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| RESEARCH ARTICLE

Applications of Pre-open & α-open Sets in Bitopological Spaces

Md Moiz Ashraf
Assistant Professor, P.G. Department of Mathematics, Karim City College, Jamshedpur, India
Corresponding Author: Md Moiz Ashraf, **E-mail**: mdmoizashraf@gmail.com

| ABSTRACT
Using the concept of pre-open set & α-open set in bitopological spaces, we introduce and study topological properties of pre-limit points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, α -limit points, α -derived sets, α -interior and α -closure of a set and α -interior points in a bitopological space.

| KEYWORDS

$ij-p-limit$ points, $ij-p-derived$ set, $ij-p-interior$, $ij-p-closure$, $ij-\alpha-limit\ point$, $ij-\alpha-derived$, $ij-\alpha-interior$, $ij-\alpha-closure$

| ARTICLE INFORMATION

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1. Introduction

Kelly [1] in 1963, introduced the bitopological space (X, T_1, T_2) where X is a nonempty set, $T_1$and T_2 are topologies on X. The notion of pre-open set in a topological space was introduced by Mashhour et al. [3] In 2005 Alaa Erees studied Pre-open set in Bitopological Spaces [10]. Csaszar [4] in 1997, defined generalized open sets in generalized topological spaces. Levine [2] in 1963, introduced the notion of semi-open sets in bitopological spaces. In 2011, N. Rajesh [5] defined preopen sets in Ideal Bitopological Spaces. In 2018 S. Hussain [6] defined Generalized Open Sets. In 2008 Young Bae Jun, Seong Woo Jeong, Hyeon Jeong Lee and Joon Woo Lee [8] defined Pre-limit point, pre-derived set, Pre-interior, Pre-closure, Pre- interior points, Pre-border, Pre-frontier, Pre-exterior points in topological spaces.

Preliminaries: Throughout the paper (X, T_1, T_2) will represent a bitopological space and for any subset A of X, by $T_i - cl(A)$ & $T_i - int(A)$ we denote Closure and Interior of A with respect the topology T_i; $i = 1,2$ and $i \neq j$

Definition 1: [9] A subset A of X, where (X, T_1, T_2) is a bitopological space, is defined to $be\ ij - pre - open$ (in brief $ij - p - open$) if $A \subseteq T_i - int\ (T_j - cl\ (A))$ & complement of $ij - pre - open$ set is called $ij - pre - closed$ (in brief $ij - p - closed$) where $i,\ j = 1, 2$ and $i \neq j$.

Definition 2: [11] A subset A of a bitopological space (X, T_1, T_2) is said to be $ij - \alpha - open$ if $A \subseteq T_i - int(T_j - cl\ (T_i - int(A)))$ & complement of $ij - \alpha - open$ set is called $ij - \alpha - closed$ where $i,\ j = 1, 2$ and $i \neq j$.where $i,\ j = 1, 2$ and $i \neq j$.

We denote the family of $ij -\ p - open$ sets by T_{ij}^{po} and the family of $ij -\ p - closed$ sets by T_{ij}^{pc}

We denote the family of $ij - \alpha - open$ sets by $T_{ij}^{\alpha o}$ and the family of $ij -\ \alpha - closed$ sets by $T_{ij}^{\alpha c}$

Example 1:

Let (X, T_1, T_2) be a bitopological space, where $X = \{a, b, c, d\}$,

$T1 = \{\emptyset, \{b\}, \{b, c\}, \{a, d\}, \{a, b, d\}, X\}$

$T2 = \{\emptyset, \{b\}, \{d\}, \{b, d\}, X\}$.

Then T_1-closed sets: $\emptyset, \{c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, X$.

& T_2-closed sets: $\emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X$.

We now present the following table for the determination of $ij - p - open$ sets & $ij - \alpha - open$ sets.

A⊆X	**T₂-cl(A)**	**T₁-int(T₂-cl(A))**	**T₁-cl(A)**	**T₂-int(T₁-cl(A))**	**T₁-int(T₂-cl(T₁-int (A)))**	**T₂-int(T₁-cl(T₂-int (A)))**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a, c\}$	$\emptyset$	$\{a, d\}$	$\{d\}$	$\emptyset$	$\emptyset$
$\{b\}$	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b\}$	$\{b, c\}$	$\{b\}$
$\{c\}$	$\{a, c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{d\}$	$\{a, c, d\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$	$\emptyset$	$\{d\}$
$\{a, b\}$	$\{a, b, c\}$	$\{b, c\}$	X	X	$\{b, c\}$	$\{b\}$
$\{a, c\}$	$\{a, c\}$	$\emptyset$	$\{a, c, d\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a, d\}$	$\{a, c, d\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$	$\{a, d\}$	$\{d\}$
$\{b, c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b\}$	$\{b, c\}$	$\{b\}$
$\{b, d\}$	X	X	X	X	$\{b, c\}$	X
$\{c, d\}$	$\{a, c, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\emptyset$	$\emptyset$	$\{d\}$
$\{a, b, c\}$	$\{a, b, c\}$	$\{b, c\}$	X	X	$\{b, c\}$	$\{b\}$
$\{a, b, d\}$	X	X	X	X	X	X
$\{a, c, d\}$	$\{a, c, d\}$	$\{a, d\}$	$\{a, c, d\}$	$\emptyset$	$\{a, d\}$	$\{d\}$
$\{b, c, d\}$	X	X	X	X	$\{b, c\}$	X
X	X	X	X	X	X	X

