \frac{1}{2}\sum_{A\in V(\Gamma_n)} x(A) = \frac{1}{2}\sum_{k=1}^{n-1}\sum_{r=0}^{k-1}\binom{n}{k}\binom{k}{r}\sqrt{\frac{(2^{n-k+r}-1)+(2^k-1)-2}{(2^{n-k+r}-1).(2^k-1)}}.$$

In the following theorem, we give the Randić index of Γ_n for $n \geq 3$.

Theorem 2.2. For $n \geq 3$, we have $R(\Gamma_n) = \frac{1}{2}\sum_{k=1}^{n-1}\sum_{r=0}^{k-1}\binom{n}{k}\binom{k}{r}\sqrt{\frac{1}{(2^{n-k+r}-1).(2^k-1)}}$.

Proof. Let $n \geq 3$ and $A \in V(\Gamma_n)$ where $|A| = k$ for $1 \leq k \leq n-1$. Moreover, let

$$W_r = \{\bar{A} \cup B : A - \bar{A} \cup B \in E(\Gamma_n)\, , \emptyset = B \subsetneq A \ \ and \ \ |B| = r\}$$

for $0 \le r \le k-1$. Let $N(A)$ be the set of all adjacent vertices of A in Γ_n. It is easy to see that $W_a \cap W_b = \emptyset$ where $a \neq b$ and $N(A) = \cup_{r=0}^{k-1} W_r$. It is clear that $|W_r| = \binom{k}{r}$, and if $Q \in W_r$, then $deg_{(\Gamma_n)}(Q) = 2^{n-k+r} - 1$. It follows that $y(A) = \sum_{r=0}^{k-1} \binom{k}{r} \sqrt{\frac{1}{(2^{n-k+r}-1).(2^k-1)}}$. Since there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$, we conclude that

$$R(\Gamma_n) = \frac{1}{2} \sum_{A \in V(\Gamma_n)} y(A) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} \binom{k}{r} \sqrt{\frac{1}{(2^{n-k+r}-1).(2^k-1)}}.$$

Theorem 2.3. For $n \ge 3$, we have $H(\Gamma_n) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} \binom{k}{r} \frac{2}{(2^{n-k+r}-1)+(2^k-1)}$.

Proof. As in the proofs of Theorem 2.1 and Theorem 2.2, it is easy to see that

$$z(A) = \sum_{r=0}^{k-1} \binom{k}{r} . \frac{2}{(2^{n-k+r}-1)+(2^k-1)}.$$

It follows that $H(\Gamma_n) = \frac{1}{2} \sum_{A \in V(\Gamma_n)} z(A) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} \binom{k}{r} \frac{2}{(2^{n-k+r}-1)+(2^k-1)}$.

Example 2.4. Let $\Gamma = \Gamma_5$ and A be the adjacency matrix of Γ. Then, we have

$$A = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&0&0&1&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&1&0&1&0&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&1&0&0&1&1 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1&1&0&1 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&1&0&1&0&1&0&1 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&1&1&0&0&1 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&1&1&1&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&1&0&1&1&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&1&1&0&1&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0 \\
0&0&0&0&1&0&0&0&1&0&0&1&0&1&1&0&0&1&0&1&1&0&1&1&1&0&1&1&1&1 \\
0&0&0&1&0&0&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&1&1&0&1&1&1 \\
0&0&1&0&0&0&1&0&0&1&0&0&1&1&0&1&0&0&1&1&0&1&1&0&1&1&1&0&1&1 \\
0&1&0&0&0&1&0&0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&1&0&1&1&1&1&0&1 \\
1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&1&1&1&1&0
\end{bmatrix}.$$

Γ is isomomorphic to the following graph in Figure 1.

