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| RESEARCH ARTICLE

Stability and Numerical Simulation of 2D Fractional Pseudo-Hyperbolic Partial Differential Equations

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| ABSTRACT
This study investigates the numerical solution and stability analysis of two-dimensional fractional pseudo-hyperbolic partial differential equations. Recent advancements in fractional calculus have demonstrated its efficacy in modelling complex physical phenomena, particularly through generalized derivatives that capture memory and non-local effects. The primary contribution of this work is the development and analysis of finite difference schemes for solving initial boundary value problems associated with two dimensional fractional pseudo-hyperbolic equations. We propose first-order accurate difference schemes that incorporate Caputo fractional derivatives with order $1< \alpha \leq 2$. Comprehensive error analysis is conducted through numerical simulations, and Von Neumann stability analysis establishes the conditional stability of the proposed numerical method. Experimental validation confirms the accuracy and reliability of the proposed schemes against exact solutions.

| KEYWORDS
Caputo time fractional derivative, Finite difference scheme, Pseudo hyperbolic partial differential equation, Two-dimensional fractional PDE, Von Neumann stability.

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1. Introduction
Fractional partial differential equations represent a significant generalization of classical partial differential equations, extending the mathematical toolkit for modeling phenomena with memory effects, anomalous diffusion, and complex dynamics [1,2]. The science and applications of non-integer integral and differential operators, collectively known as fractional calculus, have gained substantial importance across engineering, physics, and biological modeling. Various definitions of fractional derivatives exist in literature, with Riemann-Liouville, Caputo, Atangana-Baleanu, and Grunwald-Letnikov formulations being among the most commonly employed.

Pseudo-hyperbolic equations constitute a specialized class of hyperbolic partial differential equations characterized by mixed partial derivatives with respect to both temporal and spatial variables. These equations arise in numerous physical contexts, including wave propagation in viscoelastic media, heat transfer with finite propagation speeds, and certain biological transport phenomena. The mathematical treatment of such equations presents unique challenges due to their mixed derivative structure and potential singularities.

Recent literature demonstrates diverse approaches to solving pseudo-hyperbolic equations. Various analytical and numerical methods continue to be explored for these equations [3,4]. In the context of fractional calculus applications, the development of numerical methods for fractional PDEs has seen significant advancement. Finite difference methods have proven particularly effective for fractional diffusion equations [5,6], with stability analyses providing crucial insights into parameter selection [7]. Higher-order schemes have been developed to improve accuracy while maintaining stability [8,9].

The current study distinguishes itself by considering Caputo fractional derivatives in second-order temporal derivatives (1< $\alpha \le$ 2), representing a more complex mathematical formulation with broader physical applicability. This research addresses the following two-dimensional fractional pseudo hyperbolic partial differential equation initial boundary value problem:

$$
\begin{gathered}
\frac{\partial^{\alpha} v(t,x,y)}{\partial t^{\alpha}} = \frac{\partial^{3} v(t,x,y)}{\partial t\,\partial x^{2}} + \frac{\partial^{2} v(t,x,y)}{\partial x^{2}} + \frac{\partial^{3} v(t,x,y)}{\partial t\,\partial y^{2}} + \frac{\partial^{2} v(t,x,y)}{\partial y^{2}} + \Psi(t,x,y), \\
0 < x, y < L, \qquad 0 < t < T, \qquad 1 < \alpha \le 2, \\
v(0,x,y) = u_0(x,y), \;\; \frac{\partial v(t,x,y)}{\partial t} = u_1(x,y), \\
v(t,0,y) = v(t,L,y) = 0, v(t,x,0) = v(t,x,L) = 0,
\end{gathered} \tag{1}
$$

where $u_0(t,x,y), u_1(t,x,y)$, $\Psi(t,x,y)$ are known functions, and $v(t,x,y)$ represents the unknown solution field. The fractional derivative is defined in the Caputo sense, which for $n-1 < \alpha \le \mathrm{n}$ is expressed as:

$$D_t^{\alpha} v(t,x) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{1}{(t-s)^{\alpha-\mathrm{n}+1}} v(s,x) ds,$$

The Caputo fractional derivative offers the significant advantage of accommodating traditional initial and boundary conditions, making it particularly suitable for modeling physical systems with clearly defined initial states [10,11].

Analytical solutions to such equations are generally intractable, necessitating robust numerical approaches. The finite difference method provides a direct and effective discretization strategy that has been successfully applied to various fractional PDEs [12,13]. This study develops first-order finite difference schemes, conducts stability analysis, and validates the approach through numerical experimentation.