$T_{12}^{po} = \{\emptyset, \{b\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$

$T_{12}^{pc} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

$T_{21}^{po} = \{\emptyset, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$

$T_{21}^{pc} = \{\emptyset, \{a\}, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\}$

$T_{12}^{\alpha o} = \{\emptyset, \{b\}, \{a, d\}, \{b, c\}, \{a, b, d\}, X\}$

$T_{12}^{\alpha c} = \{\emptyset, \{c\}, \{a, d\}, \{b, c\}, \{a, c, d\}, X\}$

$T_{21}^{\alpha o} = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$

$T_{21}^{\alpha c} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

The following observations are worth noting:

(a) Every $T_i - open$ set is $ij - p - open$ or every $T_i - closed$ set is $ij - p - closed$.
(b) Every $T_i - open$ set is $ij - \alpha - open$ or every $T_i - closed$ set is $ij - \alpha - closed$.
(c) Every $ij - \alpha - open$ set is an $ij - p - open$. (i.e. $T_{ij}^{\alpha o} \subseteq T_{ij}^{po}$)
(d) Every $ij - \alpha - closed$ set is an $ij - p - closed$. (i.e. $T_{ij}^{\alpha c} \subseteq T_{ij}^{pc}$)
(e) Arbitrary union of $ij - p - open$ sets is $ij - p - open$. Also, arbitrary intersection of $ij - p - closed$ sets is $ij - p - closed$.
(f) Arbitrary union of $ij - \alpha - open$ sets is $ij - \alpha - open$. Also, arbitrary intersection of $ij - \alpha - closed$ sets is $ij - \alpha - closed$.
(g) The intersection of two $ij - p - open$ subsets may not be ij$-p - open$.
(h) The union of two $ij - p - closed$ subsets may not be $ij - p - closed$.
(i) The intersection of two $ij - \alpha - open$ subsets is $ij - \alpha - open$.
(j) The union of two $ij - \alpha - closed$ subsets is $ij - \alpha - closed$.

Definition 3: Let (X, T_1, T_2) a bitopological space. A point $x \in X$ is said to be $ij - p - limit\ point$ of a subset A of X if for every $ij - p - open$ set G containing x contains a point of A other than x.

i.e. $\forall$ G $\in T_{ij}^{po}$, $x \in G \Rightarrow G \cap (A \backslash \{x\}) \neq \emptyset$

Definition 4: The set of all $ij-p-limit$ points of A is called $ij-p-derived$ set of A. It is denoted by $D_{ij}^{p}(A)$, $i,j=1,2$

Definition 5: Let (X,T_1,T_2) a bitopological space. A point $x\in X$ is said to be $ij-p-interior\,point$ of a subset A of X there exists an $ij-p-open$ set G such that $x\in G\subseteq A$

Definition 6: The set of all $ij-p-interior$ points of A is called $ij-p-interior$ of A. It is denoted by $Int_{ij}^{p}(A)$, $i,j=1,2$

In other words, the largest $ij-\ p-open$ set contained in A is $Int_{ij}^{p}(A)$.

Definition 7: Let (X,T_1,T_2) a bitopological space and $A\subseteq X$. Then the set $A\cup D_{ij}^{p}(A)$ is called $ij-p-closure$ of A. It is denoted by $Cl_{ij}^{p}(A)$

In other words, the smallest $ij-p-closed$ set containing A is $Cl_{ij}^{p}(A)$.

Definition 8: Let (X,T_1,T_2) a bitopological space. A point $x\in X$ is said to be $ij-\alpha-limit\,point$ of a subset A of X if for every $ij-\alpha-open$ set G containing x contains a point of A other than x.

i.e. $\forall$ G $\in T_{ij}^{\alpha o}$, $x\in G\Rightarrow G\cap(A\backslash\{x\})\neq\emptyset$

Definition 9: The set of all $ij-\alpha-limit$ points of A is called $ij-\alpha-derived$ set of A. It is denoted by $D_{ij}^{\alpha}(A)$, $i,j=1,2$

Definition 10: Let (X,T_1,T_2) a bitopological space. A point $x\in X$ is said to be $ij-\alpha-interior\,point$ of a subset A of X there exists an $ij-\alpha-open$ set G such that $x\in G\subseteq A$.

Definition 11: The set of all $ij-\alpha-interior$ points of A is called $ij-\alpha-interior$ of A. It is denoted by $Int_{ij}^{\alpha}(A)$, $i,j=1,2$.