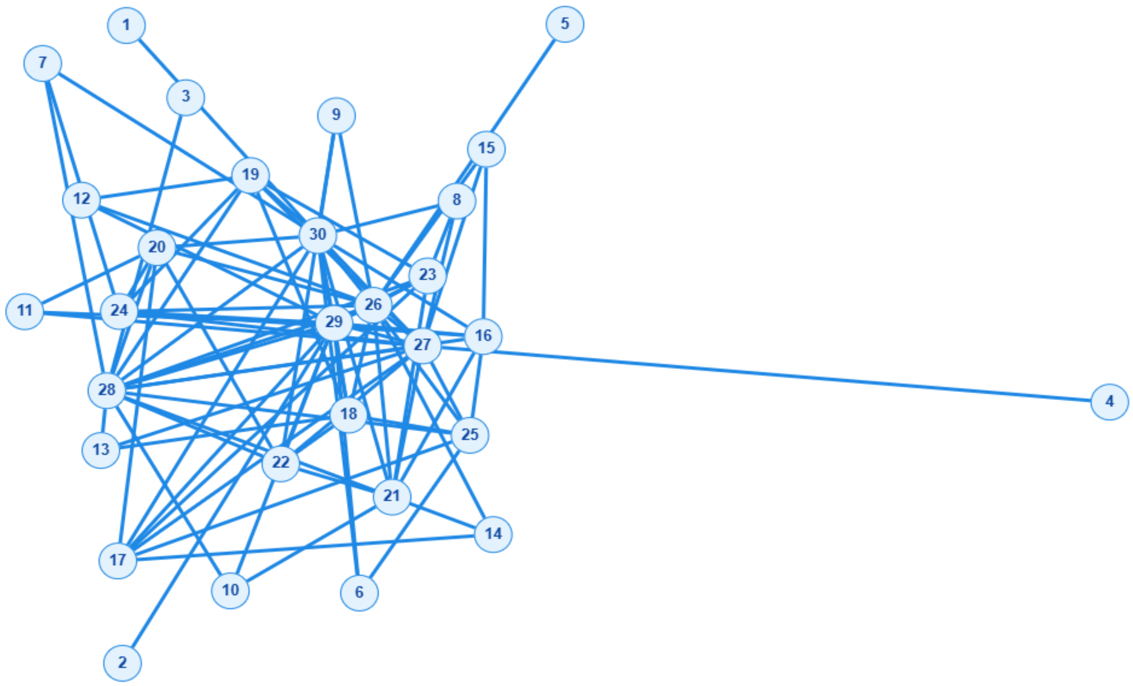

Figure 1

Moreover,

$$ABC(\Gamma) = \frac{1}{2}\sum_{k=1}^{4}\sum_{r=0}^{k-1}\binom{5}{k}\binom{k}{r}\sqrt{\frac{(2^{5-k+r}-1)+(2^{k}-1)-2}{(2^{5-k+r}-1).(2^{k}-1)}} = \frac{5\sqrt{14}}{\sqrt{15}} + \frac{10\sqrt{8}}{\sqrt{21}} + \frac{80}{3\sqrt{5}} + \frac{60}{\sqrt{21}} + \frac{15\sqrt{12}}{7} + \frac{20\sqrt{7}}{15}$$

$$R(\Gamma) = \frac{1}{2}\sum_{k=1}^{4}\sum_{r=0}^{k-1}\binom{5}{k}\binom{k}{r}\sqrt{\frac{1}{(2^{5-k+r}-1).(2^{k}-1)}} = \frac{5}{\sqrt{15}} + \frac{10}{\sqrt{21}} + \frac{20}{\sqrt{45}} + \frac{30}{\sqrt{105}} + \frac{59}{21}$$

$$H(\Gamma) = \frac{1}{2}\sum_{k=1}^{4}\sum_{r=0}^{k-1}\binom{5}{k}\binom{k}{r}\frac{2}{(2^{5-k+r}-1)+(2^{k}-1)} = 2 + \frac{26}{9} + \frac{30}{11} + \frac{5}{8} + \frac{15}{7}.$$

3. Conclusions

Many topological indices have been introduced, and several of them have been found to have various applications. The atom-bond connectivity index and Randić index have received considerable attention in mathematics and chemistry. In this paper, we study the atom-bond connectivity index, Randić index, and harmonic index of zero-divisor graphs of finite free semilattices.

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References

[1] Anderson, D. F. & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring, J. Algebra, 217 (2), 434-447. https://doi.org/10.1006/jabr.1998.7840
[2] Beck, I. (1988). Coloring of commutative rings. J. Algebra, 116 (1), 208-226. https://doi.org/10.1016/0021-8693(88)90202-5
[3] DeMeyer, F. & McKenzie, T. & Schneider, K. (2002). The zero-divisor graph of a commutative semigroup. Semigroup Forum, 65, 206-214. https://doi.org/10.1007/s002330010128
[4] Estrada, E., Torres, L., Rodriguez, L. and Gutman, I. (1998) An Atom-Bond Connectivity Index: Modelling the Enthalpy of Formation of Alkanes. Indian Journal of Chemistry, 37A, 849-855.

[5] Fajtlowicz, S. (1987). On Conjectures of Graffiti-II. Congr. Numer., 60, 187-197.

[6] Howie, J. M. (1995). Fundamentals of Semigroup Theory. New York, NY, USA: Oxford University Press.

[7] Randic, M. (1975) On Characterization of Molecular Branching. Journal of the American Chemical Society, 97, 6609-6615. http://dx.doi.org/10.1021/ja00856a001

[8] Thulasiraman, K. & Arumugam, S. & Brandstädt, A. & Nishizeki, T. (2015). Handbook of Graph Theory, Combinatorial Optimization, and Algorithms. CRC Press: Boca Raton, FL, USA.

[9] Toker, K. (2016). On the zero-divisor graphs of finite free semilattices. Turkish Journal of Mathematics, 40 (4): 824-831. https://doi.org/10.3906/mat-1508-38