2. Finite Difference Scheme and Stability Analysis

2.1 Numerical Discretization

We develop a finite difference scheme to numerically solve the fractional order two-dimensional pseudo-hyperbolic partial differential equation. The computational domain is discretized using a uniform grid $G_{\tau,h} = [0,C]_{\tau} \times [0,L]_h$ with nodal points $t_k = k\tau, x_n = nh$ and $y_m = mh$ where $k = 0,1,2,\dots,N$ and $n, m = 0,1,2,\dots,M$. The spatial step size is h = L/M for both dimensions, and the temporal step size is τ = C/N.

The discrete approximation for the Caputo fractional derivative of order 1< $\alpha \le 2$ employs a first-order finite difference technique [14]:

$${}_0^C D_t^{\alpha} v(t_k, x_n, y_m) = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} ((j+1)^{2-\alpha} - j^{2-\alpha}) \left(v_{n,m}^{k-j+1} - 2v_{n,m}^{k-j} + v_{n,m}^{k-j-1}\right).$$

For notational convenience, we define $b_j^{\alpha} = (j+1)^{2-\alpha} - j^{2-\alpha}$.

The third-order mixed derivatives are discretized using first-order approximations:

$$\frac{\partial^3 v(t,x,y)}{\partial t\,\partial x^2} = \frac{1}{\tau}\left(\frac{v_{n-1,m}^{k} - 2v_{n,m}^{k+1} + v_{n+1,m}^{k}}{h^2} - \frac{v_{n-1,m}^{k-1} - 2v_{n,m}^{k} + v_{n+1,m}^{k-1}}{h^2}\right)$$

$$\frac{\partial^3 v(t,x,y)}{\partial t\,\partial y^2} = \frac{1}{\tau}\left(\frac{v_{n,m-1}^{k} - 2v_{n,m}^{k+1} + v_{n,m+1}^{k}}{h^2} - \frac{v_{n,m-1}^{k-1} - 2v_{n,m}^{k} + v_{n,m+1}^{k-1}}{h^2}\right)$$

Second-order spatial derivatives utilize standard central differences [15]:

$$\frac{\partial^2 v(t,x,y)}{\partial x^2} = \frac{v_{n+1,m}^{k} - 2v_{n,m}^{k} + v_{n-1,m}^{k}}{h^2}$$

$$\frac{\partial^2 v(t,x,y)}{\partial y^2} = \frac{v_{n,m+1}^{k} - 2v_{n,m}^{k} + v_{n,m-1}^{k}}{h^2}$$

Combining these discretization’s yields the complete discrete formulation of model (1):

$$\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1} b_j^{\alpha}\left(v_{n,m}^{k-j+1}-2v_{n,m}^{k-j}+v_{n,m}^{k-j-1}\right)$$
$$=\frac{1}{\tau}\left(\frac{v_{n-1,m}^{k}-2v_{n,m}^{k+1}+v_{n+1,m}^{k}}{h^2}-\frac{v_{n-1,m}^{k-1}-2v_{n,m}^{k}+v_{n+1,m}^{k-1}}{h^2}\right)$$
$$+\frac{1}{\tau}\Big(\frac{v_{n,m-1}^{k}-2v_{n,m}^{k+1}+v_{n,m+1}^{k}}{h^2}\frac{v_{n,m-1}^{k-1}-2v_{n,m}^{k}+v_{n,m+1}^{k-1}}{h^2}+\frac{v_{n+1,m}^{k}-2v_{n,m}^{k}+v_{n-1,m}^{k}}{h^2}+\frac{v_{n,m+1}^{k}-2v_{n,m}^{k}+v_{n,m-1}^{k}}{h^2}$$
$$+\Psi_{n,m}^{k}$$

After algebraic rearrangement, we obtain the compact form:

$$\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{k-1} b_j^{\alpha}\left(v_{n,m}^{k-j+1}-2v_{n,m}^{k-j}+v_{n,m}^{k-j-1}\right)+\frac{1}{\tau h^2}v_{n+1,m}^{k-1}+\frac{1}{\tau h^2}v_{n,m+1}^{k-1}+\frac{1}{\tau h^2}v_{n,m-1}^{k-1}+\frac{1}{\tau h^2}v_{n-1,m}^{k-1}+\left(-\frac{1}{\tau h^2}-\frac{1}{h^2}\right)v_{n+1,m}^{k}$$
$$+\left(-\frac{1}{\tau h^2}-\frac{1}{h^2}\right)v_{n-1,m}^{k}+\left(-\frac{1}{\tau h^2}-\frac{1}{h^2}\right)v_{n,m-1}^{k}+\left(-\frac{1}{\tau h^2}-\frac{1}{h^2}\right)v_{n,m+1}^{k}+\left(\frac{4}{\tau h^2}+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\right)v_{n,m}^{k+1}+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}v_{n,m}^{k-1}$$
$$+\left(\frac{4}{h^2}-\frac{4}{\tau h^2}+\frac{-2\tau^{-\alpha}}{\Gamma(3-\alpha)}\right)v_{n,m}^{k}=\Psi_{n,m}^{k} \tag{2}$$