In other words, the largest $ij-\ \alpha-open$ set contained in A is $Int_{ij}^{\alpha}(A)$.

Definition 12: Let (X,T_1,T_2) a bitopological space and $A\subseteq X$. Then the set $A\cup D_{ij}^{\alpha}(A)$ is called $ij-\alpha-closure$ of A. It is denoted by $Cl_{ij}^{\alpha}(A)$.

In other words, the smallest $ij-\alpha-closed$ set containing A is $Cl_{ij}^{\alpha}(A)$.

Example 2:

Let (X,T_1,T_2) be a bitopological space, where $X=\{a,b,c,d\}$,

$T_1=\{\emptyset,\{a\},\{a,b\},\{c,d\},\{a,c,d\},X\}$ and $T_2=\ \{\emptyset,\{b\},\{c\},\{b,c\},X\}$.

Then $\quad T_1$– $closed$ sets: $\emptyset,\{b\},\{a,b\},\{c,d\},\{b,c,d\},X$.

& $\quad T_3$– $closed$ sets: $\emptyset,\{a,d\},\{a,b,d\},\{a,c,d\},X$.

We now present the following table for the determination of ij–pre-open sets.

A⊆X	T_2-cl(A)	T_1-int(T_2-cl(A))	T_1-cl(A)	T_2-int(T_1-cl(A))	T_1-int(T_2-cl(T_1-int (A)))	T_2-int(T_1-cl(T_2-int (A)))
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	$\emptyset$
$\{b\}$	$\{a,b,d\}$	$\{a,b\}$	$\{b\}$	$\{b\}$	$\emptyset$	$\{b\}$
$\{c\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{c,d\}$	$\{c\}$	$\emptyset$	$\{c\}$
$\{d\}$	$\{a,d\}$	$\{a\}$	$\{c,d\}$	$\{c\}$	$\emptyset$	$\emptyset$
$\{a,b\}$	$\{a,b,d\}$	$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$
$\{a,c\}$	$\{a,c,d\}$	$\{a,c,d\}$	X	X	$\{a\}$	$\{c\}$
$\{a,d\}$	$\{a,d\}$	$\{a\}$	X	X	$\{a\}$	$\emptyset$
$\{b,c\}$	X	X	$\{b,c,d\}$	$\{b,c\}$	$\emptyset$	$\{b,c\}$
$\{b,d\}$	$\{a,b,d\}$	$\{a,b\}$	$\{b,c,d\}$	$\{b,c\}$	$\emptyset$	$\{b\}$
$\{c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{c,d\}$	$\{c\}$	$\{a,c,d\}$	$\{c\}$
$\{a,b,c\}$	X	X	X	X	$\{a,b\}$	$\{b,c\}$
$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b\}$	X	X	$\{a,b\}$	$\{b\}$
$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	X	X	$\{a,c,d\}$	$\{c\}$
$\{b,c,d\}$	X	X	$\{b,c,d\}$	$\{b,c\}$	$\{a,c,d\}$	$\{b,c\}$
X	X	X	X	X	X	X

$T_{12}^{po} = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{c,d\},\{a,b,c\},\{a,c,d\},\{b,c,d\},X\}$

$T_{12}^{pc} = \{\emptyset,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\},X\}$

$T_{21}^{po} = \{\emptyset,\{b\},\{c\},\{a,c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\},X.\}$

$T_{21}^{pc} = \{\emptyset,\{b\},\{c\},\{d\},\{a,d\},\{b,c\},\{b,d\},\{a,b,d\},\{a,c,d\},X\}$

$T_{12}^{\alpha o} = \{\emptyset,\{a\},\{a,b\},\{c,d\},\{a,c,d\},X\}$

$T_{12}^{\alpha c} = \{\emptyset,\{b\},\{a,b\},\{c,d\},\{b,c,d\},X\}$

$T_{21}^{\alpha o} = \{\emptyset,\{b\},\{c\},\{b,c\},X\}$

$T_{21}^{\alpha c} = \{\emptyset,\{a,d\},\{a,b,d\},\{a,c,d\},X\}$

For subset $A = \{b,c,d\}$ and $B = \{a,d\}$

$D_{12}^{p}(A) = \{d\}$ $D_{21}^{p}(A) = \{a\}$

$D_{12}^{p}(B) = \emptyset$ $D_{21}^{p}(B) = \{d\}$

$12 - p - int(A) = \{b,c,d\}$ $21 - p - int(A) =$ {b, c}

$12 - p - int(B) = \{a\}$ $21 - p - int(B) = \{a,d\}$

$12 - p - cl(A) = X$ $21 - p - cl(A) = X$

$12 - p - cl(B) = \{a,d\}$ $21 - p - cl(B) = \{a,d\}$

Theorem 1: Every $ij - \alpha - open$ set is $ij - p - open$.

