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| RESEARCH ARTICLE

A Product Theorem for Boundary Tightness

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| ABSTRACT
Tightness at the boundary is a quantitative measure of the number of points of an open set that are needed to witness each point in the closure. It was recently proved a product theorem in the case that one of the two factors is compact and has small ordinary tightness. The problem of determining whether the same holds true for boundary tightness without restrictions in the product remains open for compact spaces with small boundary tightness. In this paper, we prove a generalization under the additional hypothesis that X is compact with boundary tightness at most an infinite cardinal κ and Y is a continuous image of a product of compact spaces each of which has tightness at most κ. The proof relies on the small-subproduct property of boundary tightness [8], Malykhin's [5] theorem on tightness of compact products, and the product theorem with compact factors of Krupski and Kucharski [3]. Thus, as an application, our result holds for dyadic compacta and provides examples for which both factors have uncountable tightness but their product has countable boundary tightness.

| KEYWORDS

boundary tightness; tightness; product spaces; compact spaces; dyadic compacta; continuous images
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1. Introduction

The products are an example of a natural test for cardinal characteristics in general topology. Even though all the factors have a particular closure characteristic, the product behaves differently from them. The concept of boundary tightness is a cardinal characteristic created to evaluate the closure property based on open sets. It can be much smaller than the ordinary tightness; therefore, results related to tightness cannot automatically answer boundary tightness results.

The area of cardinal invariants offers an overall method for comparing closure properties [2]. The product behavior has been one of the important problems for properties defined using sequences: Franklin [1] considered spaces for which sequences determine closure, while Simon [7] found a compact Fréchet space whose square is not a Fréchet space. This justifies the need for assumptions on the product beyond those assumed on each factor.

The notion of boundary tightness of a Tychonoff space X, denoted by $tb(X)$, is defined to be the smallest infinite cardinal κ such that every point in the closure of any open set is obtainable from a closed subset of this open set of cardinality at most κ. That is,

$$\mathrm{tb}(X) = \min\{\kappa \geq \omega : \mathrm{cl}(U) = U\{\mathrm{cl}(A) : A \subseteq U \text{ and } |A| \leq \kappa\} \text{ for every open set } U \subseteq X\}.$$

It is clear from the definitions that $tb(X) \leq t(X)$. This is because the ordinary definition of tightness holds for any subset of the space, whereas boundary tightness checks open subsets alone. In other words, countable tightness implies countable boundary tightness. However, the converse is not always true, and the distinction is useful in products alone.

Tkachuk [8] invented the concept of boundary tightness and established the reduction principle, which states that the product has boundary tightness at most κ iff each subproduct generated by no more than κ coordinates has that property. Tkachuk also

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showed that closed and open surjections do not raise boundary tightness and the continuous image of a product of compact spaces of countable tightness is of countable boundary tightness. Krupski and Kucharski [3] proved the following theorem: if Q is compact and $t(Q) \leq \kappa$ and X has $tb(X) \leq \kappa$, then $tb(Q \times X) \leq \kappa$. This article also notes the open problem of whether every product of two compact spaces with boundary tightness at most κ is of the same tightness.

Some other concepts that help to understand the background include the following: O-tightness was defined by Tkachenko [9] as part of his research about product spaces and C-embedded subspaces. The properties of κ-Fréchet-Urysohn spaces were investigated by Liu and Ludwig [4] as regards product spaces, and by Sakai [6] in the case of spaces of continuous functions.

The theorem stated below applies these results to weaken the hypothesis of small tightness for one factor and replace it with an appropriate hypothesis concerning the representation of the other factor. The compact factor for which it is sufficient to assume only small boundary tightness can be rather large. The second factor must be representable as a continuous image of a product of compact spaces, where each of the factors has tightness at most the same cardinal. All dyadic compact spaces are of this form. The proof is given below in full.

This contribution is a conditional theorem about permanence rather than an answer to the open problem of the compact product. The contribution lies in the exact set of products for which the weaker hypothesis $tb(X) \leq \kappa$ is valid.