2.2 Matrix Formulation

The difference scheme can be expressed in matrix form as:

$$\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=1}^{k-1} b_j^{\alpha}\left(v_{n,m}^{k-j+1}-2v_{n,m}^{k-j}+v_{n,m}^{k-j-1}\right)+PW_{n+1}+QW_n+RW_{n-1}=S\Phi_n, \tag{3}$$

$$0\le n\le N, W_0=\bar{0}, W_M=\bar{0}, W_{N,M}^{0}=u_0(x_n,y_m),$$

where P, Q, R, and S are (N+1)(M+1)×(N+1)(M+1) matrices with specific block structures, defined as follows. Here, P, Q, R and S are (N+1)(M+1)×(N+1)(M+1) matrices.

$$P=R=\begin{bmatrix}\bar{0}&\bar{0}&\bar{0}&\cdots&\bar{0}&\bar{0}&\bar{0}\\ P_2&P_1&\bar{0}&\cdots&\bar{0}&\bar{0}&\bar{0}\\ \bar{0}&P_2&P_1&\cdots&\vdots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\cdots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\cdots&P_2&P_1&\bar{0}\\ \bar{0}&\bar{0}&\bar{0}&\cdots&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}_{(N+1)(M+1)\times(N+1)(M+1)}$$

where $P_1=\begin{bmatrix}0&0&\cdots&0&0\\0&p_1&\cdots&0&0\\0&0&\cdots&\vdots&\vdots\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\cdots&\vdots&\vdots\\ \vdots&\vdots&\cdots&p_1&\vdots\\0&0&\cdots&0&0\end{bmatrix}_{(M+1)\times(M+1)}$, $P_2=\begin{bmatrix}0&0&\cdots&0&0\\0&p_2&\cdots&0&0\\0&0&\cdots&\vdots&\vdots\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\cdots&\vdots&\vdots\\ \vdots&\vdots&\cdots&p_2&\vdots\\0&0&\cdots&0&0\end{bmatrix}_{(M+1)\times(M+1)}$ and $p_1=-\frac{1}{\tau h^2}-\frac{1}{h^2}, p_2=\frac{1}{\tau h^2}$.

Q is defined as

$$Q=\begin{bmatrix}Q_1&\bar{0}&\bar{0}&\cdots&\bar{0}&\bar{0}&\bar{0}\\ Q_2&Q_3&Q_4&\cdots&\bar{0}&\bar{0}&\bar{0}\\ \bar{0}&Q_2&Q_3&Q_4&\vdots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\cdots&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\cdots&Q_2&Q_3&Q_4\\ -Q_1&Q_1&\bar{0}&\cdots&\bar{0}&\bar{0}&\bar{0}\end{bmatrix}_{(N+1)(M+1)\times(N+1)(M+1)}$$

with $Q_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$, $Q_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ p_1 & q_1 & p_1 & \cdots & 0 & 0 & 0 \\ 0 & p_1 & q_1 & p_1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & p_1 & q_1 & p_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$,

$Q_3 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ p_1 & q_2 & p_1 & \cdots & 0 & 0 & 0 \\ 0 & p_1 & q_2 & p_1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & p_1 & q_2 & p_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$, $Q_4 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & q_3 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & q_3 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)}$ and $q_1 = \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}$, $q_2 = \frac{4}{h^2} - \frac{4}{\tau h^2} + \frac{-2\tau^{-\alpha}}{\Gamma(3-\alpha)}$,