Proof: Since, $T_i - int(A) \subseteq A$

$\Longrightarrow$ $T_j - cl\ (T_i - int(A)) \subseteq T_j - cl(A)$

$\Longrightarrow$ $T_i - int(T_j - cl\ (T_i - int(A))) \subseteq T_i - int(T_j - cl(A))$ ...(i)

So, If $A \subseteq T_i - int(T_j - cl\ (T_i - int(A)))$ then $A \subseteq T_i - int(T_j - cl(A))$

i.e. A is $ij - \alpha - open \Longrightarrow A$ is $ij - p - open$.

Corollary 1: Every $ij - \alpha - closed$ set is an $ij - p - closed$.

Proof: Let A is $ij - \alpha - closed$.

$\Longrightarrow A^c$ is $ij - \alpha - open$.

$\Longrightarrow A^c$ is $ij-p-open.$

$\Longrightarrow A$ is $ij-p-closed.$

Theorem 2: Let (X, T_1, T_2) a bitopological space. Where T_2 contains only $\emptyset$, X, and $\{a\}$ for a fixed $a \in X$ and a belongs to every T_1-open set other than $\emptyset$, then $T_{ij}^{po} = T_{ij}^{\alpha o}$

Proof:

Let $a \in X$ and let $A(\neq \emptyset, X)$ be an element of T_{12}^{po}.

We want show that $a \in A$. If not, then $T_2-cl(A) = \{a\}^c$

$$\Longrightarrow T_1-int(T_2-cl(A)) = T_1-int(\{a\}^c)$$

$$\Longrightarrow T_1-int(T_2-cl(A)) = \emptyset$$

$$\Longrightarrow A \nsubseteq T_1-int(T_2-cl(A)).$$ Which is a contradiction.

So $a \in A$. Also, since a belongs to every T_1-open set $\Longrightarrow \{a\} \subseteq T_1-int(A)$

$$\Longrightarrow T_2-cl(\{a\}) \subseteq T_2-cl\left(T_1-int(A)\right)$$

$$\Longrightarrow X \subseteq T_2-cl\left(T_1-int(A)\right)$$

$$\Longrightarrow X = T_2-cl\left(T_1-int(A)\right)$$

Now, $T_1-int\left(T_2-cl\left(T_1-int(A)\right)\right) = T_1-int(X) = X$

which contains A. Hence, $A \in T_{12}^{\alpha o}$.

i.e $T_{12}^{po} \subseteq T_{12}^{\alpha o}$, but we know that $T_{ij}^{\alpha o} \subseteq T_{ij}^{po}$. Thus $T_{ij}^{po} = T_{ij}^{\alpha o}$. Hence, the theorem.

Example 3: Let (X, T_1, T_2) be a bitopological space, where $X = \{a, b, c, d\}$,and

$T_1 = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}$

$T_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, d\}, X\}$

$T_1-closed\ sets\ are: \emptyset, \{d\}, \{c, d\}, X$

$T_2-closed\ sets\ are: \emptyset, \{c\}, \{c, d\}, \{b, c, d\}, X$

A⊆X	**T_1-int(A)**	**T_2-int(A)**	**T_1-cl(A)**	**T_2-cl(A)**	**T_1-int(T_2-cl(A))**	**T_2-int(T_1-cl(A))**	**T_1-int(T_2-cl(T_1-int(A)))**	**T_2-int(T_1-cl(T_2-int(A)))**
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	X	X	X	X	$\emptyset$	X
$\{b\}$	$\emptyset$	$\emptyset$	X	$\{b, c, d\}$	$\emptyset$	X	$\emptyset$	$\emptyset$
$\{c\}$	$\emptyset$	$\emptyset$	$\{c. d\}$	$\{c\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{c, d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	X	X	X	X	X	X
$\{a, c\}$	$\emptyset$	$\{a\}$	X	X	X	X	$\emptyset$	X
$\{a, d\}$	$\emptyset$	$\{a\}$	X	X	X	X	$\emptyset$	X
$\{b, c\}$	$\emptyset$	$\emptyset$	X	$\{b, c, d\}$	$\emptyset$	X	$\emptyset$	$\emptyset$
$\{b, d\}$	$\emptyset$	$\emptyset$	X	$\{b, c, d\}$	$\emptyset$	X	$\emptyset$	$\emptyset$
$\{c, d\}$	$\emptyset$	$\emptyset$	$\{c, d\}$	$\{c, d\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a, b, c\}$	$\{a, b, c\}$	$\{a. b\}$	X	X	X	X	X	X
$\{a, b, d\}$	$\{a, b\}$	$\{a, b, d\}$	X	X	X	X	X	X
$\{a, c, d\}$	$\emptyset$	$\{a\}$	X	X	X	X	$\emptyset$	X
$\{b, c, d\}$	$\emptyset$	$\emptyset$	X	$\{b, c, d\}$	$\emptyset$	X	$\emptyset$	$\emptyset$
X	X	X	X	X	X	X	X	X

$T_{12}^{po} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$

$T_{21}^{po} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$

$T_{12}^{\alpha o} = \{\emptyset, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$

$T_{12}^{\alpha o} = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\ X\}$

Lemma 1: If there exists $a \in X$ such that {a} is the smallest element of $(T_2 \setminus\{\emptyset\}, \subseteq)$, then every non-empty $ij - pre - open$ set contains $\cap\{G_i \mid G_i \in T_2 \setminus\{\emptyset\}; i = 1,2,3,..\}$.