2. Notation and established facts

2.1. Cardinal closure invariants

All spaces below are nonempty and Tychonoff. The closure of a set A in X is denoted by $\mathrm{cl}_X(A)$. Ordinary tightness $t(X)$ is the least infinite cardinal κ such that for every subset A of X and every point x in its closure, some subset of A of size at most κ still has x in its closure. Boundary tightness restricts A to be open, as in the definition in Section 1. Thus

$$\mathrm{tb}(X) \leq t(X).$$

A space has countable boundary tightness when $\mathrm{tb}(X) \leq \omega$. This property is also called having accessible boundaries of open sets. A dyadic compact space is a continuous image of a Cantor cube, that is, a product of copies of the two-point discrete space.

2.2. Three product and image theorems

The proof uses the following published results. They are stated here in the precise form needed.

Theorem 2.1 (Tkachuk's small-subproduct criterion [8]). Let $\{Z_j : j \in J\}$ be a family of spaces and let κ be infinite. Then

$$\mathrm{tb}(\textstyle\prod_{j\in J} Z_j) \leq \kappa \text{ if and only if } \mathrm{tb}(\prod_{j\in A} Z_j) \leq \kappa \text{ for every } A \subseteq J \text{ with } |A| \leq \kappa.$$

Theorem 2.2 (Malykhin's compact-product theorem [5]). If at most κ compact spaces are given and each has tightness at most κ, their product has tightness at most κ. The result applies also to the empty product, which is a singleton.

Theorem 2.3 (Krupski–Kucharski product theorem [3]). If Q is compact, $t(Q) \leq \kappa$, and X is Tychonoff with $\mathrm{tb}(X) \leq \kappa$, then

$$\mathrm{tb}(Q \times X) \leq \kappa.$$

Theorem 2.4 (Tkachuk's closed-map theorem [8]). If a space Z maps onto W by a closed continuous surjection, then

$$\mathrm{tb}(W) \leq \mathrm{tb}(Z).$$

Theorem 2.1 and Theorem 2.4 are from Tkachuk [8]. Theorem 2.3 is Theorem 4.1 of Krupski and Kucharski [3]. The compact-product tightness assertion in Theorem 2.2 is Malykhin's classical theorem [5].

3. A preliminary product bound

We first record the consequence of the small-subproduct criterion that controls the product presenting the second factor.

Proposition 3.1. Let κ be infinite and let $\{K_i : i \in I\}$ be compact spaces with $t(K_i) \leq \kappa$ for every i. Then

$$\mathrm{tb}(\textstyle\prod_{i\in I} K_i) \leq \kappa.$$

Proof. Fix $J \subseteq I$ with $|J| \leq \kappa$. The product over J is compact, and Theorem 2.2 gives

$$t(\textstyle\prod_{i\in J} K_i) \leq \kappa.$$

By the definition of boundary tightness,

$$tb(\prod_{i\in J} K_i) \leq t(\prod_{i\in J} K_i) \leq \kappa.$$

This holds for every subfamily indexed by at most κ coordinates. Theorem 2.1 now yields the stated bound for the full product. End of proof.

The proposition includes Tkachuk's [8] countable-boundary-tightness conclusion for products of compact spaces of countable tightness. The cardinal form is convenient here because it lets the compact factor in the main theorem be tested on all small subproducts at once.

4. Main theorem

Theorem 4.1. Let κ be an infinite cardinal. Suppose that X is compact and has boundary tightness at most κ. Suppose also that Y is a continuous image of

$$P = \prod_{i\in I} K_i,$$

where every K_i is compact and has tightness at most κ. Then

$$tb(X \times Y) \leq \kappa.$$

Proof. Choose a continuous surjection $f : P \to Y$. We first show that

$$tb(X \times P) \leq \kappa.$$

View $X \times P$ as a product with coordinate set $\{*\} \sqcup I$, placing X at the distinguished coordinate $*$ and K_i at coordinate i. By Theorem 2.1, it is enough to check every subproduct with at most κ coordinates.

First consider a subproduct that does not include $*$. It has the form

$$Q_J = \prod_{i\in J} K_i, \qquad \text{where } J \subseteq I \text{ and } |J| \leq \kappa.$$