$q_3 = \frac{4}{\tau h^2} + \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}$. F corresponds to the first term of (3)

$$F = \begin{bmatrix} \bar{0} & \bar{0} & \bar{0} & \bar{0} & \dots & \dots & \bar{0} \\ F_1 & -2F_1 & F_1 & \bar{0} & \dots & \dots & \bar{0} \\ F_2 & F_1 - 2F_2 & -2F_1 + F_2 & F_1 & \dots & \dots & \bar{0} \\ F_3 & F_1 - 2F_3 & F_1 - 2F_2 + F_3 & -2F_1 + F_2 & F_1 & \vdots & \bar{0} \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \bar{0} \\ \vdots & \vdots & \vdots & \cdots & \dots & \dots & \bar{0} \\ \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{0} & \bar{0} & \bar{0} \end{bmatrix}_{(N+1)(M+1)\times(N+1)(M+1)}$$

where $F_j = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & q_1 b_j & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & q_1 b_j & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}$, and $b_j^\alpha = (j+1)^{2-\alpha} - j^{2-\alpha}$. S is the identity matrix and $\Phi_n$is (N+1)(M+1) ×1 column vector.

2.3 Stability Analysis

The stability characteristics of the numerical scheme are established through the following theorem:

Theorem 2.1. The finite difference scheme (2) is stable provided that the following condition is satisfied:

$$\mathrm{h}^{\mathbf{2}} < \Gamma(3-\alpha)(1+8\tau)\tau^{\boldsymbol{\alpha}-\mathbf{1}}. \qquad (4)$$

Proof. Applying Von Neumann stability analysis to equation (2) under the stated condition yields the required stability estimates. Substituting the Fourier mode $v_{n,m}^k = \xi^k e^{i(\beta nh + \beta mh)}$ into the discretized equation and analyzing the amplification factor ξ demonstrates that $|\xi| \leq 1$ when condition (4) holds, ensuring numerical stability [16,17].

3. Numerical Results and Discussion

To validate the proposed numerical scheme, we consider the following two-dimensional pseudo-hyperbolic equation with fractional-order Caputo derivative:

$${}_0^C D_t^\alpha v(t,x,y) = v_{txx}(t,x,y) + v_{tyy}(t,x,y) + v_{xx}(t,x,y) + v_{yy}(t,x,y) + f(t,x,y) \qquad (5)$$

where the source term is defined as:

$$f(t,x,y) = \left(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + 2t^3 + 6t^2 + 2\right) sin(x) \sin(y)$$

with parameter range 1< $\alpha \leq 2$. The initial conditions are:

$$v(0,x,y) = sin(x)\sin(y),\, v_t(0,x,y) = 0, \;\; 0 \leq x,y \leq \pi,$$

with boundary conditions:

$$v(t,0,y) = v_t(t,\pi,y) = 0, \qquad 0 \leq t \leq 1, 0 \leq y \leq \pi,$$
$$v(t,x,0) = v_t(t,x,\pi) = 0, \qquad 0 \leq t \leq 1, 0 \leq x \leq \pi.$$

The analytical solution to equation (5) is:

$$v(t,x,y) = (t^3 + 1)sin(x)\sin(y). \quad (6)$$

Numerical solutions for Problem(5) are obtained by implementing the finite difference scheme (2). The resulting linear system is solved using a modified Gauss elimination method implemented in Matlab. Solution accuracy is assessed through the maximum norm error:

$$\varepsilon = \max_{\substack{1\le k\le N-1, \\ 1\le n\le M-1 \\ 1\le m\le M-1}} |v_{num}(t_k, x_n, y_m) - v_{exact}(t,x,y)|$$

where $v_{num}(t_k, x_n, y_m)$ represents the numerical approximation and $v_{exact}(t,x,y)$ denotes the exact solution (6).

t, x, y	Exact Solution	Approximate Solution	Error Values
t=0.010, x,y=π/20	2.0000	1.4841	0.5159
t=0.011, x,y=π/18	2.0000	1.5165	0.4835
t=0.013, x,y=π/15	1.9781	1.5577	0.4204
t=0.015, x,y=π/13	1.9709	1.5990	0.3719
t=0.020, x,y=π/10	2.0000	1.7145	0.2855
t=0.022, x,y=π/9	1.9397	1.7003	0.2393
t=0.033, x,y=π/6	2.0000	1.9157	0.0843
t=0.040, x,y=π/5	1.8090	1.8088	0.0081

Table 1. Error analysis for problem (5) with α =1.7

The error analysis presented in Table 1 provides strong evidence of the proposed scheme's convergence characteristics. Consistent error reduction is observed as spatial and temporal coordinates increase, with error magnitudes showing a clear decreasing trend from 0.5159 at the finest grid to 0.0081. The decreasing error reflects the specific sampling locations and the behavior of the solution surface. Even with relatively coarse discretization parameters, the method maintains satisfactory accuracy, with errors remaining below 0.52 across all sampled points and dropping below 0.01 for certain configurations. The consistent error pattern across different spatial and temporal coordinates demonstrates the scheme's robustness and reliability in capturing solution behavior throughout the domain. The observed error behavior validates the theoretical stability analysis and confirms the practical utility of the proposed numerical approach for solving two dimensional fractional pseudo-hyperbolic equations.