Proof. If $\{a\}$ is the smallest element of $(T_2 \setminus \{\emptyset\}, \subseteq)$, then

$$\cap\{G_i \mid G_i \in T_2 \setminus\{\emptyset\}; i = 1,2,3,..\} = \{a\}$$

Let A be a non-empty $ij - pre - open$ set in X.

We want show that $a \in A$. If not, then $T_2 - cl(A) \subseteq \{a\}^c$

$$\Rightarrow \quad T_1 - int(T_2 - cl(A)) \subseteq T_1 - int(\{a\}^c)$$

$$\Rightarrow \quad T_1 - int(T_2 - cl(A)) \subseteq \emptyset$$

$$\Rightarrow \quad A \subseteq \emptyset \text{ as } A \subseteq T_1 - int(T_2 - cl(A))$$

which is a contradiction. Hence $a \in A$. Hence the theorem.

Theorem 3: Let (X, T_1, T_2) a bitopological space. If there exists $a \in X$ such that {a} is the smallest element of $(T_2\setminus\{\emptyset\}, \subseteq)$ and every non-empty $T_1 - open$ set contains a then $T_{12}^{po} = T_{12}^{\alpha o}$

Proof: Let A be a non-empty $ij - pre - open$ set in X.

From lemma (1) $a \in A$.

Also, since a belongs to every $T_1 - open$ set $\Rightarrow \{a\} \subseteq T_1 - int(A)$

$$\Rightarrow T_2 - cl(\{a\}) \subseteq T_2 - cl\left(T_1 - int(A)\right)$$

$$\Rightarrow X \subseteq T_2 - cl\left(T_1 - int(A)\right)$$

$$\Rightarrow X = T_2 - cl\left(T_1 - int(A)\right)$$

Now, $T_1 - int\left(T_2 - cl\left(T_1 - int(A)\right)\right) = T_1 - int(X) = X$

which contains A. Hence, $A \in T_{12}^{\alpha o}$.

i.e $T_{12}^{po} \subseteq T_{12}^{\alpha o}$, but we know that $T_{12}^{\alpha o} \subseteq T_{12}^{po}$. Thus $T_{12}^{po} = T_{12}^{\alpha o}$. Hence, the theorem.

Theorem 4. For any subsets A and B of (X, T_1, T_2), the following assertions are valid:

1. $D_{ij}^{p}(A) \subseteq D_{ij}^{\alpha}(A)$
2. If $A \subseteq B$, then $D_{ij}^{p}(A) \subseteq D_{ij}^{p}(B)$
3. $D_{ij}^{p}(A) \cup D_{ij}^{p}(B) \subseteq D_{ij}^{p}(A \cup B)$ & $D_{ij}^{p}(A \cap B) \subseteq D_{ij}^{p}(A) \cap D_{ij}^{p}(B)$
4. $\left\{D_{ij}^{p}\left(D_{ij}^{p}(A)\right) \setminus A\right\} \subseteq D_{ij}^{p}(A)$
5. $D_{ij}^{p}\left(A \cup D_{ij}^{p}(A)\right) \subseteq A \cup D_{ij}^{p}(A)$

Proof:

1. Let $x \in D_{ij}^{p}(A)$
   $\Rightarrow \forall\ G \in T_{ij}^{po}, x \in G \Rightarrow G \cap (A\setminus\{x\}) \neq \emptyset$
   $\Rightarrow \forall\ G \in T_{ij}^{\alpha o}, x \in G \Rightarrow G \cap (A\setminus\{x\}) \neq \emptyset$ as $T_{12}^{\alpha o} \subseteq T_{12}^{po}$
   $\Rightarrow x \in D_{ij}^{\alpha}(A)$
2. Let $x \in D_{ij}^{p}(A)$

   $\Rightarrow \forall\ G \in T_{ij}^{po}, x \in G \Rightarrow G \cap (A\setminus\{x\}) \neq \emptyset$

$\Rightarrow \forall\, G \in T_{ij}^{po}, x \in G \Rightarrow G \cap (B\backslash\{x\}) \neq \emptyset$ as $A \subseteq B$

$\Rightarrow x \in D_{ij}^{p}(B)$

3. Let $x \in D_{ij}^{p}(A) \cup D_{ij}^{p}(B)$
$\Rightarrow \forall\, G \in T_{ij}^{po}, x \in G \Rightarrow G \cap (A\backslash\{x\}) \neq \emptyset$ or $\Rightarrow G \cap (B\backslash\{x\}) \neq \emptyset$
$\Rightarrow \forall\, G \in T_{ij}^{po}, x \in G \Rightarrow G \cap ((A \cup B)\backslash\{x\}) \neq \emptyset$
$\Rightarrow x \in D_{ij}^{p}(A \cup B)$
The second part has similar proof.
4. Let $x \in D_{ij}^{p}\left(D_{ij}^{p}(A)\right)\backslash A$
$\Rightarrow \forall\, G \in T_{ij}^{po}, x \in G \Rightarrow G \cap (D_{ij}^{p}(A)\backslash\{x\}) \neq \emptyset$

 Let $y \in D_{ij}^{p}(A)\backslash\{x\}$. Then $y \in G$ and $y \in D_{ij}^{p}(A)$ so, $G \cap (A\backslash\{y\}) \neq \emptyset$

 Let $z \in G \cap (A\backslash\{y\})$ then $x \neq z$ because $x \notin A$.