By Theorem 2.2, $t(Q_J) \leq \kappa$. Therefore

$$tb(Q_J) \leq t(Q_J) \leq \kappa.$$

Now consider a subproduct that includes $*$. It is homeomorphic to $X \times Q_J$ for a set $J \subseteq I$ of size at most κ. The space Q_J is compact and has tightness at most κ. Apply Theorem 2.3 with Q_J as the compact factor and X as the factor of boundary tightness at most κ. Since products are unchanged up to homeomorphism when the factors are interchanged,

$$tb(X \times Q_J) = tb(Q_J \times X) \leq \kappa.$$

Thus every subproduct over at most κ coordinates has boundary tightness at most κ. Theorem 2.1 gives

$$tb(X \times P) \leq \kappa.$$

Finally, define the product map

$$g = id_X \times f : X \times P \to X \times Y.$$

It is a continuous surjection. The domain is compact, and the range is Hausdorff because X and Y are Tychonoff. Hence g is closed. By Theorem 2.4,

$$tb(X \times Y) \leq tb(X \times P) \leq \kappa,$$

which proves the theorem. End of proof.

5. Consequences and examples

5.1. Dyadic compact factors

Corollary 5.1. Let X be compact with $tb(X) \leq \kappa$, where κ is infinite. If Y is dyadic compact, then

$$tb(X \times Y) \leq \kappa.$$

Proof. By the definition of dyadic compactness, there is a continuous surjection from a Cantor cube 2^I onto Y. Each factor of that cube is compact and has tightness ω, which is at most κ. Theorem 4.1 applies. End of proof.

In particular, if X has countable boundary tightness and Y is dyadic compact, then the product also has countable boundary tightness. No assumption that Y itself has countable tightness is required.

5.2. A product with uncountable tightness in both factors

Example 5.2. Let

$$X = [0,1]^{\omega_1} \text{ and } Y = 2^{\omega_1}.$$

Each factor is a product of separable spaces. Tkachuk's [8] product theorem therefore gives

$$tb(X) \leq \omega \text{ and } tb(Y) \leq \omega.$$

Both factors nevertheless have uncountable tightness. Here is a direct verification for X. Let p be the point whose every coordinate equals 1, and let A consist of the points with finite nonempty support, where support is the set of coordinates at which the value is nonzero. Every basic neighborhood of p restricts only finitely many coordinates, so it contains a point of A whose values are 1 on those coordinates. Thus

$$p \in cl_X(A).$$

If B is a countable subset of A, the union of the supports of its elements is countable. Choose $\alpha < \omega_1$ outside this union. Every element of B has value 0 at α, so the neighborhood

$$U_\alpha = \{x \in X : x(\alpha) > 1/2\}$$

contains p and misses B. Consequently p is not in the closure of any countable subset of A, and $t(X) > \omega$. The identical construction in the Cantor cube, with finite-support points, gives $t(Y) > \omega$.

Since Y is dyadic compact, Corollary 5.1 now gives

$$tb([0,1]^{\omega_1} \times 2^{\omega_1}) \leq \omega,$$

even though each factor has uncountable tightness. This example separates product boundary tightness from ordinary product tightness and shows why the representation hypothesis in Theorem 4.1 is useful.

5.3. Relation to the general product question

Theorem 4.1 is about a special class of compact spaces Y, i.e., those spaces which appear as images of compact products with tightness bounded by κ. This theorem does not guarantee that any arbitrary compact space with $tb(Y) \leq \kappa$ can be represented in this way. Thus, this theorem does not resolve the open problem related to products of compact spaces.

This restriction is real and must remain in the statement while using the theorem. The theorem gives a verifiable sufficient condition such that one can establish a compact-factor representation for Y, and from there, the researcher uses boundary tightness of X to control the product.

6. Discussion and further questions

The proof is divided into two parts. Tkachuk's [8] subproduct lemma reduces the product $X \times P$ to no more than κ coordinates. Malykhin's [5] theorem controls subproducts that do not involve X by ordinary tightness. Krupski–Kucharski theorem [3] controls the remaining subproducts, which contain X under a weaker condition of $tb(X) \leq \kappa$. Then compactness ensures that the quotient product map is closed so that tightness can be transferred from $X \times P$ to $X \times Y$.

A number of natural questions emerge from this reasoning. The first is whether the representation property for Y can be relaxed while retaining compactness of X and $tb(X) \leq \kappa$. The second is whether there exist compact Y such that $tb(Y) \leq \kappa$ but for which this theorem is not applicable, although the product is boundary tight at most κ. The third is whether the assumption of compactness for X can be replaced with one that will make the product projection closed.

This holds for any infinite cardinal κ without invoking any further axiom of set theory. This is a conditional result proved by using known theorems; the proof does not solve the unrestricted product problem.

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