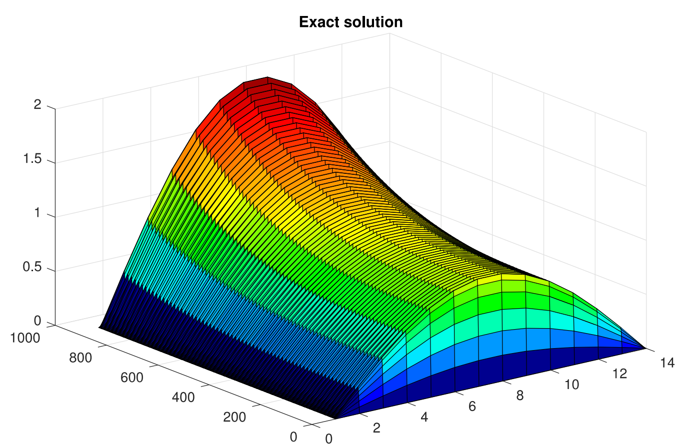

Figure 1. Exact solution surface for α =1.7, t=0.015, and x, y= π/13.

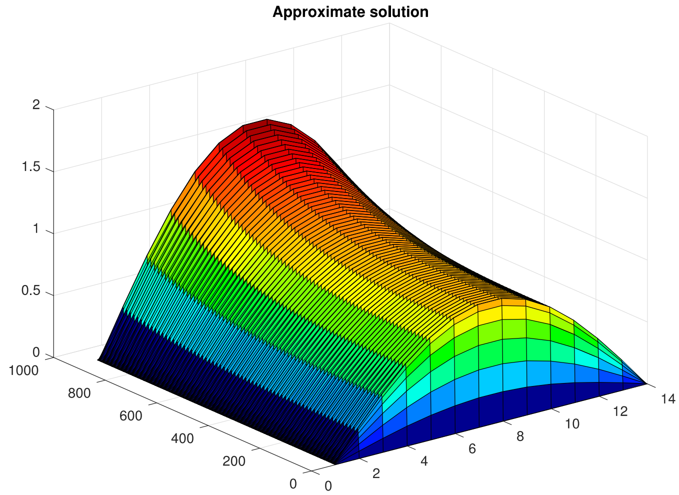

Figure 2. Approximate solution surface for α =1.7, t=0.015, and x, y= π/13.

Figure 1 and 2 provide visual confirmation of the numerical scheme's accuracy. The exact solution surface displays the characteristic sinusoidal behavior in both spatial dimensions, modulated by the temporal polynomial factor $(t^3 + 1)$. The approximate solution closely replicates this structure, with visible but minimal deviations that correspond to the error magnitudes reported in Table 1. The approximate solution maintains the smoothness properties of the exact solution, without introducing oscillations or discontinuities that might indicate numerical instability. The visual similarity between figures extends throughout the entire domain, confirming that the numerical method provides globally accurate approximations rather than local accuracy at specific points. The visual comparison supports the quantitative error analysis, indicating that the finite difference scheme accurately reproduces both the qualitative behavior and quantitative values of the exact solution.

4. Conclusion

This study has presented a comprehensive numerical framework for solving two-dimensional fractional pseudo-hyperbolic partial differential equations with Caputo time derivatives of order $1< \alpha \leq 2$. We developed a first-order accurate finite difference scheme that effectively handles the two dimensional structure of pseudo-hyperbolic equations while incorporating fractional temporal derivatives. The scheme's formulation accounts for the non-local nature of fractional derivatives through summation terms that capture memory effects.

Through Von Neumann analysis, we established the conditional stability of the proposed numerical method that ensures reliable numerical solutions.

Numerical experiments conducted on a test problem demonstrated the scheme's accuracy and convergence characteristics. Error analysis revealed consistent error reduction and satisfactory accuracy.

The successful application of finite difference methods to fractional pseudo-hyperbolic equations expands the available toolkit for solving these challenging problems, complementing existing approaches like finite element and spectral methods. The demonstrated accuracy and convergence support the use of this approach for practical engineering and scientific applications where fractional pseudo-hyperbolic models are applicable.

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