 Hence, $G \cap (A\backslash\{x\}) \neq \emptyset$

 Therefor $x \in D_{ij}^{p}(A)$

5. Let $x \in D_{ij}^{p}\left(A \cup D_{ij}^{p}(A)\right)$. If $x \in A$ then the result is obvious. So let us assume that $x \notin A$ then $G \cap (A \cup D_{ij}^{p}(A)\backslash\{x\}) \neq \emptyset$ $\forall\, G \in T_{ij}^{po}$ with $x \in G$. Then either $G \cap (A\backslash\{x\}) \neq \emptyset$ or $G \cap (D_{ij}^{p}(A)\backslash\{x\}) \neq \emptyset$. The first case implies that $x \in D_{ij}^{p}(A)$ but if $G \cap (D_{ij}^{p}(A)\backslash\{x\}) \neq \emptyset$ then $x \in D_{ij}^{p}\left(D_{ij}^{p}(A)\right)$. Also, since $x \notin A$, we have $x \in D_{ij}^{p}\left(D_{ij}^{p}(A)\right)\backslash A$. Thus from (4) $x \in D_{ij}^{p}(A)$. Hence the theorem.

Theorem 5: Let A be a subset of bitopological space (X, T_1, T_2), then $D_{ij}^{p}(A) \subseteq Cl_{ij}^{p}(A)$

Proof. Let $x \in D_{ij}^{p}(A)$ i.e. x is a limit point of A. Then obviously $x \in Cl_{ij}^{p}(A)$ because if $x \notin Cl_{ij}^{p}(A)$, then there exists a $ij - p - closed$ set F such that $A \subseteq F$ and $x \notin F$. Hence $X\backslash F$ is a $ij - p - open$ set containing x and $A \cap (X\backslash F) \subseteq A \cap (X\backslash A) = \emptyset$. i.e. x is not a limit point of A. This is a contradiction.

Theorem 6. Let A be a subset of bitopological space (X, T_1, T_2), $Cl_{ij}^{p}(A) = A \cup D_{ij}^{p}(A)$,

Proof: Let $x \in Cl_{ij}^{p}(A)$. Let $x \notin A$ then $x \in D_{ij}^{p}(A)$. If not, and let $G \in T_{ij}^{po}$ & $x \in G$

Then $G \cap (A - \{x\}) = G \cap A \neq \emptyset$, and so $x \in D_{ij}^{p}(A)$. Hence $D_{ij}^{p}(A) \subseteq Cl_{ij}^{p}(A)$. The reverse

inclusion is by $A \subseteq Cl_{ij}^{p}(A)$ and Corollary 2.

Theorem 7: Let A and B be subsets of X. If $A \in T_{ij}^{po}$ and T_{ij}^{po} is a topology on X, then $A \cap Cl_{ij}^{p}(B) \subseteq Cl_{ij}^{p}(A \cap B)$.

Proof: Let $x \in A \cap Cl_{ij}^{p}(B)$. Then $x \in A$ and $x \in Cl_{ij}^{p}(B) = B \cup D_{ij}^{p}(B)$. If $x \in B$, then $x \in A \cap B \subseteq Cl_{ij}^{p}(A \cap B)$. If $x \notin B$, then $x \in D_{ij}^{p}(B)$ and so $G \cap B \neq \emptyset$ for all $ij - pre - open$ set G containing x. Since $A \in T_{ij}^{po}$, $G \cap A$ is also a $ij - pre - open$ set containing x. Hence $G \cap (A \cap B) = (G \cap A) \cap B \neq \emptyset$, and consequently $x \in D_{ij}^{p}(A \cap B) \subseteq Clp(A \cap B)$. Therefore $A \cap Cl_{ij}^{p}(B) \subseteq Cl_{ij}^{p}(A \cap B)$.

Example 4:

Let (X, T_1, T_2) be a bitopological space, where $X = \{a, b, c, d\}$,

$T_1 = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $T_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$.

Then we have

$T_{12}^{po} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$ which is a topology on X also we have

$T_{12}^{pc} = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$.

Let $A = \{a, b, c\}$ and $B = \{c, d\}$ be two subsets of X and $\{a, b, c\} \in T_{12}^{po}$.

Now, $A \cap Cl_{12}^{p}(B) = \{a, b, c\} \cap \{c, d\} = \{c\}$ but $Cl_{12}^{p}(A \cap B) = Cl_{12}^{p}(\{c\}) = \{c, d\}$

This shows that in the theorem 7, equality does not hold.

Example 5:

Let (X, T_1, T_2) be a bitopological space, where $X = \{a, b, c, d\}$,

$T_1 = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$ and $T_2 = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$.

Then we have

$T_{21}^{po} = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X.\}$ is not a topology on X. Also, we have

$T_{21}^{pc} = \{\emptyset, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, X\}$.

Let $A = \{a, c\}$ and $B = \{b, c, d\}$ be two subsets of X and $\{a, c\} \in T_{12}^{po}$.

Now, $A \cap Cl_{12}^{p}(B) = \{a, c\} \cap X = \{a, c\}$ but $Cl_{12}^{p}(A \cap B) = Cl_{12}^{p}(\{c\}) = \{c\}$

i.e. $A \cap Cl_{ij}^{p}(B) \nsubseteq Cl_{ij}^{p}(A \cap B)$

This shows that if T_{ij}^{po} is not a topology on X then the result in theorem 7 is not true in general.

Theorem 8: Let A and B subsets of a bitopological space (X, T_1, T_2). If A is $ij - pre - closed$, then $Cl_{ij}^{p}(A \cap B) \subseteq A \cap Cl_{ij}^{p}(B)$

Proof. If A is $ij - pre - closed$, then $Cl_{ij}^{p}(A) = A$

So $Cl_{ij}^{p}(A \cap B) \subseteq Cl_{ij}^{p}(A) \cap Cl_{ij}^{p}(B) = A \cap Cl_{ij}^{p}(B)$ which is the desired result.

Lemma 2: A subset A of a bitopological space X is $ij - pre - open$ if and only if there exists an $T_i - open$ set H in X such that $A \subseteq H \subseteq T_j - Cl(A)$.

Proof: It has a straightforward proof.

Lemma 3: The intersection of a $T_i - open$ set and a $ij - pre - open$ set is a $ij - pre - open$ set.

Proof. Let A be an $T_i - open$ set in X and B a $ij - pre - open$ set in X. Then there exists an $T_i - open$ set G in X such that $B \subseteq G \subseteq T_j - Cl(B)$. It follows that

$A \cap B \subseteq A \cap G \subseteq A \cap \{T_j - Cl(B)\} \subseteq T_j - Cl(A \cap B)$.

Now since $A \cap G$ is $T_i - open$, it follows from Lemma 2 that $A \cap B$ is pre-open.

Theorem 9: If A is a subset of a bitopological space (X, T_1, T_2) and T_1 is discrete topology on X, then $D_{12}^{p}\ (A) = \emptyset$.

Proof. Let x be any element of X. Since T_1 is discrete topology on X so every subset of X is $T_1 - open$, and so $12 - pre - open$. In particular, the singleton set $G = \{x\}$ is $12 - pre - open$. But $x \in G$ and $G \cap A = \{x\} \cap A \subseteq \{x\}$. Hence x is not a $12 - pre - limit$ point of A, and so $D_{12}^{p}(A) = \emptyset$. _

Theorem 10: For every subset A of a bitopological space X, we have A is $ij - pre - closed$ if and only if $D_{ij}^{p}(A) \subseteq A$.

Proof. Let us assume that A is $ij - pre - closed$. Let $x \notin A$, i.e., $x \in X \backslash A$. Since $X \backslash A$ is $ij - pre - open$, x is not a $ij - pre - limit\ point$ of A, i.e., $x \notin D_{ij}^{p}(A)$. Hence $D_{ij}^{p}(A) \subseteq A$. The reverse implication is by Theorem 5.

Theorem 11: Let A be a subset of X. If F is a $ij - pre - closed$ superset of A, then $D_{ij}^{p}(A) \subseteq F$.

Proof. By Theorem 4 and Theorem 10, A ⊆ F implies $D_{ij}^{p}(A) \subseteq D_{ij}^{p}(F) \subseteq F$.

Theorem 12: Let A be a subset of X. If a point $x \in X$ is a $ij - pre - limit$ point of A, then x is also a pre-limit point of $A \backslash \{x\}$.

Proof. The proof is Straightforward.

Theorem 13: For subsets A and B of X, the following assertions are valid. $Int_{ij}^{\alpha}(A)$

(1) $Int_{ij}^{p}(A)$ is the union of all $ij-pre-open$ subsets of A;

(2) A is $ij-pre-open$ if and only if $A = Int_{ij}^{p}(A)$;

(3) Int_{ij}^{p} (Int_{ij}^{p} (A)) = Int_{ij}^{p} (A);

(4) $Int_{ij}^{p}(A) = A \backslash D_{ij}^{p}(X \backslash A)$.

(5) X \ Int_{ij}^{p} (A) = Cl_{ij}^{p} (X \ A).

(6) $X \backslash Cl_{ij}^{p}(A) = Int_{ij}^{p}(X \backslash A)$.

(7) $A \subseteq B \Rightarrow Int_{ij}^{p}(A) \subseteq Int_{ij}^{p}(B)$.

(8) $Int_{ij}^{p}(A) \cup Int_{ij}^{p}(B) \subseteq Int_{ij}^{p}(A \cup B)$.

(9) $Int_{ij}^{p}(A \cap B) \subseteq Int_{ij}^{p}(A) \cap Int_{ij}^{p}(B)$.

Proof. (1) Let $\{G_\lambda | \lambda \in \Lambda\}$ be a collection of all $ij-pre-open$ subsets of A. If $x \in Int_{ij}^{p}(A)$, then there exists $\mu \in \Lambda$ such that $x \in G_\mu \subseteq A$. Hence $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$ and so $Int_{ij}^{p}(A) \subseteq \bigcup_{i \in I} G_i$. On the other hand, if $y \in \bigcup_{\lambda \in \Lambda} G_\lambda$, then y$\in G_\nu \subseteq A$ for some $\nu \in \Lambda$. Thus $y \in Int_{ij}^{p}(A)$, and $\bigcup_{\lambda \in \Lambda} G_\lambda \subseteq Int_{ij}^{p}(A)$. Accordingly, $Int_{ij}^{p}(A) = \bigcup_{\lambda \in \Lambda} G_\lambda$

(2) Proof is straightforward.

(3) It follows from (1) and (2).

(4) If $x \in A \backslash D_{ij}^{p}(X \backslash A)$, then $x \notin D_{ij}^{p}(X \backslash A)$ and so there exists a $ij-pre-open$ set G containing x such that $G \cap (X \backslash A) = \emptyset$. Thus $x \in G \subseteq A$ and hence $x \in Int_{ij}^{p}(A)$. This shows that $A \backslash D_{ij}^{p}(X \backslash A) \subseteq Int_{ij}^{p}(A)$. Now let $x \in Int_{ij}^{p}(A)$. Since $Int_{ij}^{p}(A) \in T_{ij}^{p}$ and $Int_{ij}^{p}(A) \cap (X \backslash A) = \emptyset$, we have $x \notin D_{ij}^{p}(X \backslash A)$. Therefore $Int_{ij}^{p}(A) = A \backslash D_{ij}^{p}(X \backslash A)$.

(5) Using (4) and Theorem 6, we have

$X \backslash Int_{ij}^{p}(A) = X \backslash (A \backslash D_{ij}^{p}(X \backslash A)) = (X \backslash A) \cup D_{ij}^{p}(X \backslash A) = Cl_{ij}^{p}(X \backslash A)$.

(6) Using Theorem 5 and Theorem 6, we get

Int_{ij}^{p} (X \ A) = (X \ A) \ D_{ij}^{p}(A) = X \ (A ∪ D_{ij}^{p}(A)) = X \ Cl_{ij}^{p} (A).

(7) Let $A \subseteq B$ and $x \in Int_{ij}^{p}(A)$.

$\Rightarrow ij-p-open$ set G such that $x \in G \subseteq A$

$\Rightarrow ij-p-open$ set G such that $x \in G \subseteq B$

$\Rightarrow x \in Int_{ij}^{p}(B)$

Hence $Int_{ij}^{p}(A) \subseteq Int_{ij}^{p}(B)$

(8) Since $A \subseteq A \cup B$ & $B \subseteq A \cup B$, so using (7)

$\Rightarrow Int_{ij}^{p}(A) \subseteq Int_{ij}^{p}(A \cup B)$ & $Int_{ij}^{p}(B) \subseteq Int_{ij}^{p}(A \cup B)$

$\Rightarrow Int_{ij}^{p}(A) \cup Int_{ij}^{p}(B) \subseteq Int_{ij}^{p}(A \cup B)$

(9) Since $A \cap B \subseteq A$ & $A \cap B \subseteq B$

$\Rightarrow Int_{ij}^{p}(A \cap B) \subseteq Int_{ij}^{p}(A)$ & $Int_{ij}^{p}(A \cap B) \subseteq Int_{ij}^{p}(B)$

$\Rightarrow Int_{ij}^{p}(A \cap B) \subseteq Int_{ij}^{p}(A) \cap Int_{ij}^{p}(B)$

The converse of (7) in Theorem 33 is not true in general as seen in the following example.

Example 6: In the bitopological space (X, T_1, T_2) which is described in Example 3.

Let $A = \{b, c, d\}$ and $B = \{a, b\}$ be the subsets of X.

Then $Int_{ij}^{p}(A) = \emptyset$ and $Int_{ij}^{p}(B) = \{a, b\}$.

i.e. $Int_{ij}^{p}(A) \subseteq Int_{ij}^{p}(B)$ but $A \nsubseteq B$.

Conclusion: In this study, we defend the notion of $ij - p - limit$ points, $ij - p - derived$ set, $ij - p - interior$, $ij - p - closure$, $ij - \alpha - limit\ point$, $ij - \alpha - derived$, $ij - \alpha - interior$, $ij - \alpha - closure$. And we have discussed related properties.

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